Functional Analysis, BSM, Spring 2012

Exercise sheet: Compact operators

Solutions

1. Let $\varepsilon = c/2$. Then any open ball of radius ε can clearly contain at most one x_n. Thus we need infinitely many open balls of radius ε to cover the sequence (x_n). It follows that X has no finite ε-lattice, so X is not totally bounded.

For the converse, suppose that X is not totally bounded. It means that there exists $\varepsilon > 0$ for which X has no finite ε-lattice. Pick an arbitrary $x_1 \in X$. Then $B_\varepsilon(x_1) \neq X$, otherwise $\{x_1\}$ would be a finite ε-lattice. So we can pick $x_2 \in X \setminus B_\varepsilon(x_1)$. Then $B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \neq X$, otherwise $\{x_1, x_2\}$ would be a finite ε-lattice. So we can pick $x_3 \in X \setminus B_\varepsilon(x_1) \setminus B_\varepsilon(x_2)$. If we keep doing this, we get an infinite sequence $x_1, x_2, \ldots$ such that $d(x_n, x_m) \geq \varepsilon$ for $n \neq m$.

2. a-b) The vectors $x_n = (0, 0, \ldots, 0, 1/2, 0, 0, \ldots)$ are in the unit ball of both ℓ_∞ and ℓ_1. If $n \neq m$, then

$$\|x_n - x_m\|_\infty = 1/2 \text{ and } \|x_n - x_m\|_1 = 1.$$

By Exercise 1 it follows that the unit balls are not totally bounded.

c) We claim that for any $\varepsilon > 0$ there exist $x_1, x_2, \ldots \in X$ such that $\|x_n\| = 1$ for all n and $\|x_n - x_m\| > 1 - \varepsilon$ if $n \neq m$. We construct such a sequence recursively. If we already have $x_1, \ldots, x_k$, then consider the linear subspace spanned by these vectors. It is a closed proper subspace (because it is finite dimensional), so by Riesz-lemma there exists x_{k+1} with $\|x_{k+1}\| = 1$ such that the distance of x_{k+1} from this closed subspace is more than $1 - \varepsilon$. In particular, for any $i = 1, \ldots, k$ we have $\|x_{k+1} - x_i\| > 1 - \varepsilon$.

3. The image of the open unit ball is the following set:

$$\left\{ (\beta_1, \beta_2, \ldots) : |\beta_n| < \frac{1}{n} \right\}.$$

We need to show that this set is totally bounded. Fix a positive integer N and consider sequences $(\beta_1, \beta_2, \ldots)$ with the following properties:

- if $j > N$, then $\beta_j = 0$;
- if $j \leq N$, then $\beta_j = a_j/N + ib_j/N$, where a_j, b_j are integers between $-N$ and N.

There are finitely many such sequences and the $2/N$-balls around these sequences cover $T(B_1(0))$.

The point spectrum $\sigma_p(T)$ is clearly $\{1, 1/2, 1/3, 1/4, \ldots\}$. Since the image of a bounded sequence $(\alpha_1, \alpha_2, \ldots)$ is a sequence converging to 0, we have $\operatorname{ran} T \subset c_0$. Since c_0 is closed, it follows that $\operatorname{cl}(\operatorname{ran} T) \subset c_0$, which means that $\operatorname{ran} T$ is not dense. So $0 \in \sigma_r(T) \subset \sigma(T)$. For a compact operator the spectrum is the same as the point spectrum except maybe the point 0, so we get that $\sigma(T) = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$ and $\sigma_r(T) = \{0\}$.

4. Similar arguments show that they are also compact.

5. a) We need to show that for any positive ε the image $T(B_1(0))$ has a finite ε-lattice. There exists N such that $\|T_N - T\| < \varepsilon/2$. Since T_N is compact, $T_N(B_1(0))$ has a finite $\varepsilon/2$-lattice: $y_1, \ldots, y_m \in Y$ such that

$$T_N(B_1(0)) \subset \bigcup_{i=1}^{m} B_{\varepsilon/2}(y_i).$$

We claim that $\{y_1, \ldots, y_m\}$ is an ε-lattice for $T(B_1(0))$. Pick an arbitrary $x \in B_1(0)$. There exists $1 \leq i \leq m$ for which $T_N x \in B_{\varepsilon/2}(y_i)$. Since $\|T_N x - Tx\| \leq \|T_N - T\| \|x\| < \varepsilon/2$, it follows that $Tx \in B_\varepsilon(y_i)$.

b) The image of the unit ball is contained by the ball in $\operatorname{ran} T$ of radius $\|T\|$. If $\operatorname{ran} T$ is finite dimensional, then this ball is totally bounded.

c) Consider the operators

$$T_n : (\alpha_1, \alpha_2, \ldots) \mapsto \left(\alpha_1, \frac{\alpha_2}{2}, \ldots, \frac{\alpha_n}{n}, 0, 0, \ldots \right).$$

They clearly have finite rank and $\|T - T_n\| = 1/(n+1)$. Using part a) and b) we conclude that T is compact.

