Functional Analysis, BSM, Spring 2012

Exercise sheet: L_p spaces

1. Let $f, g \in \mathcal{L}_\infty(\mu)$. Show that $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

2. W8P1. (5 points) Let $f \in \mathcal{L}_\infty(\mu)$. Show that

$$\|f\|_\infty = \min\left\{C \in \mathbb{R} : |f(x)| \leq C \text{ for } \mu\text{-almost all } x \in X\right\}.$$

3. Prove that the space $(L_\infty(\mathbb{R}), \|\cdot\|_\infty)$ is complete.

4. Show that there is a nonzero bounded linear functional Λ on $L_\infty(\mathbb{R})$ such that $\Lambda f = 0$ for any continuous function f.

5. W8P2. (10 points) We say that a Banach space X is *uniformly convex* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1 \text{ and } \left\|\frac{x+y}{2}\right\| > 1 - \delta \text{ imply } \|x - y\| < \varepsilon.$$

(It can be proved that every uniformly convex Banach space is reflexive.)
a) Show that $L_1(\mathbb{R})$ is not uniformly convex.
b) Show that $L_\infty(\mathbb{R})$ is not uniformly convex.
c) Show that $L_2(\mathbb{R})$ is uniformly convex.

6.* Prove that $C[0,1]$ is dense in $L_1[0,1]$.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`