Functional Analysis, BSM, Spring 2012

Exercise sheet: L_p spaces

Solutions

1. Suppose that $|f| \leq C_1$ μ-almost everywhere and $|g| \leq C_2$ μ-almost everywhere. It means that the *exceptional sets*

$$\{x \in X : |f(x)| > C_1\} \text{ and } \{x \in X : |g(x)| > C_2\}$$

have measure zero. So their union

$$E = \{x \in X : |f(x)| > C_1\} \cup \{x \in X : |g(x)| > C_2\}$$

also has measure zero. However, for any $x \notin E$ we have $|(f+g)(x)| \leq |f(x)| + |g(x)| \leq C_1 + C_2$. Thus $|f+g| \leq C_1 + C_2$ holds μ-almost everywhere and the statement easily follows

2. We need to prove that $|f| \leq \|f\|_\infty$ μ-almost everywhere. Let $C_n = \|f\|_\infty + 1/n$. We know that $|f| \leq C_n$ holds μ-almost everywhere. It means that the exceptional set

$$E_n = \{x \in X : |f(x)| > C_n\}$$

has measure zero. Then the countable union

$$E = \bigcup_{n=1}^{\infty} E_n$$

also has measure zero. If $x \notin E$, then $|f(x)| \leq C_n$ for all n, thus $|f(x)| \leq \|f\|_\infty$ as claimed.

3. We prove the statement for any space $L_2(\mu)$. We need to show that every Cauchy sequence (f_n) in $L_2(\mu)$ is convergent. Let

$$E_{n,m} = \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}.$$

By the previous exercise $\mu(E_{n,m}) = 0$. Hence

$$E = \bigcup_{n,m} E_{n,m}$$

also has measure zero. For any $x \notin E$ we have that $f_n(x)$ is a Cauchy sequence. Since $\mathbb{C}$ is complete, it follows that $f_n(x)$ converges to some $f(x)$. For $x \in E$ we define $f(x)$ arbitrarily. It suffices to show that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Since (f_n) is Cauchy, for any positive ε there exists N such that $\|f_n - f_m\|_\infty < \varepsilon$ if $m, n > N$. For $x \notin E$ it follows that $|f_n(x) - f_m(x)| < \varepsilon$ if $m, n > N$. Now let us fix $n > N$ and let m go to infinity; we get that $|f_n(x) - f(x)| \leq \varepsilon$. Since this is true for any $x \notin E$, we obtain $\|f_n - f\|_\infty \leq \varepsilon$ if $n > N$.

4. Let $X = L_\infty(\mathbb{R})$, let $Y \leq X$ be the space of continuous functions and $g \in X$ be (the equivalence class of) the function

$$g(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

Let Y_1 be the subspace spanned by Y and g. For a continuous function $f \in Y$ and a scalar α we set $\Lambda(f + \alpha g) = \alpha$. We claim that Λ is a bounded linear functional on Y_1. Since

$$\alpha = \lim_{x \to 0+} (f + \alpha g)(x) - \lim_{x \to 0-} (f + \alpha g)(x),$$

we get that

$$|\Lambda(f + \alpha g)| = |\alpha| \leq 2\|f + \alpha g\|_\infty.$$

Thus $\|\Lambda\| \leq 2$. By the Hahn-Banach theorem Λ can be extended to a bounded linear functional on X.

Second solution: Since the uniform limit of continuous functions is continuous, we have that Y is a closed subspace of X. It is clearly a proper subspace, too. By Extra problem 3a it follows that there exists a nonzero linear functional vanishing on Y.

5. a,b) It suffices to show that there exist distinct vectors $x \neq y$ such that

$$\|x\| = \|y\| = \left\| \frac{x+y}{2} \right\| = 1.$$

For $L_1(\mathbb{R})$ consider the functions

$$f_1(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and } f_2(t) = \begin{cases} 1, & \text{if } 2 \leq t \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\|f_1\|_1 = \|f_2\|_1 = \|(f_1+f_2)/2\|_1 = 1$.

For $L_\infty(\mathbb{R})$ let

$$g_1(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and } g_2(t) = \begin{cases} 1, & \text{if } -1 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|g_1\|_\infty = \|g_2\|_\infty = \|(g_1+g_2)/2\|_\infty = 1$.

c) We show that every Hilbert space H is uniformly convex. The parallelogram law yields that

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\| \frac{x+y}{2} \right\|^2 < 2+2-4(1-\delta) = 4\delta.$$

We get the desired inequality by setting $\delta = \varepsilon/4$.

6.* *Sketch of the proof:* we need to show that any function $f \in L_1[0,1]$ can be approximated arbitrarily closely (in the L_1-distance) by continuous functions.

- The characteristic function $\chi_{(a,b)}$ of an open interval (a,b) can clearly be approximated. So is the characteristic function of the finite union of open intervals.
- An open set U is the disjoint union of countably many open intervals. Thus χ_U can be approximated, too.
- Using the regularity of the Lebesgue measure it follows that χ_B can also be approximated for any Borel set B.
- It follows that step functions can also be approximated, which easily implies the general case.