An Almost Optimal Bound on the Number of Intersections of Two Simple Polygons

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2020, GeoSzem-CoGe virtual reality

Problem statement

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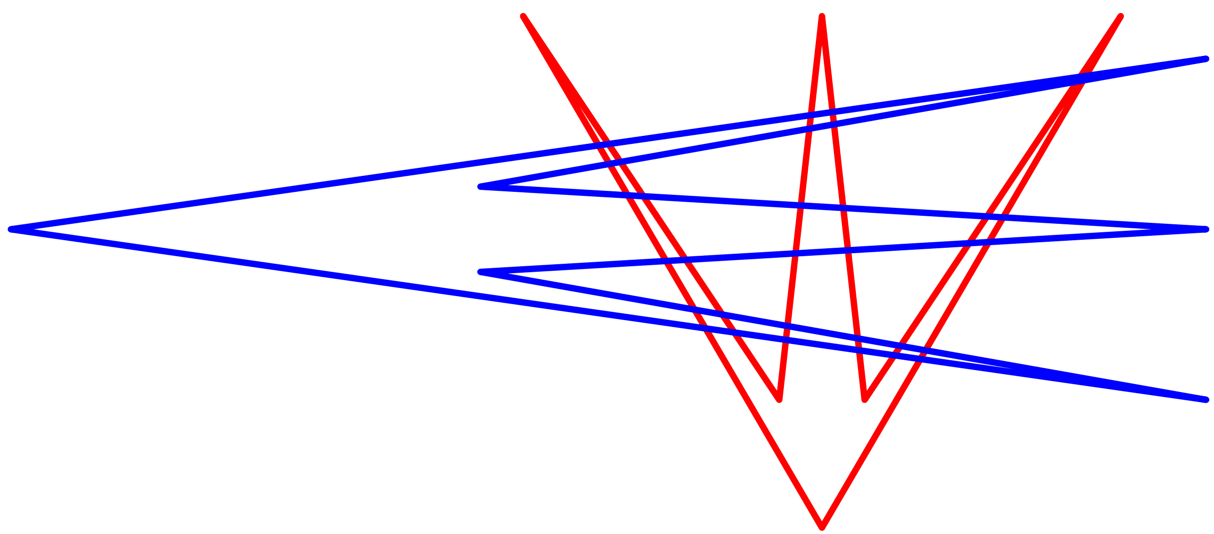
<u>when n and m are both even</u>:



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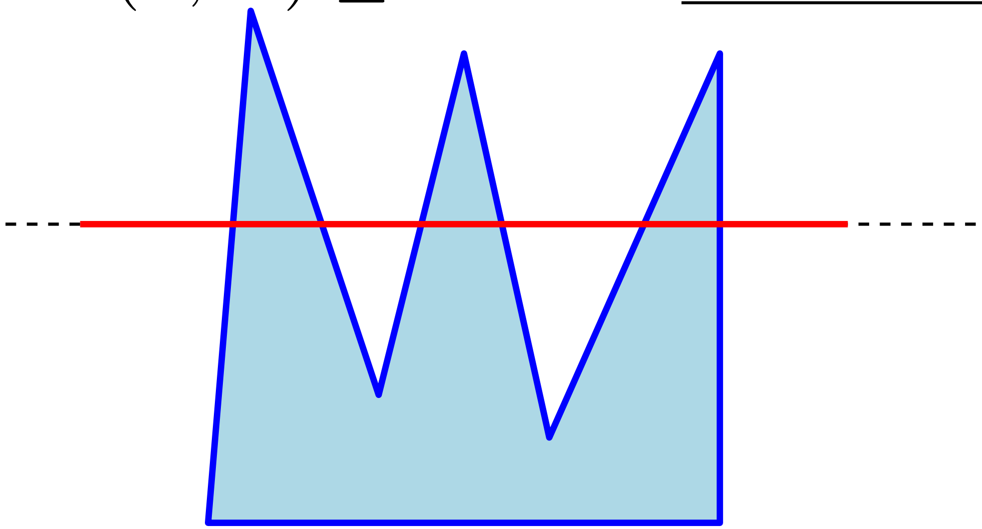

