Functional Analysis, BSM, Spring 2012

Exercise sheet: Inner product spaces

Inner product space: a vector space H (over $K = \mathbb{R}$ or $\mathbb{C}$) equipped with an inner product $(\cdot, \cdot)\colon H \times H \to K$ satisfying

- $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$ and $(\alpha x, y) = \alpha(x, y)$ for any scalar $\alpha \in K$ and vectors $x, x_1, x_2, y \in H$;
- $(y, x) = \overline{(x, y)}$;
- $(x, x) > 0$ for any $0 \neq x \in H$.

It follows that $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$ and $(x, \alpha y) = \overline{\alpha}(x, y)$.
Induced norm: $\|x\| \overset{\text{def}}{=} \sqrt{(x, x)}$. We proved that this is a norm on H.
Cauchy inequality: $|(x, y)| \leq \|x\|\|y\|$.
Hilbert space: an inner product space is called Hilbert space if it is complete (as a normed space with the induced norm).
Orthogonality: x and y are orthogonal if $(x, y) = 0$. We write $x \perp y$.
The *orthogonal complement* of a set $M \subset H$ is defined as $M^\perp = \{x \in H : x \perp m \text{ for all } m \in M\}$.
Weak convergence: We say that (x_n) weakly converges to x ($x_n \xrightarrow{\text{w}} x$ in notation) if for any $y \in H$ we have $(x_n, y) \to (x, y)$.

1. Let H be an inner product space; suppose that $x_n \to x$ and $y_n \to y$. Show that $(x_n, y_n) \to (x, y)$.
2. a) Show that in an inner product space the *parallelogram law* is satisfied:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

b) Prove the *polarisation formula*: for real spaces

$$(x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right);$$

while for complex spaces

$$(x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right).$$

(This means that the inner product can be recovered from the norm.)
3. W9P1. (5 points) For which $1 \leq p \leq \infty$ is the ℓ_p-norm induced by an inner product?
4. W9P2. (5 points) Let H be an inner product space; $x_n, x \in H$. Prove that the following two assertions are equivalent:

- $x_n \to x$;
- $\|x_n\| \to \|x\|$ and $x_n \xrightarrow{\text{w}} x$.

5.* Let $(X, \|\cdot\|)$ be a real normed space that satisfies the parallelogram law. Prove that the norm $\|\cdot\|$ is induced by an inner product. Prove the statement for complex normed spaces as well.
6. W9P3. (5 points) Let H be an inner product space. As a normed space H has a completion $\widetilde{H}$, which is a Banach space. Show that $\widetilde{H}$ is a Hilbert space.
7. W9P4. (5 points) Let H be an inner product space and $M \subset H$ arbitrary subset.
a) Show that $M^\perp$ is a closed linear subspace of H.
b) Show that $M^\perp = (\operatorname{cl}(\operatorname{span} M))^\perp$, where $\operatorname{span} M$ is the linear subspace spanned by M.
c) Show that $M \subset \left(M^\perp\right)^\perp$.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`