Functional Analysis, BSM, Spring 2012

Exercise sheet: spectrum; polar decomposition
Solutions

1. By definition, $\lambda \in \sigma(T)$ if and only if $\lambda I - T$ is invertible. We have seen that if S is invertible, then so is S^*. Since $(S^*)^* = S$, this means that S is invertible if and only if S^* is invertible. Using $(\lambda I - T)^* = \overline{\lambda} I - T^*$ the statement follows.

2. a) In a complex Hilbert space every positive operator is self-adjoint, and hence normal. Thus $\sigma(T) = \sigma_{ap}(T)$. Suppose that λ is an approximate eigenvalue of T, that is, there exist $x_n \in X$ such that $\|x_n\| = 1$ for all n and $\|Tx_n - \lambda x_n\| \to 0$. It follows that

$$|(Tx_n - \lambda x_n, x_n)| \leq \|Tx_n - \lambda x_n\| \|x_n\| \to 0.$$

Since

$$(Tx_n - \lambda x_n, x_n) = (Tx_n, x_n) - (\lambda x_n, x_n) = (Tx_n, x_n) - \lambda,$$

we get that $(Tx_n, x_n) \to \lambda$. Since T is positive, (Tx_n, x_n) is a nonnegative real number for all n, thus so is λ.
b) The same proof yields that every approximate eigenvalue of a self-adjoint operator T is real. (We need to use that if T is self-adjoint, then (Tx, x) is real for any $x \in H$.)

3. If T is unitary, then so is T^*. In particular, $\|T\| = \|T^*\| = 1$. Therefore both $\sigma(T)$ and $\sigma(T^*)$ are contained in the closed unit disk $\{\lambda : |\lambda| \leq 1\}$.

On the other hand, if T is unitary, then it is invertible and $T^{-1} = T^*$. Now suppose that there exists $\lambda \in \sigma(T)$ with $|\lambda| < 1$. Then $\lambda^{-1} \in \sigma(T^{-1}) = \sigma(T^*)$. Since $|\lambda^{-1}| > 1$, this contradicts that $\sigma(T^*)$ is contained in the closed unit disk.

4. Recall that $(\ker S)^{\perp} = \mathrm{cl}(\mathrm{ran}\, S^*)$. Using this for $S = \lambda I - T$:

$$\lambda \in \sigma_p(T) \Leftrightarrow \ker(\lambda I - T) \neq \{0\} \Leftrightarrow \mathrm{cl}(\mathrm{ran}(\lambda I - T)^*) \neq H \Leftrightarrow \mathrm{ran}(\overline{\lambda} I - T^*) \text{ is not dense.}$$

5. Recall that

$$\sigma_r(T) = \{\lambda : \ker(\lambda I - T) = \{0\} \text{ and } \mathrm{ran}(\lambda I - T) \text{ is not dense}\}.$$

If T is normal, then $\ker T = \ker T^* = (\mathrm{ran}\, T)^{\perp}$. It follows that for a normal operator the kernel is trivial if and only if the range is dense. Since $\lambda I - T$ is normal, we conclude that $\sigma_r(T) = \emptyset$.

6. Recall that S is invertible if and only if S is bounded below and $\mathrm{ran}\, S$ is dense. If $\lambda \in \sigma(T)$, then $\lambda I - T$ is not invertible, so either $\lambda I - T$ is not bounded below (i.e., λ is an approximate eigenvalue of T), or $\mathrm{cl}(\mathrm{ran}(\lambda I - T)) \neq H$. Since $\mathrm{cl}(\mathrm{ran}(\lambda I - T)) = (\ker(\lambda I - T)^*)^{\perp} = (\ker(\overline{\lambda} I - T^*))^{\perp}$, the latter means that $\ker(\overline{\lambda} I - T^*) \neq \{0\}$ (i.e., $\overline{\lambda}$ is an eigenvalue of T^*).

7. a) If R is the right shift operator, then $R^* R = I$, so $|R| = I$ and we must have $U = R$ in the polar decomposition.
b) If L is the left shift operator, then

$$L^* L : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (0, \alpha_2, \alpha_3, \ldots).$$

The square root of this operator is itself. This means that we can choose U to be the left shift.