6. For any $\varepsilon > 0$ we construct a finite ε-lattice of $T(B_1(0))$. It suffices to do that when $\varepsilon = 1/N$ for some positive integer N. Consider integer sequences $a_0, a_1, \ldots, a_N$ with the following property: $a_0 = 0$ and a_{k+1} is equal to $a_k - 1$, a_k or $a_k + 1$. There are finitely many sequences with this property. For any such sequence we consider the following function f: let $f(k\varepsilon) = a_k \varepsilon$ and let f be linear on each interval $[k\varepsilon, (k+1)\varepsilon]$. It is not hard to see that these functions form a finite ε-lattice for $T(B_1(0))$.

7.* We claim that the functions in $T(B_1(0))$ are uniformly equicontinuous, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $g \in T(B_1(0))$ and any $x_1, x_2 \in [0,1]$ with $|x_1 - x_2| < \delta$ we have $|g(x_1) - g(x_2)| < \varepsilon$. Since $[0,1] \times [0,1]$ is compact, the kernel function k is uniformly continuous. We will only need the weaker property that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|k(x_1, y) - k(x_2, y)| < \varepsilon$ if $|x_1 - x_2| < \delta$. It follows that for any $f \in B_1(0)$ and $x_1, x_2 \in [0,1]$ with $|x_1 - x_2| < \delta$:

$$|Tf(x_1) - Tf(x_2)| = \left| \int_0^1 (k(x_1, y) - k(x_2, y)) f(y) \, \mathrm{d}y \right| \leq \int_0^1 |k(x_1, y) - k(x_2, y)| |f(y)| \, \mathrm{d}y \leq \varepsilon.$$

We will also need that the functions in $T(B_1(0))$ are uniformly bounded. This is easy: k is continuous function on a compact set, so $|k(x,y)|$ has a maximum C. Then for any $f \in B_1(0)$ we have $|Tf(x)| \leq C$ for all x.

Using uniform equicontinuity and uniform boundedness one can construct a finite ε-lattice as in the solution of the previous exercise.

8. a) First suppose that S is compact. Since T is bounded,

$$ST\left(B_1^X(0)\right) \subset S\left(B_{\|T\|}^Y(0)\right) = \|T\| \cdot S\left(B_1^Y(0)\right),$$

which is totally bounded, because S is compact.

Now we suppose that T is compact. It means that for any $\varepsilon > 0$ there exists a finite ε-lattice of $T(B_1(0))$: $\{y_1, \ldots, y_m\} \subset Y$. It is easy to see that $\{Sy_1, \ldots, Sy_m\} \subset Z$ is an $\|S\|\varepsilon$-lattice for $ST(B_1(0))$. This implies that $ST(B_1(0))$ is totally bounded, ST is compact.

b) Let $X = \ell_\infty$ and let

$$T : (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots) \mapsto (0, \alpha_1, 0, \alpha_3, 0, \alpha_5, \ldots).$$

Since the image of the unit ball contains the unit ball of an infinite dimensional subspace, T cannot be compact. However, it is easy to see that $T^2 = 0$, so T^2 is compact.

9. Assume that T is surjective. Then by the open mapping theorem T is open. It means that $T(B_1(0))$ is an open set. In particular, it contains an open ball around 0. Since X is infinite dimensional, such a ball is not totally bounded, contradiction.

We have $0 \in \sigma_p(T)$ for $T : (\alpha_1, \alpha_2, \ldots) \mapsto (\alpha_1, 0, 0, \ldots)$.

Let

$$T : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto \left(\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, \frac{\alpha_4}{4}, \ldots\right)$$

as in Exercise 3. If we consider T as an $\ell_\infty \to \ell_\infty$ operator, then $0 \in \sigma_r(T)$. If we consider it as an $\ell_1 \to \ell_1$ operator, then $0 \in \sigma_c(T)$.

10. Assume that T is bounded below, that is, there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for any $x \in X$. It follows that $B_c^Y(0) \cap \operatorname{ran} T$ is contained by $T(B_1(0))$. (For any $y \in \operatorname{ran} T$ there is $x \in X$ for which $Tx = y$. If $\|y\| < c$, then $\|x\| \leq \|Tx\|/c = \|y\|/c < 1$.) If T has infinite rank (i.e., $\operatorname{ran} T$ is infinite dimensional), then $B_c^Y(0) \cap \operatorname{ran} T$ is not totally bounded by part c) of Exercise 2. Since $B_c^Y(0) \cap \operatorname{ran} T \subset T(B_1(0))$, this contradicts the compactness of T.

11. We claim that T is compact if and only if τ is a constant function. If τ is constant, then so is every function $f \circ \tau$, which means that $\operatorname{ran} T$ is one-dimensional, so T is indeed compact.

Now suppose that τ is not constant. It follows that there are disjoint closed intervals $I_1, I_2, \ldots \subset [0,1]$ such that their images $J_n = \tau(I_n)$ are pairwise disjoint. There clearly exist functions $f_n \in C[0,1]$ with the following properties:

- $0 \leq f_n(x) \leq 1$ for all $x \in [0,1]$;
- f_n vanishes outside J_n;
- there exists $x_n \in J_n$ for which $f_n(x_n) = 1$.

On the one hand, each f_n is in the closed unit ball of $C[0,1]$. On the other hand, if $n \neq m$, then the distance of Tf_n and Tf_m is 1. It follows that $T(\bar{B}_1(0))$ is not totally bounded, T is not compact.