Functional Analysis, BSM, Spring 2012

Exercise sheet: Baire category theorem and its consequences

Solutions

1. Let $q_1, q_2, \ldots$ be an enumeration of the rationals and let $F_n = \{q_n\}$. These are clearly closed sets containing no balls. However, their union is the whole space.

2. A subset of a metric space is dense if it has a point in every ball. So if G_n is open and dense, then its complement $F_n = X \setminus G_n$ is closed and contains no ball. By Baire category theorem it follows that $\bigcup_{n=1}^{\infty} F_n \neq X$, which yields that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$.

To prove that $\bigcap_{n=1}^{\infty} G_n$ is dense, we need to show that it has a point in every ball. Let us consider the closed ball $F = \bar{B}_r(x)$, which can be viewed as a complete metric space itself (with the original metric). The sets $G'_n = F \cap G_n$ are open and dense in F. So using the first part of the solution, we conclude that $\bigcap_{n=1}^{\infty} G'_n \neq \emptyset$, that is, $F \cap \bigcap_{n=1}^{\infty} G_n \neq \emptyset$. In other words, $\bigcap_{n=1}^{\infty} G_n$ has a point in the ball $F = \bar{B}_r(x)$.

For a non-complete metric space in which this is not true, consider the example in the previous exercise and let $G_n = X \setminus F_n = \mathbb{Q} \setminus \{q_n\}$.

3. Since $Y \neq X$, there exists $x \in X \setminus Y$. Since Y is a linear subspace, $y + \alpha x$ is outside Y for any $y \in Y$ and $\alpha > 0$. However, if α is small enough, then $y + \alpha x$ is clearly in the ball $B_r(y)$, which means that Y cannot contain this ball.

4. Let Y be a finite dimensional subspace of X. Assume that it is not closed, so there exists a sequence $x_1, x_2, \ldots \in Y$ converging to $x \notin Y$. Let us consider the linear subspace X' spanned by Y and x. Then X' is finite dimensional, and Y is clearly not closed in X' either. So it suffices to prove the statement in the finite dimensional X'.

We assume that X was finite dimensional in the first place. We need to use the fact that every finite dimensional normed space is complete. (See the extra problems.) Then both X and Y are complete, and a complete subspace of a complete space must be closed.

5. Assume that a complete normed space has a countably infinite basis: $x_1, x_2, \ldots$. Let F_n denote the linear subspace spanned by $x_1, \ldots, x_n$. It is closed (since it is finite dimensional) and contains no ball (since it is a proper linear subspace). Since $x_1, x_2, \ldots$ is a basis, $\bigcup_{n=1}^{\infty} F_n = X$, which contradicts Baire category theorem.

6. This follows from the previous exercise, since the vectors $e_n = (0, 0, \ldots, 0, 1, 0, 0, \ldots)$ form a countably infinite basis for X.

7. Assume that $\|T_1 x\|_Y, \|T_2 x\|_Y, \ldots$ is bounded for any x, that is, $\exists C_x$ such that $\|T_n x\|_Y \leq C_y$ for all n. By the uniform boundedness principle it follows that $\exists C$ such that $\|T_n\| \leq C$ for all n, which contradicts $\|T_n\| \to \infty$.

8. Let Λ_n be the following linear functional on c_0:

$$\Lambda_n(\beta_1, \beta_2, \ldots) = \alpha_1 \beta_1 + \cdots + \alpha_n \beta_n.$$

It is easy to see that Λ_n is bounded with $\|\Lambda_n\| = |\alpha_1| + \cdots + |\alpha_n|$. We know that for any fixed $x = (\beta_1, \beta_2, \ldots) \in c_0$ the sequence $\Lambda_n x$ is convergent, hence bounded. By the uniform boundedness principle this means that there exists C such that $\|\Lambda_n\| = |\alpha_1| + \cdots + |\alpha_n| \leq C$ for all n. It follows that $|\alpha_1| + |\alpha_2| + \cdots \leq C < \infty$.

A direct proof: assume that $|\alpha_1| + |\alpha_2| + \cdots = \infty$. We need to construct a sequence (β_n) converging to 0 for which $\sum_{n=1}^{\infty} \alpha_n \beta_n$ is not convergent. First assume that $\alpha_n \geq 0$. We can choose $1 = k_1 < k_2 < k_3 < \ldots$ such that for any $i \geq 1$ we have

$$\sum_{k_i \leq j < k_{i+1}} \alpha_j \geq 1.$$

For any $k_i \leq j < k_{i+1}$ set $\beta_j = 1/i$. Then

$$\sum_{k_i \leq j < k_{i+1}} \alpha_j \beta_j \geq \frac{1}{i}.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i} = \infty$, we get that $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, so it is not convergent.

In the general case (when α_n can be negative), we choose the sign of β_n to be the same as the sign of α_n.

9. Let Λ_n denote the bounded linear functional $f(x_n, \cdot) : Y \to \mathbb{R}$, that is, $\Lambda_n y = f(x_n, y)$. For a fixed y, the linear functional $f(\cdot, y) : X \to \mathbb{R}$ is bounded, that is, there exists C_y such that

$$|f(x, y)| \leq C_y \|x\|.$$

Since $\|x_n\| \to 0$, there exists M such that $\|x_n\| \leq M$ for all n. It follows that

$$|\Lambda_n y| = |f(x_n, y)| \leq C_y \|x_n\| \leq M \cdot C_y.$$

Thus the uniform boundedness principle yields that there exists C such that $\|\Lambda_n\| \leq C$ for all n. So

$$|f(x_n, y_n)| = |\Lambda_n y_n| \leq \|\Lambda_n\| \|y_n\| \leq C \|y_n\| \to 0 \text{ as } n \to \infty.$$

10. Consider the identity map on X:

$$T(x) = x \text{ for } x \in X.$$

We can view this map as a linear operator from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$. Since $\|x\|_2 \leq C\|x\|_1$, we have $\|T\|_{1,2} \leq C < \infty$. So T is a bounded operator; it is clearly bijective, so by the inverse mapping theorem $T^{-1} = T$ as an $(X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ operator is also bounded. With $D = \|T^{-1}\|_{2,1}$ we get the desired inequality $\|x\|_1 \leq D\|x\|_2$.

11. Consider the vector space ℓ_1 with the ℓ_1 and the ℓ_∞ norms. For any $x \in \ell_1$ it clearly holds that $\|x\|_\infty \leq \|x\|_1$. However, for $x_n = (1, 1, \ldots, 1, 0, 0, \ldots) \in \ell_1$ we have $\|x\|_\infty = 1$, but $\|x\|_1 = n$.

12. We will apply the uniform boundedness principle to the dual space X^*. The role of T_n will be played by $\hat{x}_n \in X^{**}$. Recall that $\hat{x}_n$ is defined as the bounded linear functional on X^* for which $\hat{x}_n \Lambda = \Lambda x_n$ $(\Lambda \in X^*)$. The assumption that (Λx_n) is bounded means that for any *vector* Λ in our space X^* the sequence $\hat{x}_n \Lambda$ is bounded. Using the uniform boundedness principle we get that there exists C such that $\|\hat{x}_n\| \leq C$ for all n. However, we proved that $\|x_n\| = \|\hat{x}_n\|$, we are done. (Note that we did not use the completeness of X. We needed that the dual space X^* is complete, which is true even if X is not.)