Functional Analysis, BSM, Spring 2012

Exercise sheet: metric spaces and convergence

Let X be a set and $d : X \times X \to \mathbb{R}$ a distance function with the following properties:

- $d(x, y) \geq 0$ for any $x, y \in X$;
- $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) = d(y, x)$ for any $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$.

(X, d) is called a *metric space*; d is called a *metric* on X.

A sequence $x_1, x_2, \ldots \in X$ is said to *converge* to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ (that is, for any $\varepsilon > 0$ there exists an N such that $d(x_n, x) < \varepsilon$ for $n \geq N$). In notation: $x_n \xrightarrow{d} x$. A sequence $x_1, x_2, \ldots \in X$ is *convergent* if there exists an $x \in X$ for which $x_n \xrightarrow{d} x$. In certain topological spaces it can happen that a sequence converges to two different points. However, this cannot happen in a metric space (see Exercise 3).

A sequence $x_1, x_2, \ldots \in X$ is said to be a *Cauchy sequence* if for any $\varepsilon > 0$ there exists an N such that $d(x_m, x_n) < \varepsilon$ for $m, n \geq N$. We proved that every convergent sequence is a Cauchy sequence. We also showed an example for a Cauchy sequence that is not convergent. A metric space in which every Cauchy sequence converges is said to be *complete*.

A metric space (X, d) is called *totally bounded* if for any $\varepsilon > 0$ there exists a finite ε-lattice (i.e., a finite set of points $\{x_1, \ldots, x_N\} \subset X$ such that for any $x \in X$ there is an x_i with $d(x, x_i) < \varepsilon$).

A subset $S \subset X$ is said to be *dense* if every $x \in X$ is the limit point of some sequence in S, that is, $\forall x \in X\ \exists x_1, x_2, \ldots \in S$ such that $x_n \xrightarrow{d} x$. A metric space (X, d) is called *separable* if it has a countable dense subset.

1. Show that $|d(x_1, y) - d(x_2, y)| \leq d(x_1, x_2)$.

2. Consider the set of infinite 0-1 sequences

$$X = \{(a_1, a_2, \ldots) : a_i \in \{0, 1\}\}$$

with the following metric:

$$d\left((a_1, a_2, \ldots), (b_1, b_2, \ldots)\right) = 1/k,$$

where k is the smallest positive integer for which $a_k \neq b_k$. (If there is no such k, that is, $a_i = b_i$ for each i, then the two sequences are the same. In that case, let their distance be 0.) Prove that (X, d) is totally bounded. Also prove that it is separable.

3. Suppose that for a sequence $x_1, x_2, \ldots \in X$ we have $x_n \xrightarrow{d} x$ and $x_n \xrightarrow{d} y$. Prove that $x = y$.

4. Suppose that $x_n \xrightarrow{d} x$. Prove that $d(x_n, y) \to d(x, y)$ for any $y \in X$.

5. Suppose that $x_n \xrightarrow{d} x$ and $y_n \xrightarrow{d} y$. Prove that $d(x_n, y_n) \to d(x, y)$. Use this fact to give another solution for Exercise 3.

6. Show that a Cauchy sequence with a convergent subsequence is convergent. More precisely, let $x_1, x_2, \ldots \in X$ be a Cauchy sequence and suppose that there are positive integers $k_1 < k_2 < k_3 < \ldots$ such that $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ converge to some $x \in X$. Prove that $x_1, x_2, \ldots$ converge to x.

7. Consider $\ell_\infty = \{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{C}$ and $\alpha_1, \alpha_2, \ldots$ is a bounded sequence$\}$ with the following metric:

$$d\left((\alpha_1, \alpha_2, \ldots), (\beta_1, \beta_2, \ldots)\right) = \sup_{n \in \mathbb{N}} |\alpha_n - \beta_n|.$$

Show that this is indeed a metric and prove that this metric space is not separable.

8. Let (X, d) be a metric space. We define another distance function d' on X:

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that d' is also a metric on X. Prove that $x_n \xrightarrow{d} x \Leftrightarrow x_n \xrightarrow{d'} x$. (Note that $d'(x, y) < 1$ for any $x, y \in X$.)

9.* Let (X, d) be a **complete** metric space. A map $f : X \to X$ is called a *contraction* if there exists a $0 < q < 1$ such that

$$d\left(f(x), f(y)\right) \leq q d(x, y).$$

Prove that for every contraction f there exists a unique point $x \in X$ with $f(x) = x$.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`