Functional Analysis, BSM, Spring 2012

Exercise sheet: infinite dimensional vector spaces

Let V and W be vector spaces over the same field F ($\mathbb{R}$ or $\mathbb{C}$). Let $T : V \to W$ be an *operator* or *linear transformation* (i.e., $T(v_1 + v_2) = Tv_1 + Tv_2$ and $T(\alpha v) = \alpha T(v)$ for any $v_1, v_2, v \in V$ and $\alpha \in F$). The *kernel* (or *null space*) of T is the set of those vectors $v \in V$ for which $Tv = 0$:

$$\ker T = \{v \in V : Tv = 0\}.$$

In fact, $\ker T$ is always a *(linear) subspace* of V. We say that a subset $U \subset V$ is a linear subspace of V if it is closed under addition (i.e., if $u_1, u_2 \in U$, then $u_1 + u_2 \in U$) and multiplication with scalars (i.e., if $\alpha \in F$ and $u \in U$, then $\alpha u \in U$). In notation: $U \leq V$. Note that a linear subspace $U \leq V$ is itself a vector space (with the same operations as V has). So it makes sense to talk about a basis of U or the dimension of U.

1. Show that $\ker T \leq V$.

2. Since $T0 = 0$, the null vector 0 is always in the kernel. Show that T is injective if and only if $\ker T = \{0\}$.

We showed that

$$\ell_\infty = \{(\alpha_1, \alpha_2, \ldots) : \alpha_i \in \mathbb{C} \text{ and } \alpha_1, \alpha_2, \ldots \text{ is a bounded sequence}\}$$

is a vector space over $\mathbb{C}$; let $T : \ell_\infty \to \ell_\infty$ be the left shift operator:

$$T(\alpha_1, \alpha_2, \alpha_3, \ldots) = (\alpha_2, \alpha_3, \alpha_4, \ldots).$$

We saw that the set of eigenvalues for T is $\{\lambda : |\lambda| \leq 1\}$.

3. What is the kernel of T? What is dimension of this kernel?

4. Show that there is no operator $S : \ell_\infty \to \ell_\infty$ such that $S \circ T = \mathrm{id}$.

5. Show that there is an operator $S : \ell_\infty \to \ell_\infty$ with the property $T \circ S = \mathrm{id}$.

6. Can you find another operator with the same property? Try to find as many as you can. (Don't forget that they have to be linear!)

7. Show that any such operator S is injective.

8. Can you find such an operator S with the additional property that S has

a) no eigenvalue;

b) an eigenvalue λ with $0 < |\lambda| < 1$;

c) exactly one eigenvalue;

d) at least two different eigenvalues;

e) at least a thousand different eigenvalues;

f)* infinitely many different eigenvalues?

From this point on, let the underlying field be $\mathbb{R}$. Consider the vector space of continuous functions on the interval $[0, 1]$:

$$C[0,1] = \{f : f : [0,1] \to \mathbb{R} \text{ continuous}\}.$$

For a fixed continuous function $h : [0,1] \to \mathbb{R}$ let T_h denote the $C[0,1] \to C[0,1]$ map which takes $f \in C[0,1]$ to $hf \in C[0,1]$. It can be seen easily that $T_h : C[0,1] \to C[0,1]$ is a linear transformation.

9. What is $T_g \circ T_h$?

10. Prove that $\ker T_h = \{0\}$ if

a) $h(x) = x + 1$;

b) $h(x) = x - 1/2$.

11. Show a continuous function h such that $\ker T_h \neq \{0\}$. Describe $\ker T_h$ for this h.

12. Show a continuous function h such that T_h has at least two different eigenvalues.

13. Show a continuous function h such that T_h has infinitely many different eigenvalues.

Solutions can be found on: `www.renyi.hu/~harangi/bsm/`