

INTERSECTION OF CONTINUA AND RECTIFIABLE CURVES

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ABSTRACT. We prove that for any non-degenerate continuum $K \subseteq \mathbb{R}^d$ there exists a rectifiable curve such that its intersection with K has Hausdorff dimension 1. This answers a question of B. Kirchheim.

1. INTRODUCTION

A topological space K is called a *continuum* if it is compact and connected. The following question was asked by B. Kirchheim [4].

Question 1.1. *Does there exist a non-degenerate curve (or more generally, a continuum) $K \subseteq \mathbb{R}^d$ such that every rectifiable curve intersects K in a set of Hausdorff dimension less than 1?*

The motivation behind this question was that in [3, Examples (b), p. 208.] Gromov implicitly suggested that such curves exist. In this paper we answer Question 1.1 in the negative.

Theorem 3.4. For any non-degenerate continuum $K \subseteq \mathbb{R}^d$ there exists a rectifiable curve such that its intersection with K has Hausdorff dimension 1.

Remark 1.2. Finding a 1-dimensional intersection is the best we can hope for, since any purely unrectifiable curve K in the plane (e.g., the Koch snowflake curve) has the property that the intersection of K and a rectifiable curve has zero \mathcal{H}^1 measure.

2. PRELIMINARIES

The diameter and the boundary of a set A are denoted by $\text{diam } A$ and ∂A , respectively. For $A \subseteq \mathbb{R}^d$ and $s \geq 0$ the *s-dimensional Hausdorff measure* is defined as

$$\mathcal{H}^s(A) = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^s(A), \text{ where}$$
$$\mathcal{H}_\delta^s(A) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } A_i)^s : A \subseteq \bigcup_{i=1}^{\infty} A_i, \forall i \text{ diam } A_i \leq \delta \right\}.$$

Then the *Hausdorff dimension* of A is

$$\dim_H A = \sup\{s \geq 0 : \mathcal{H}^s(A) > 0\}.$$

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Let $A \subseteq \mathbb{R}^d$ be non-empty and bounded, and let $\delta > 0$. Set

$$N(A, \delta) = \min \left\{ k : A \subseteq \bigcup_{i=1}^k A_i, \forall i \text{ diam } A_i \leq \delta \right\}.$$

The *upper Minkowski dimension* of A is defined as

$$\overline{\dim}_M(A) = \limsup_{\delta \rightarrow 0^+} \frac{\log N(A, \delta)}{-\log \delta}.$$

If $A \subseteq \mathbb{R}^d$ is non-empty and bounded, then it follows easily from the above definitions that

$$\dim_H A \leq \overline{\dim}_M(A).$$

For more information on these concepts see [2] or [5].

A continuous map $f: [a, b] \rightarrow \mathbb{R}^d$ is called a *curve*. Its *length* is defined as

$$\text{length}(f) = \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| : n \in \mathbb{N}^+, a = x_0 < \dots < x_n = b \right\}.$$

If $\text{length}(f) < \infty$, then f is said to be *rectifiable*. We say that f is *naturally parametrized* if for all $x, y \in [a, b]$, $x \leq y$ we have

$$\text{length}(f|_{[x, y]}) = |x - y|.$$

We simply write $\Gamma = f([a, b])$ instead of f if the parametrization is obvious or not important for us. For every non-degenerate rectifiable curve Γ we have $0 < \mathcal{H}^1(\Gamma) < \infty$, so $\dim_H \Gamma = 1$. If $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [a, b]$, then f is called *1-Lipschitz*. Every naturally parametrized curve is clearly 1-Lipschitz.

3. THE PROOF

First we need some lemmas. The following lemma is probably known, but we could not find a reference, so we outline its proof.

Lemma 3.1. *If a non-empty bounded set $A \subseteq \mathbb{R}^d$ has upper Minkowski dimension less than 1, then a rectifiable curve covers A .*

Proof. We can assume that A is compact and $A \subseteq [0, 1]^d$, since we can take its closure and transform it into the unit cube with a similarity, this does not change the upper Minkowski dimension of the set and the fact whether it can be covered by a rectifiable curve.