8.* In a complex Hilbert space every positive operator is self-adjoint. So if $A_1 A_2$ is positive, then $A_1 A_2 = (A_1 A_2)^* = A_2^* A_1^* = A_2 A_1$. For the converse, suppose that A_1 and A_2 commute. Let $B_1 = \sqrt{A_1}$; B_1 is a positive operator for which $A_1 = B_1^2$. We also know from the square root lemma that B_1 commutes with A_2. Therefore

$$(A_1 A_2 x, x) = (B_1 B_1 A_2 x, x) = (B_1 A_2 x, B_1^* x) = (A_2 B_1 x, B_1 x) \geq 0.$$

9. We may assume that $\|A\| \le 1$ and $\|A_n\| \le 1$ for all n. Then with the notations of the proof of the square root lemma:

$$\sqrt{A} = I - \sum_{k=1}^{\infty} c_k (I-A)^k \text{ and } \sqrt{A_n} = I - \sum_{k=1}^{\infty} c_k (I-A_n)^k \text{ for all } n.$$

Recall that c_k are positive real numbers with $\sum_{k=1}^{\infty} c_k = 1$. Therefore

$$\left\|\sqrt{A} - \sqrt{A_n}\right\| \le \sum_{k=1}^{\infty} c_k \left\|(I-A)^k - (I-A_n)^k\right\|.$$

Now let $\varepsilon > 0$ be arbitrary. We choose K such that

$$\sum_{k=K+1}^{\infty} c_k < \varepsilon/4.$$

It is easy to see that for any fixed k, $\left\|(I-A)^k - (I-A_n)^k\right\| \to 0$ as $n \to \infty$. So there exists N such that $\left\|(I-A)^k - (I-A_n)^k\right\| \le \varepsilon/(2K)$ for all $1 \le k \le K$ and $n > N$. Since $\|(I-A)^k - (I-A_n)^k\| \le 2$,

$$\left\|\sqrt{A} - \sqrt{A_n}\right\| \le \sum_{k=1}^{K} \left\|(I-A)^k - (I-A_n)^k\right\| + \sum_{k=K+1}^{\infty} 2c_k < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for any $n > N$.

10. In view of the previous exercise, it suffices to show that $\|T_n^* T_n - T^*T\| \to 0$. We know that

$$\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \to 0.$$

So it is enough to prove that if $\|A_n - A\| \to 0$ and $\|B_n - B\| \to 0$, then $\|A_nB_n - AB\| \to 0$, but this is clear, since

$$\|A_nB_n - AB\| \le \|A_nB_n - AB_n\| + \|AB_n - AB\| = \|(A_n - A)B_n\| + \|A(B_n - B)\| \le \\ \|A_n - A\|\|B_n\| + \|A\|\|B_n - B\| \to 0.$$

11.* The statement will easily follow from the following lemma.
Lemma: Let $A_n, A \in B(H)$. Suppose that $A_n \in B(H)$ is invertible for each n and $\|A_n - A\| \to 0$ as $n \to \infty$. If A is not invertible, then A is not bounded below.
Proof of the lemma: Assume that A is bounded below, but not invertible. It means that $\operatorname{ran} A$ cannot be dense, thus $(\operatorname{ran} A)^\perp$ is not trivial: there exists $x \neq 0$ such that x is orthogonal to $\operatorname{ran} A$. We know that A_n is surjective for each n, so there exists $x_n \neq 0$ such that $A_n x_n = x$. Then

$$(A - A_n)x_n = Ax_n - A_nx_n = Ax_n - x.$$

Since Ax_n is in $\operatorname{ran} A$, it is orthogonal to x, so

$$\|(A - A_n)x_n\|^2 = \|Ax_n\|^2 + \|x\|^2 \ge \|Ax_n\|^2.$$

Therefore

$$\frac{\|Ax_n\|}{\|x_n\|} \le \frac{\|(A - A_n)x_n\|}{\|x_n\|} \le \|A - A_n\| \to 0$$

as $n \to \infty$, which contradicts that A is bounded below.

Now let λ be an arbitrary element of the boundary of the spectrum. Since the spectrum is closed, this means that $\lambda \in \sigma(T)$, but there exist $\lambda_n \notin \sigma(T)$ such that $\lambda_n \to \lambda$. So we can use the lemma with $A_n = \lambda_n I - T$ and $A = \lambda I - T$. We get that $\lambda I - T$ is not bounded below, that is, λ is an approximate eigenvalue. This is what we wanted to prove.