Tatuzawa's theorem for Rankin–Selberg *L*-functions

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Establishing zero-free regions for automorphic L-functions is a central problem of number theory. In 2023, we established [7] a new zero-free region for all GL_1-twists of $\mathrm{GL}_n \times \mathrm{GL}_m$ Rankin–Selberg L-functions, generalizing Siegel's celebrated work [25] on Dirichlet L-functions. We report about our recent strengthening of this result [8], which generalizes Tatuzawa's refinement [26] of Siegel's theorem. Throughout, F will be a number field with adele ring $\mathbb{A}_F$. We denote by $\mathfrak{F}_n$ the set of unitary cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_F)$, and for each $\pi \in \mathfrak{F}_n$, we denote by $C(\pi) \geq 3$ the analytic conductor of π as defined in [7, §3.1] and [8, §2.1].

We recall first some classical results for Dirichlet L-functions (corresponding to $n = m = 1$ and $F = \mathbb{Q}$). The standard zero-free region for these L-functions and the rarity of exceptions is due to de la Vallée Poussin [20], Gronwall [6], Landau [15], and Titchmarsh [28]. We state the result with explicit constants, as can be derived from the work of McCurley [18, Theorem 1].

Theorem 1. *If χ is a primitive Dirichlet character, then $L(\sigma + it, \chi)$ has at most one zero (necessarily real and simple) in the region*

$$\sigma \geq 1 - \frac{1}{10 \log(C(\chi)(|t| + 3))}.$$

If the exceptional zero exists, then $\chi^2 = 1$. Moreover, if χ_1 and χ_2 are two distinct primitive Dirichlet characters, then $L(s, \chi_1)L(s, \chi_2)$ has at most one real zero (counted with multiplicity) in the interval

$$\sigma \geq 1 - \frac{1}{10 \log(C(\chi_1)C(\chi_2))}.$$

Concerning the location of the possible exceptional zero, we can derive an elegant result from Tatuzawa's theorem [26, Theorem 1] with the help of Rademacher's convexity bound [21, Theorem 3] and Cauchy's formula for $L'(s, \chi)$.

Theorem 2. *If $\varepsilon \in (0, 1)$, then the following holds for all but one primitive quadratic Dirichlet character χ:*

$$L(\sigma, \chi) \neq 0, \qquad \sigma \geq 1 - \frac{\varepsilon^3}{240} C(\chi)^{-\varepsilon}.$$

Zero-free regions have been established for various families of automorphic L-functions [13, 24, 19, 9, 10, 1, 22, 23, 2, 4, 5, 3, 11, 14, 17, 30, 12, 29, 27]. In order to state some of these results, we need further notation. Given $(\pi, \chi) \in \mathfrak{F}_n \times \mathfrak{F}_1$, we denote by $\pi \otimes \chi$ the (unique) element of $\mathfrak{F}_n$ whose space consists of the functions $g \mapsto f(g)\chi(\det g)$, where $g \mapsto f(g)$ is from the space of π. It is straightforward to check that $\pi \otimes \chi$ is isomorphic to the representation $g \mapsto \pi(g)\chi(\det g)$. It is convenient to introduce $\mathfrak{F}_n^*$, the set of those elements of $\mathfrak{F}_n$ whose central character is trivial on the diagonally embedded positive reals. For each $\pi \in \mathfrak{F}_n$, there exists a unique pair $(\pi^*, t_\pi) \in \mathfrak{F}_n^* \times \mathbb{R}$ such that $\pi = \pi^* \otimes |\cdot|^{it_\pi}$.

The next theorem combines some important results of Brumley, Humphries, and Thorner (see [2], [16, Theorem A.1], [3, Theorem A.1], [11, Theorem 2.1]).

Theorem 3. *There exists an effective constant* $c_1 = c_1(n, m, [F : \mathbb{Q}]) > 0$ *such that if* $(\pi, \rho) \in \mathfrak{F}_n^* \times \mathfrak{F}_m^*$*, then* $L(\sigma + it, \pi \times \rho)$ *has no zero in the region*

$$\sigma \geq 1 - c_1(C(\pi)C(\rho))^{-n-m}(|t| + 1)^{-nm}.$$

Moreover, if $\pi = \tilde{\pi}$ *or* $\rho = \tilde{\rho}$ *or* $\rho = \tilde{\pi}$*, then* $L(\sigma + it, \pi \times \rho)$ *has at most one zero (necessarily real and simple) in the region*

$$\sigma \geq 1 - c_1/\log(C(\pi)C(\rho)(|t| + 3)).$$

If the exceptional zero exists, then $(\pi, \rho) = (\tilde{\pi}, \tilde{\rho})$ *or* $\rho = \tilde{\pi}$*.*

Our first new result generalizes Theorem 1, [11, Theorem 1.1], [12, Theorem 8.2].

Theorem 4 ([8, Theorem 1.6])**.** *If* $(\pi, \chi) \in \mathfrak{F}_n \times \mathfrak{F}_1^*$*, then* $L(\sigma + it, \pi \times (\tilde{\pi} \otimes \chi))$ *has at most one zero (necessarily real and simple) in the region*

$$\sigma \geq 1 - \frac{1}{903 \log(C(\pi)^{2n}C(\chi)^{n^2}(|t| + 3)^{n^2[F:\mathbb{Q}]})}.$$

If the exceptional zero exists, then $\pi \otimes \chi^2 = \pi$*. Moreover, if* $(\pi, \chi_1, \chi_2) \in \mathfrak{F}_n \times \mathfrak{F}_1^* \times \mathfrak{F}_1^*$ *satisfies* $\pi \otimes \chi_1 \neq \pi \otimes \chi_2$*, then* $L(s, \pi \times (\tilde{\pi} \otimes \chi_1))L(s, \pi \times (\tilde{\pi} \otimes \chi_2))$ *has at most one real zero (counted with multiplicity) in the interval*

$$\sigma \geq 1 - \frac{1}{158 \log(C(\pi)^{4n}C(\chi_1)^{n^2}C(\chi_2)^{n^2})}.$$

Our second new result generalizes Theorem 2 and refines the zero-free region of [7, Theorem 1.1].

Theorem 5 ([8, Theorem 1.1])**.** *Let* $(\pi, \rho, \chi) \in \mathfrak{F}_n \times \mathfrak{F}_m \times \mathfrak{F}_1$ *and* $\varepsilon > 0$*. There exist an effective constant* $c_2 = c_2(n, m, [F : \mathbb{Q}], \varepsilon) > 0$ *and a character* $\psi = \psi_{\pi,\rho,\varepsilon} \in \mathfrak{F}_1$ *such that if* $L(s, \pi \times (\rho \otimes \chi))$ *differs from* $L(s, \pi \times (\rho \otimes \psi))$*, then*

$$L(\sigma, \pi \times (\rho \otimes \chi)) \neq 0, \qquad \sigma \geq 1 - c_2(C(\pi)C(\rho)C(\chi))^{-\varepsilon}.$$

Moreover, $L(s, \pi \times (\rho \otimes \psi))$ *has at most one real zero (necessarily simple) in the interval* $\sigma \geq 1 - c_2(C(\pi)C(\rho)C(\psi))^{-\varepsilon}$*.*

The proofs of Theorems 4 and 5 are based on the zero repulsion technique of Hoffstein–Lockhart [9, Lemma 3.3] and Goldfeld–Hoffstein–Lieman [9, Appendix, Lemma]. Interestingly, Theorem 4 is used in the proof of Theorem 5.

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