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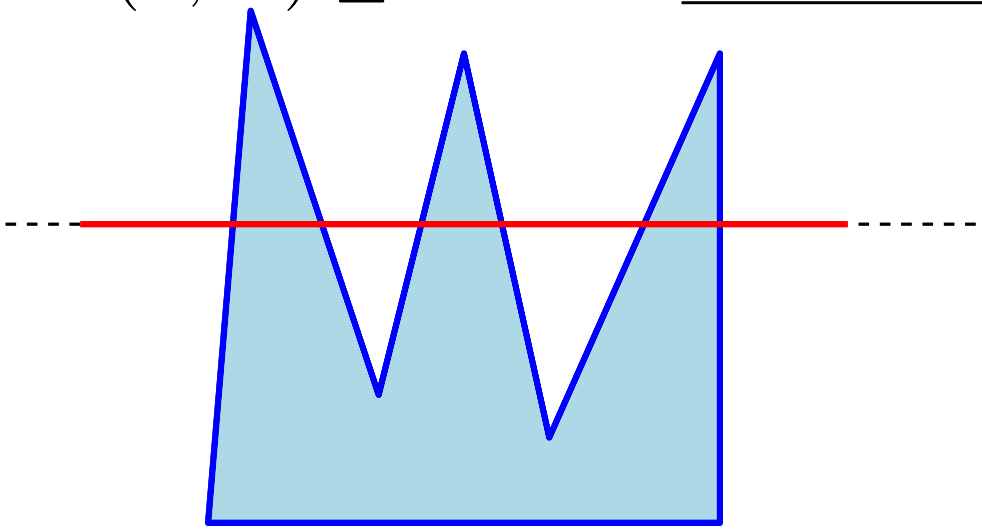

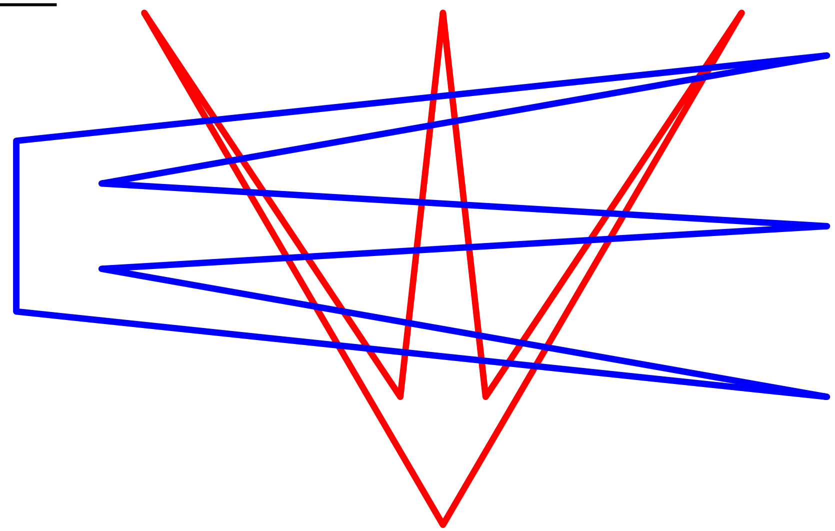

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This can be reached when m is odd and n is even.

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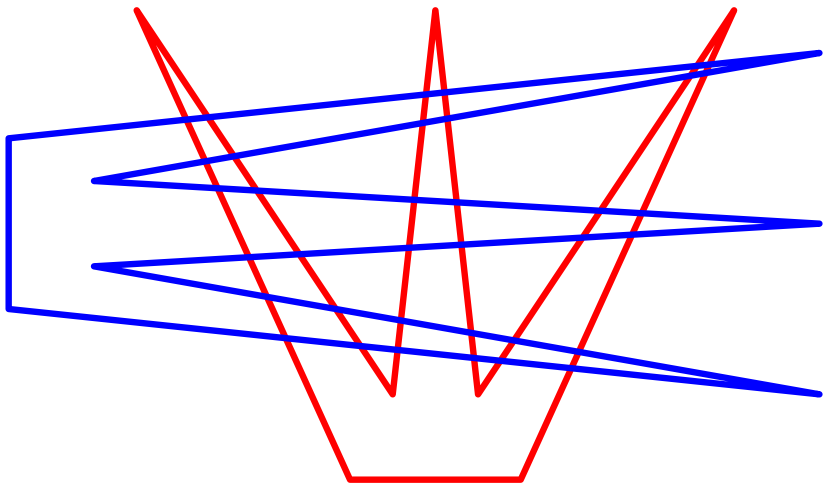

