Functional Analysis, BSM, Spring 2012

Midterm exam, March 26

1. (20 points)
Consider the following $\ell_\infty \to \ell_\infty$ operator:

$$T : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) \mapsto \left(\alpha_1, \frac{\alpha_1 + \alpha_2}{2}, \frac{\alpha_1 + \alpha_2 + \alpha_3}{3}, \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}, \ldots \right).$$

Show that T is bounded and determine $\|T\|$. Is T injective? Is T surjective?

2. (15 points)
Let $1 \leq p < q < \infty$. Show that $\ell_p \subsetneq \ell_q$.

3. (15 points)
Let (X, d) be a totally bounded metric space. Prove that any sequence in X has a subsequence that is Cauchy.

4. (15 points)
Let $(X, \|\cdot\|)$ be a normed space and let S be a basis of its dual space X^*. Prove that

$$\bigcap_{\Lambda \in S} \ker \Lambda = \{0\}.$$

5. (15 points)
Let (X, d) be a complete metric space. Suppose that X has no isolated points. Prove that X is uncountably infinite. (We say that x is an isolated point if there exists $r > 0$ such that $B_r(x) = \{x\}$.)

6. (20 points)
Consider $C[0,1]$ (the space of continuous $[0,1] \to \mathbb{R}$ functions) with the supremum norm:

$$\|f\| = \max_{x \in [0,1]} |f(x)| \text{ for any } f \in C[0,1].$$

Let T be the following $C[0,1] \to C[0,1]$ operator:

$$(Tf)(x) = xf(x).$$

Determine its point spectrum $\sigma_p(T)$, residual spectrum $\sigma_r(T)$ and spectrum $\sigma(T)$.

Extra problems:

7. Let T be as in Problem 1. Determine $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma(T)$. Is T compact?

8. Let X be a non-trivial normed space. Suppose that $ST - TS = I$ for some linear operators $S, T : X \to X$. Prove that at least one of the operators S, T is unbounded.