For every $n \in \mathbb{N}$ we divide $[0, 1]^d$ into non-overlapping cubes with edge length 2^{-n} in the natural way, and we denote the cubes that intersect A by

$$Q_{n,1}, Q_{n,2}, \dots, Q_{n,r_n},$$

where r_n is the number of such cubes. As every set with diameter at most 2^{-n} can intersect at most 3^d of the above cubes, we obtain $r_n \leq 3^d N(A, 2^{-n})$. Let us fix s such that $\overline{\dim}_M(A) < s < 1$. By the definition of upper Minkowski dimension there exists a constant $c_1 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$

$$(3.1) \quad r_n \leq c_1 \cdot 2^{sn}.$$

Let $n \in \mathbb{N}$ and $i \in \{1, \dots, r_n\}$ be arbitrarily fixed. Let $P_{n,i}$ be the vertex of $Q_{n,i}$ that is the closest to the origin. If $Q_{n+1,j_1}, \dots, Q_{n+1,j_m}$ are the next level cubes contained by $Q_{n,i}$, then consider the broken line

$$\Gamma_{n,i} = P_{n,i} P_{n+1,j_1} P_{n+1,j_2} \dots P_{n+1,j_m} P_{n,i}.$$

Thus

$$(3.2) \quad \text{length}(\Gamma_{n,i}) \leq (m+1) \text{diam } Q_{n,i} \leq 2m\sqrt{d}2^{-n}.$$

Let l_n be the sum of these lengths for all $i \in \{1, \dots, r_n\}$. Then (3.2) and (3.1) imply

$$(3.3) \quad l_n \leq 2r_{n+1}\sqrt{d}2^{-n} \leq 2c_1 \cdot 2^{s(n+1)}\sqrt{d}2^{-n} = c_2 2^{(s-1)n},$$

where $c_2 = c_1\sqrt{d}2^{s+1}$. We set

$$L_n = \sum_{k=0}^n l_k \quad \text{and} \quad L = \sum_{k=0}^{\infty} l_k.$$

Since $s < 1$, (3.3) implies $L < \infty$.

Now we define the rectifiable curve covering A . First we take the broken line $\Gamma_0 = \Gamma_{0,1}$ with its natural parametrization $g_0: [0, L_0] \rightarrow \Gamma_0$. Assume that the curves $g_k: [0, l_k] \rightarrow \Gamma_k$ are already defined for all $k < n$. At every point $P_{n,i}$, $i \in \{1, \dots, r_n\}$, we insert the broken line $\Gamma_{n,i}$ in Γ_{n-1} , so we obtain a naturally parametrized curve $g_n: [0, L_n] \rightarrow \Gamma_n$.

For every $n \in \mathbb{N}$ let us define $f_n: [0, L] \rightarrow \Gamma_n$ such that

$$f_n(x) = \begin{cases} g_n(x) & \text{if } x \in [0, L_n], \\ g_n(L_n) & \text{if } x \in [L_n, L]. \end{cases}$$

Now we prove that the sequence $\langle f_n \rangle$ uniformly converges. Let us fix $n \in \mathbb{N}$ and $x \in [0, L]$ arbitrarily. As $\sum_{n=0}^{\infty} l_n < \infty$, it is enough to prove that $|f_{n+1}(x) - f_n(x)| \leq l_{n+1}$. By construction there exists $y \in [0, L]$ such that $f_n(x) = f_{n+1}(y)$ and $|x - y| \leq l_{n+1}$. Since g_{n+1} is naturally parametrized, we obtain that

$$|f_{n+1}(x) - f_n(x)| = |f_{n+1}(x) - f_{n+1}(y)| \leq |x - y| \leq l_{n+1}.$$

Therefore $\langle f_n \rangle$ uniformly converges to some $f: [0, L] \rightarrow \mathbb{R}^d$. As a uniform limit of 1-Lipschitz functions f is also 1-Lipschitz, thus rectifiable.