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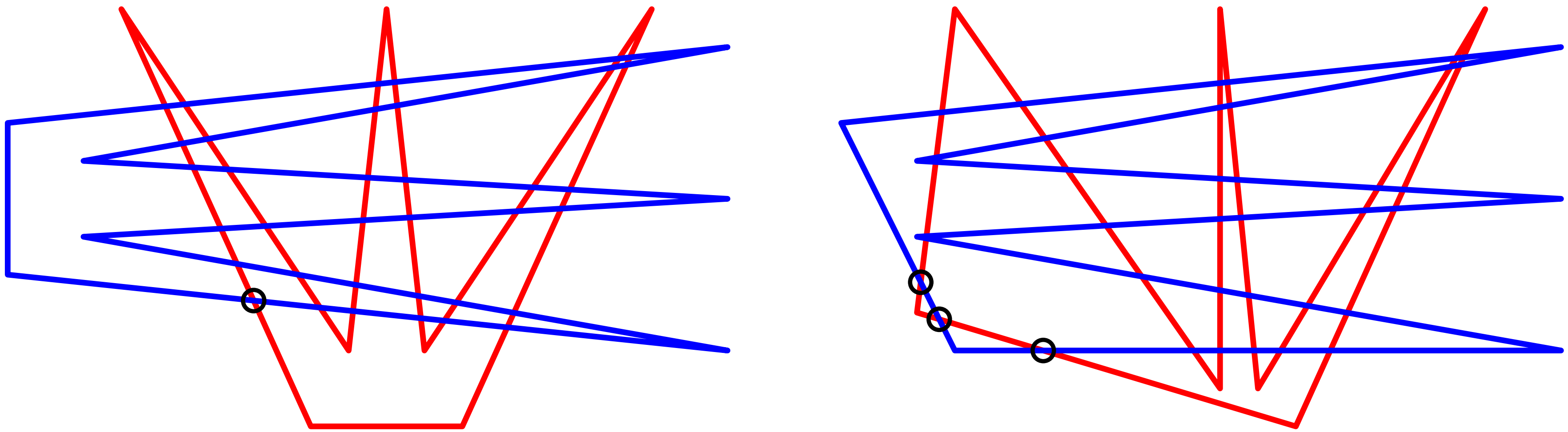

