Functional Analysis, BSM, Spring 2012

Exercise sheet: infinite dimensional vector spaces

Solutions

1. We need to show that $\ker T$ is closed under addition and multiplication with scalars.

Assume that $v_1, v_2 \in \ker T$, that is, $Tv_1 = Tv_2 = 0$. Then $T(v_1 + v_2) = Tv_1 + Tv_2 = 0 + 0 = 0$, which means that $v_1 + v_2 \in \ker T$.

If $v \in \ker T$, then for any scalar $\alpha \in F$ we have $T(\alpha v) = \alpha T(v) = 0$

2. First we show that if T is injective, then $\ker T = \{0\}$. Let v be any element in $\ker T$. Then $Tv = 0 = T0$, and injectivity implies that $v = 0$.

In the other direction, assume that $\ker T = \{0\}$. Suppose that for some $u, v \in V$ we have $Tu = Tv$. It follows that

$$T(u - v) = Tu - Tv = 0 \Rightarrow u - v \in \ker T = \{0\} \Rightarrow u - v = 0 \Rightarrow u = v.$$

3. The kernel is

$$\{(\alpha_1, 0, 0, 0, \ldots) : \alpha_1 \in \mathbb{C}\},$$

which is a one-dimensional linear subspace.

4. It is easy to check that $\ker(S \circ T) \supset \ker T$. Since $\ker \mathrm{id} = \{0\}$ and $\ker T \supsetneq \{0\}$, we get that $S \circ T$ cannot be the identity operator for any S.

5. The so-called right shift operator clearly has this property:

$$(\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (0, \alpha_1, \alpha_2, \alpha_3, \ldots).$$

6. Let L be an arbitrary linear functional on ℓ_∞ (i.e., an operator $L : \ell_\infty \to \mathbb{C}$). Then the operator

$$(\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (L(\alpha_1, \alpha_2, \alpha_3, \ldots), \alpha_1, \alpha_2, \alpha_3, \ldots)$$

satisfies $T \circ S = \mathrm{id}$. We will denote this operator by S_L. In fact, these are all the operators with the desired property.

Remark: What are the linear functionals on ℓ_∞? Of course, L can be any finite linear combination of the α_i's such as

$$L(\alpha_1, \alpha_2, \alpha_3, \ldots) = 3\alpha_4 - 5\alpha_7.$$

Some infinite linear combinations also give us linear functionals: let $(\beta_1, \beta_2, \ldots) \in \ell_1$ and set

$$L(\alpha_1, \alpha_2, \alpha_3, \ldots) = \sum_{i=1}^{\infty} \beta_i \alpha_i.$$

Actually, one can arbitrarily define L on a basis of ℓ_∞; any such function can be uniquely extended to a linear functional on ℓ_∞.

7. $\{0\} \subset \ker T \subset \ker(T \circ S) = \ker \mathrm{id} = \{0\}$. So $\ker T = \{0\}$, which is equivalent to injectivity.

8. Since T is injective, 0 cannot be an eigenvalue. Suppose that $v = (\alpha_1, \alpha_2, \ldots)$ is an eigenvector for S_L with eigenvalue $\lambda \neq 0$. We get that

$$v = (\alpha_1, \alpha_1\lambda^{-1}, \alpha_1\lambda^{-2}, \alpha_1\lambda^{-3}, \ldots).$$

We can assume that $\alpha_1 = 1$. Then v is an eigenvector if and only if

$$L(1, \lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \ldots) = \lambda.$$

So the eigenvalues for S_L are exactly those λ's that satisfy the above equation and for which $|\lambda| \geq 1$. (We need $|\lambda| \geq 1$, because otherwise the vector v would not be in ℓ_∞.)

a) L = 0; so $S_L : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (0, \alpha_1, \alpha_2, \alpha_3, \ldots)$.

b) There is no such S, because for $0 < |\lambda| < 1$ the above v is not in ℓ_∞.

c) $L(\alpha_1, \alpha_2, \ldots;) = \alpha_1$; so $S_L : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_1, \alpha_1, \alpha_2, \alpha_3, \ldots)$. The only eigenvalue is 1.