It remains to prove that $A \subseteq f([0, L])$. Let $\vec{z} \in A$. We need to show that there is $x \in [0, L]$ such that $f(x) = \vec{z}$. For every $n \in \mathbb{N}$ there exists $i_n \in \{1, \dots, r_n\}$ such that $\vec{z} \in Q_{n,i_n}$. Let $x_n \in [0, L]$ such that $f_n(x_n) = P_{n,i_n}$ for all $n \in \mathbb{N}$. By choosing a subsequence we may assume that x_n converges to some $x \in [0, L]$. Therefore

$$f(x) = \lim_{n \rightarrow \infty} f_n(x_n) = \lim_{n \rightarrow \infty} P_{n,i_n} = \vec{z}.$$

The proof is complete. \square

The next lemma is [1, Lemma 6.1.25].

Lemma 3.2. *If A is a closed subspace of a continuum X such that $\emptyset \neq A \neq X$, then for every connected component C of A we have $C \cap \partial A \neq \emptyset$.*

We also need the following technical lemma.

Lemma 3.3. *Suppose that $K \subseteq \mathbb{R}^d$ is a continuum contained by a unit cube Q and K has a point on each of two opposite sides of Q . Then for any positive integer N we can find N pairwise non-overlapping cubes Q_1, \dots, Q_N with edge length $\frac{1}{N}$ such that for each $i \in \{1, \dots, N\}$ there exists a continuum $K_i \subseteq K \cap Q_i$ with the property that K_i has a point on each of two opposite sides of Q_i .*

Proof. Let $N \in \mathbb{N}^+$ be fixed. Set $S_0 = \{0\} \times [0, 1]^{d-1}$ and for all $i \in \{1, \dots, N\}$ consider

$$S_i = \{i/N\} \times [0, 1]^{d-1} \quad \text{and} \quad T_i = [(i-1)/N, i/N] \times [0, 1]^{d-1}.$$

We may assume that $Q = [0, 1]^d$ and that the two opposite sides intersecting K are S_0 and S_N . Let $\vec{x} \in K \cap S_0$ and $\vec{y} \in K \cap S_N$.

Now we prove that for each $i \in \{1, \dots, N\}$ there is a continuum $C_i \subseteq K \cap T_i$ such that $C_i \cap S_{i-1} \neq \emptyset$ and $C_i \cap S_i \neq \emptyset$. Let C_1 be the component of $K \cap T_1$ containing \vec{x} . Applying Lemma 3.2 for $X = K$, $A = K \cap T_1$, and $C = C_1$ yields that $C_1 \cap S_1 \neq \emptyset$. Let C'_2 be the component of $K \cap (T_2 \cup \dots \cup T_N)$ containing \vec{y} . Similarly as above, we obtain $C'_2 \cap S_1 \neq \emptyset$. If we continue this process, we get the required continua C_2, \dots, C_N .