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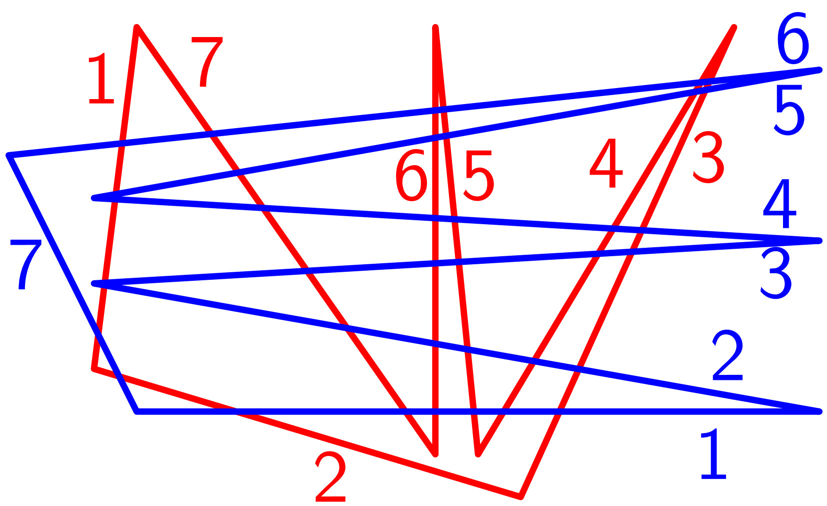
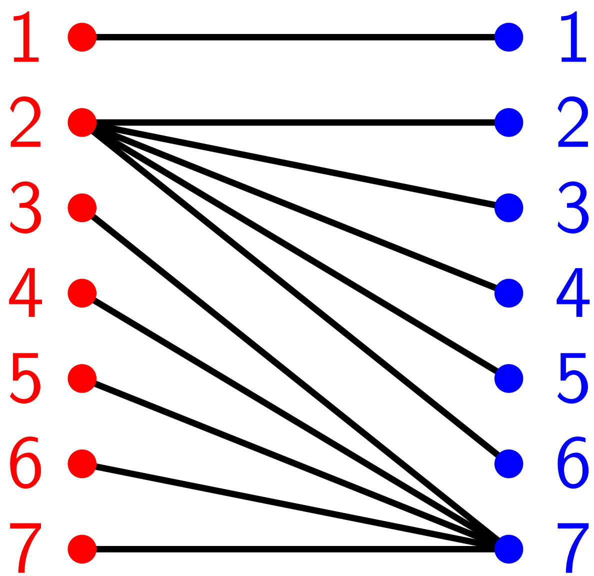
Definition The *disjointness/non-intersection graph G of polygons P and Q* is a bipartite graph with vertices corresponding to edges of the polygons and two vertices are connected whenever the corresponding edges do not intersect.





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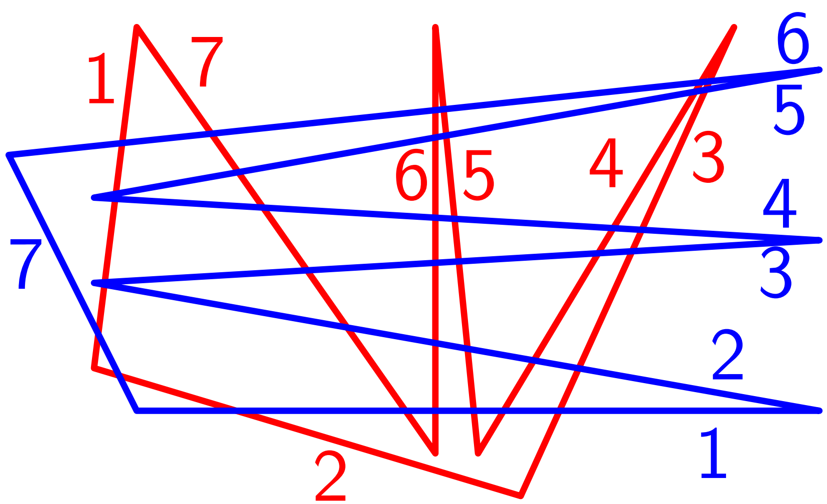

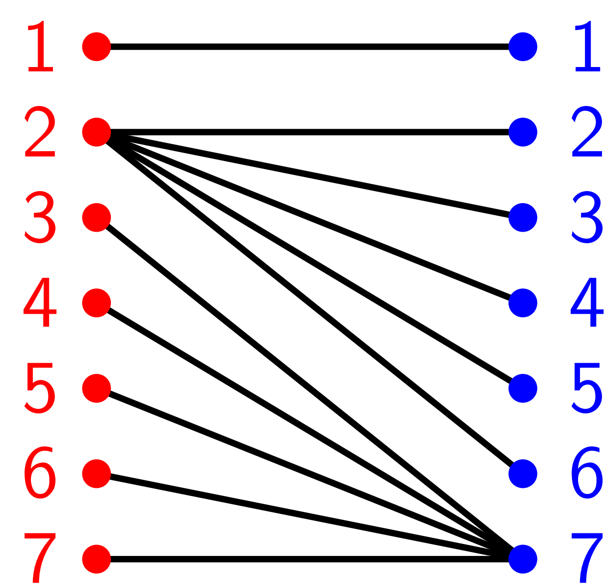

$i(n,m) = nm - e(G) \leq nm - (m + n - cc(G))$ where
$cc(G) =$ *the number of conn. components of* G.
Thus it is enough to prove that $cc(G) \leq C$.

