Functional Analysis, BSM, Spring 2012

Exercise sheet: Baire category theorem and its consequences

Baire category theorem. If (X,d) is a complete metric space and $X = \bigcup_{n=1}^{\infty} F_n$ for some closed sets F_n, then for at least one n the set F_n contains a ball, that is, $\exists x \in X, r > 0 : B_r(x) \subset F_n$.
Principle of uniform boundedness (Banach-Steinhaus). Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \|\cdot\|_Y)$ a normed space and $T_n : X \to Y$ a bounded operator for each $n \in \mathbb{N}$. Suppose that for any $x \in X$ there exists $C_x > 0$ such that $\|T_n x\|_Y \leq C_x$ for all n. Then there exists $C > 0$ such that $\|T_n\| \leq C$ for all n.
Inverse mapping theorem. Let X, Y be Banach spaces and $T : X \to Y$ a bounded operator. Suppose that T is injective ($\ker T = \{0\}$) and surjective ($\operatorname{ran} T = Y$). Then the inverse operator $T^{-1} : Y \to X$ is necessarily bounded.

1. Consider the set $\mathbb{Q}$ of rational numbers with the metric $d(x,y) = |x-y|$; $x, y \in \mathbb{Q}$. Show that the Baire category theorem does not hold in this metric space.

2. Let (X,d) be a complete metric space. Suppose that $G_n \subset X$ is a dense open set for $n \in \mathbb{N}$. Show that $\bigcap_{n=1}^{\infty} G_n$ is non-empty. Find a non-complete metric space in which this is not true.

In fact, we can say more: $\bigcap_{n=1}^{\infty} G_n$ is dense. Prove this using the fact that a closed subset of a complete metric space is also complete.

3. W4P1. (4 points) Let $(X, \|\cdot\|)$ be a normed space, $Y \leq X$, $Y \neq X$ a proper linear subspace. Prove that Y contains no ball (that is, its interior is empty).

4. W4P2. (12 points) Let $(X, \|\cdot\|)$ be a normed space. Prove that any finite dimensional linear subspace of X is closed.

5. W4P3. (8 points) Prove that the (vector space) dimension of any Banach space is either finite or uncountably infinite.

6. Let us consider the following complex vector space:

$$X = \{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{C} \text{ and } \alpha_i = 0 \text{ for all but finitely many } i\text{'s}\}.$$

Prove that there is no norm $\|\cdot\|$ on X such that $(X, \|\cdot\|)$ is complete.

7. W4P4. (3 points) Let X and Y be Banach spaces. Suppose that $\|T_n\| \to \infty$ as $n \to \infty$ for some bounded operators $T_n \in B(X,Y)$. Prove that there exists $x \in X$ such that the sequence $\|T_1 x\|_Y, \|T_2 x\|_Y, \ldots$ is unbounded.

8. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of real numbers with the following property: whenever we have a sequence β_n of real numbers converging to 0, the infinite sum $\sum_{n=1}^{\infty} \alpha_n \beta_n$ is convergent. Prove that

$$\sum_{n=1}^{\infty} |\alpha_n| < \infty.$$

(Give a direct proof and give a proof using the uniform boundedness principle.)

9. Let X and Y be real Banach spaces and let $f : X \times Y \to \mathbb{R}$ be a separately continuous bilinear mapping, that is, for any fixed $x \in X$ the function $f(x, \cdot) : Y \to \mathbb{R}$ is a bounded linear functional, and for any fixed $y \in Y$ the function $f(\cdot, y) : X \to \mathbb{R}$ is also a bounded linear functional. Prove that f is jointly continuous, that is, if $x_n \to 0$ and $y_n \to 0$, then $f(x_n, y_n) \to 0$.

10. W4P5. (8 points) Let X be a vector space. Suppose that we have two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are both Banach spaces. Prove that if there exists C such that $\|x\|_2 \leq C\|x\|_1$ for all $x \in X$, then there exists D such that $\|x\|_1 \leq D\|x\|_2$ for all $x \in X$.

11. W4P6. (5 points) Show that the statement of the previous exercise is not necessarily true if the normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are not complete.

12. W4P7. (15 points) Let (x_n) be a sequence in a Banach space X with the property that the sequence (Λx_n) is bounded for any $\Lambda \in X^*$. Prove that there exists C such that $\|x_n\| \leq C$ for all n.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`