Finally, for each $i \in \{1, \dots, N\}$ we construct a cube $Q_i \subseteq T_i$ with edge length $\frac{1}{N}$ and a continuum $K_i \subseteq Q_i$ such that K_i has a point on each of two opposite sides of Q_i . Clearly, the cubes Q_i will be pairwise non-overlapping, and it is enough to construct Q_1 and K_1 (one can get Q_i, K_i similarly). Let us consider the standard basis of \mathbb{R}^d : $\vec{e}_1 = (1, 0, \dots, 0), \dots, \vec{e}_d = (0, 0, \dots, 1)$. Set $A_1 = C_1$, $V_1 = \{0\} \times \mathbb{R}^{d-1}$, $W_1 = \{1/N\} \times \mathbb{R}^{d-1}$, $Z_1 = [0, 1/N] \times \mathbb{R}^{d-1}$, and $m(1) = 1$. Then the definitions yield that A_1 has a point on both $V_{m(1)}$ and $W_{m(1)}$. Let $j \in \{2, \dots, d\}$ and assume that A_k, V_k, W_k, Z_k , and $m(k)$ are already defined for all $k < j$ such that A_k has a point on both $V_{m(k)}$ and $W_{m(k)}$. Let $\vec{x}_j \in A_{j-1}$ be a point which has minimal j th coordinate, and let V_j be the affine hyperplane that is orthogonal to \vec{e}_j and contains \vec{x}_j . Set $W_j = V_j + \frac{1}{N}\vec{e}_j$, and let Z_j be the closed strip between V_j and W_j . If $A_{j-1} \subseteq Z_j$ then let $A_j = A_{j-1}$ and $m(j) = m(j-1)$. If $A_{j-1} \not\subseteq Z_j$ then let A_j be the component of \vec{x}_j in $A_{j-1} \cap Z_j$ and $m(j) = j$, in this case Lemma 3.2 yields $A_j \cap W_j \neq \emptyset$. Thus A_j has a point on both $V_{m(j)}$ and $W_{m(j)}$. Let $Q_1 = \bigcap_{j=1}^d Z_j$ and $K_1 = A_d$. Then $Q_1 \subseteq S_1$ is a cube with edge length $\frac{1}{N}$ and $K_1 \subseteq Q_1$ is a continuum. As K_1 has a point on both $V_{m(d)}$ and $W_{m(d)}$, we obtain that K_1 has a point on each of two opposite sides of Q_1 . The proof is complete. \square

Now we are ready to prove Theorem 3.4.

Theorem 3.4. *For any non-degenerate continuum $K \subseteq \mathbb{R}^d$ there exists a rectifiable curve such that its intersection with K has Hausdorff dimension 1.*

Proof. By considering a similar copy of K we may assume that K is contained by a unit cube Q and K has a point on each of two opposite sides of Q .

Let $\varepsilon > 0$ be arbitrary. First we prove the weaker result that there exists $A \subseteq K$ such that $1 - \varepsilon \leq \dim_H A = \overline{\dim}_M(A) < 1$. By Lemma 3.1 A is covered by a rectifiable curve. Let us fix an integer $N \geq 2$ for which $s := \frac{\log(N-1)}{\log N} \geq 1 - \varepsilon$. We construct $A \subseteq K$ such that $\dim_H A = \overline{\dim}_M(A) = s$. Set $\mathcal{I}_n = \{1, \dots, N-1\}^n$ for every $n \in \mathbb{N}^+$. Iterating Lemma 3.3 implies that for all $n \in \mathbb{N}^+$ and $(i_1, \dots, i_n) \in \mathcal{I}_n$ there are cubes $Q_{i_1 \dots i_n}$ in Q with edge length $\frac{1}{N^n}$ such that $Q_{i_1 \dots i_n} \subseteq Q_{i_1 \dots i_{n-1}}$, and there are continua $K_{i_1 \dots i_n} \subseteq K$ such that $K_{i_1 \dots i_n} \subseteq Q_{i_1 \dots i_n} \cap K_{i_1 \dots i_{n-1}}$ and $K_{i_1 \dots i_n}$ has a point on each of two opposite sides of $Q_{i_1 \dots i_n}$. Set

$$A_n = \bigcup_{i_1=1}^{N-1} \dots \bigcup_{i_n=1}^{N-1} K_{i_1 \dots i_n},$$

and let

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Clearly, $A \subseteq K$ is compact.

On the one hand, as $A \subseteq A_n$ and A_n is covered by $(N-1)^n$ many cubes of edge length $\frac{1}{N^n}$, we obtain that $N(A_n, \sqrt{d}/N^n) \leq (N-1)^n$ for all $n \in \mathbb{N}^+$. Therefore $\overline{\dim}_M(A) \leq \frac{\log(N-1)}{\log N} = s$.