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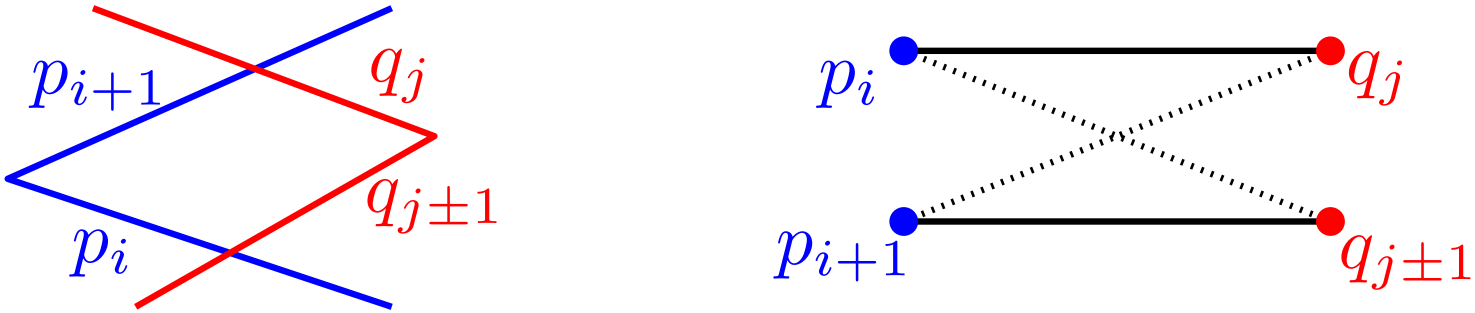

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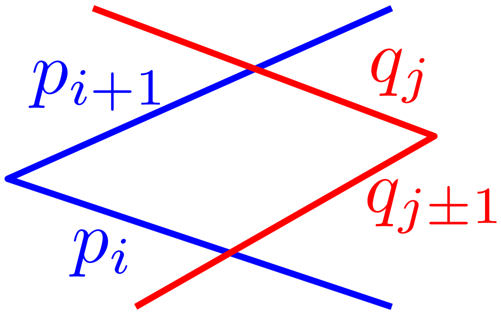
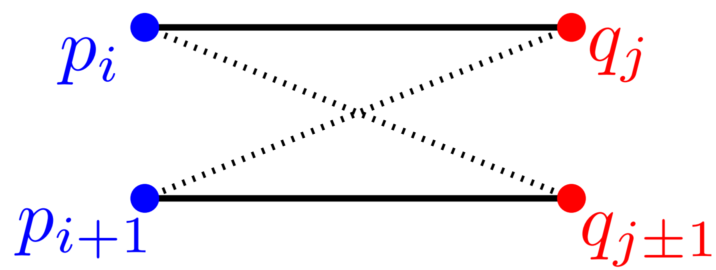
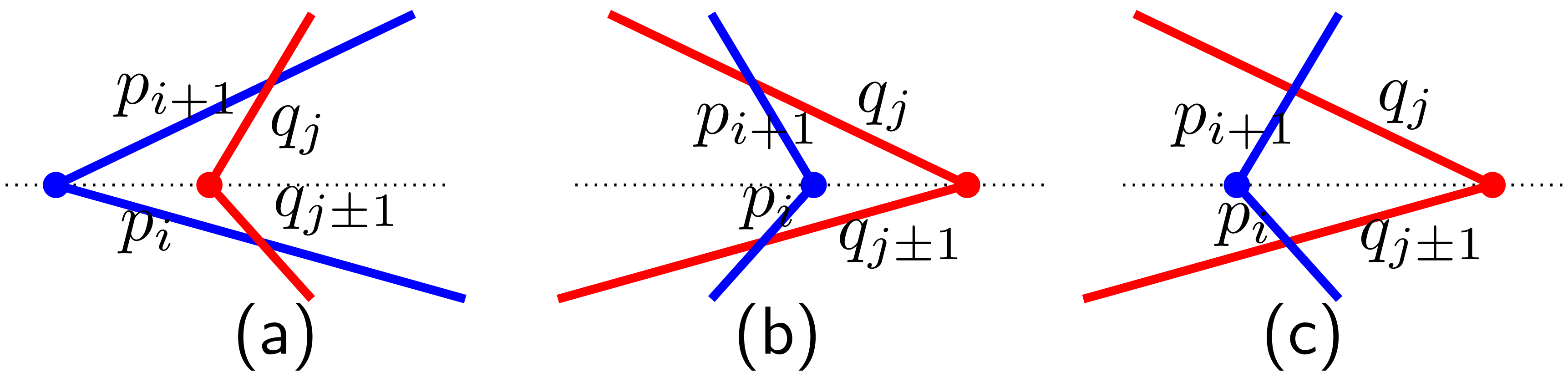
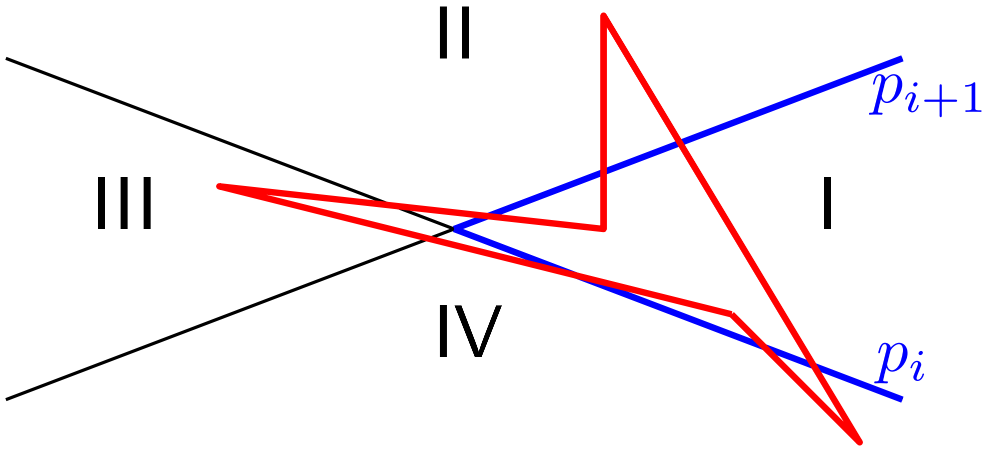