d-e) $L(\alpha_1, \alpha_2, \ldots;) = \alpha_{1000}$; so $S_L : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto (\alpha_{1000}, \alpha_1, \alpha_2, \alpha_3, \ldots)$, and our equation transforms to

$$\lambda^{-999} = \lambda \Leftrightarrow \lambda^{1000} = 1,$$

which has a thousand different complex roots (with absolute value 1).
f)* Surprisingly, L can be chosen in such a way that all the possible λ's (those with $|\lambda| \geq 1$) are eigenvalues. The reason for this is that the vectors

$$(1, \lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \ldots)$$

are linearly independent in ℓ_∞ as λ runs through $\mathbb{C} \setminus \{0\}$. Consequently, we can define L for each of these vectors independently, and it can be extended to a linear functional over ℓ_∞. So there exists a linear functional L with $L(1, \lambda^{-1}, \lambda^{-2}, \lambda^{-3}, \ldots) = \lambda$ for each $\lambda \neq 0$. It follows that the set of eigenvalues for S_L is $\{\lambda : |\lambda| \geq 1\}$.

To prove that these vectors are linearly independent, it suffices to show that any finite collection of them is independent. Let k be a positive integer, and suppose that $\lambda_1, \ldots, \lambda_k$ are distinct nonzero complex numbers. It is enough to consider the first k coordinates: the vectors $\left(1, \lambda_i^{-1}, \ldots, \lambda_i^{-(k-1)}\right)$; $i = 1, \ldots, k$ are already linearly independent, because the determinant of the corresponding matrix, which is a Vandermonde determinant, is nonzero.

9. $T_g \circ T_h = T_{gh}$.

10-13. Let h be a fixed continuous function on $[0,1]$ and let λ be any real number. We will give a necessary and sufficient condition for λ being an eigenvalue for T_h. Suppose that for some $f \in C[0,1]\{0\}$ we have

$$h(x)f(x) = \lambda f(x) \text{ for every } x \in [0,1] \Leftrightarrow (h(x) - \lambda)\, f(x) = 0 \text{ for every } x \in [0,1].$$

Let $A = \{x \in [0,1] : h(x) = \lambda\}$. The above equation says that f must be zero everywhere on $[0,1] \setminus A$. Since f is continuous, it follows that f must be zero in the closure of this set. However, the closure of this set is $[0,1]$ except when A contains an interval. Consequently, λ is an eigenvalue for T_h if and only if $h(x) = 0$ for every point x of an interval $I \subset [0,1]$.
10. The kernel is trivial if and only if 0 is not an eigenvalue. For $\lambda = 0$: if $h(x) = x + 1$, then $A = \emptyset$; if $h(x) = x - 1/2$, then $A = \{1/2\}$. Neither of them contains an interval, so 0 is not an eigenvalue in either case.
11. If $h \equiv 0$, then $\ker T_h = C[0,1]$. Another example:

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2 \\ x - 1/2 & \text{if } 1/2 < x \leq 1 \end{cases}$$

Then:

$$\ker T_h = \{f \in C[0,1] : f(x) = 0 \text{ for } 1/2 \leq x \leq 1\}.$$

12. Let h be constant λ_1 on an interval and some other constant λ_2 on an other interval, for example:

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/3 \\ x - 1/3 & \text{if } 1/3 < x \leq 2/3 \\ 1/3 & \text{if } 2/3 < x \leq 1 \end{cases}$$

13. We need to construct a continuous function on $[0,1]$ such that there are distinct real numbers $\lambda_1, \lambda_2, \ldots$ and disjoint intervals $I_1, I_2, \ldots \subset [0,1]$ in such a way that h is equal to λ_k on I_k. Set

$$I_k = \left[\frac{1}{2k}, \frac{1}{2k-1}\right] \text{ and } \lambda_k = \frac{1}{k}.$$

Set $h(0) = 0$ and extend h linearly between I_k and I_{k+1}. It is not hard to prove that h is a continuous function such that the eigenvalues for T_h are $1/k$; $k \in \mathbb{N}$.

However, there is no $h \in C[0,1]$ such that the set of eigenvalues for T_h is uncountable.