On the other hand, we prove that $\mathcal{H}^s(A) > 0$. Assume that $A \subseteq \bigcup_{j=1}^{\infty} U_j$, it is enough to prove that $\sum_{j=1}^{\infty} (\text{diam } U_j)^s \geq \frac{1}{2^d(N-1)}$. Clearly, we may assume that U_j is a non-empty open set with $\text{diam } U_j < 1$ for each j , and the compactness of A implies that there is a finite subcover $A \subseteq \bigcup_{j=1}^k U_j$. Let us fix $n_0 \in \mathbb{N}^+$ such that $\frac{1}{N^{n_0}} < \min_{1 \leq j \leq k} \text{diam } U_j$. For $j \in \{1, \dots, k\}$ let

$$t_j = \# \{(i_1, \dots, i_{n_0}) \in \mathcal{I}_{n_0} : U_j \cap K_{i_1 \dots i_{n_0}} \neq \emptyset\}.$$

Since $A \subseteq \bigcup_{j=1}^k U_j$, we have

$$(3.4) \quad \sum_{j=1}^k t_j \geq (N-1)^{n_0}.$$

Now we show that for all $j \in \{1, \dots, k\}$

$$(3.5) \quad (\text{diam } U_j)^s \geq \frac{t_j}{2^d(N-1)^{n_0+1}}.$$

Let us fix $j \in \{1, \dots, k\}$. There exists $0 \leq m < n_0$ such that $\frac{1}{N^{m+1}} \leq \text{diam } U_j < \frac{1}{N^m}$. Clearly, the number of cubes $Q_{i_1 \dots i_m}$ at level m that intersect U_j is at most 2^d . Therefore $t_j \leq 2^d(N-1)^{n_0-m}$. On the other hand, $\text{diam } U_j \geq \frac{1}{N^{m+1}}$ implies $(\text{diam } U_j)^s \geq \frac{1}{(N-1)^{m+1}}$, and (3.5) follows. By (3.4) and (3.5) we obtain

$$\sum_{j=1}^k (\text{diam } U_j)^s \geq \sum_{j=1}^k \frac{t_j}{2^d(N-1)^{n_0+1}} \geq \frac{1}{2^d(N-1)}.$$

Hence $\mathcal{H}^s(A) > 0$. Therefore $\dim_H A \geq s$, so $s \leq \dim_H A \leq \overline{\dim}_M(A) \leq s$. Thus $1 - \varepsilon \leq \dim_H A = \overline{\dim}_M(A) < 1$.

Now we are in a position to prove that there exists a rectifiable curve Γ with $\dim_H(\Gamma \cap K) = 1$. Pick an arbitrary point $\vec{x} \in K$ and let K_n be the intersection of K and the closed ball of radius $1/2^n$ centered at \vec{x} . Let C_n denote the component of K_n containing \vec{x} . Since C_n is a non-degenerate continuum by Lemma 3.2, we know that there exists $A_n \subseteq C_n$ such that $1 - \frac{1}{n} \leq \dim_H A_n = \overline{\dim}_M(A) < 1$. Therefore Lemma 3.1 implies that there exist rectifiable curves Γ_n covering A_n . We may assume that the endpoints of Γ_n are in A_n . We can also assume that the length of Γ_n is at most $1/2^n$. (Otherwise we split up Γ_n into finitely many parts, each having length at most $1/2^n$; then one of these parts intersects A_n in a set of Hausdorff dimension at least $1 - \frac{1}{n}$.) Let us concatenate the curves Γ_n with line segments. Then the full length of the line segments is at most $2 \sum_{n=1}^{\infty} \frac{1}{2^n} = 2$, the full length of the curves Γ_n is at most $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, so we get a rectifiable curve Γ that covers $\bigcup_{n=1}^{\infty} A_n$. As $\dim_H(\bigcup_{n=1}^{\infty} A_n) = 1$, the intersection $\Gamma \cap K$ has Hausdorff dimension 1. The proof is complete. \square

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