and $p_i p_{i+1}$ is either
a) *hooking* $q_j q_{j\pm1}$ or is b) *hooked by* $q_j q_{j\pm1}$ or c) both.



Proof of Lemma



Suppose $CC(p_i) \neq CC(p_{i+1})$. Then every red edge intersects at least one of p_i and p_{i+1}.

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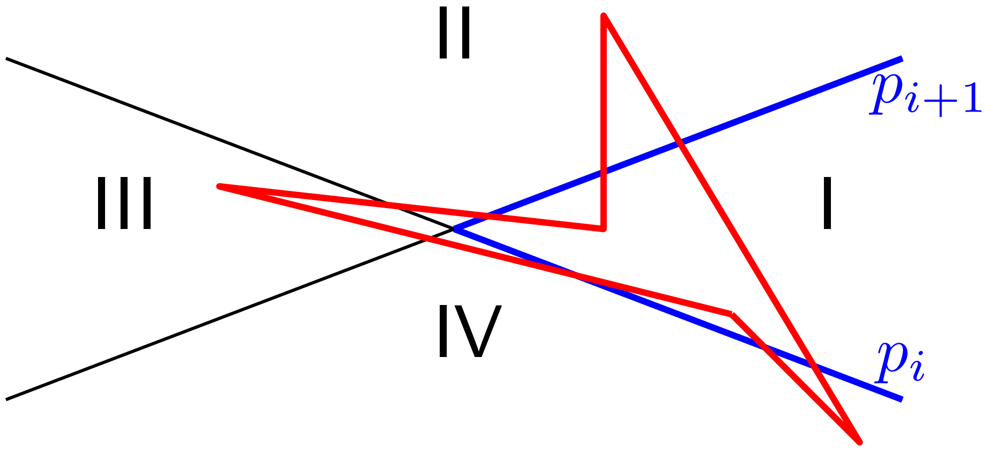

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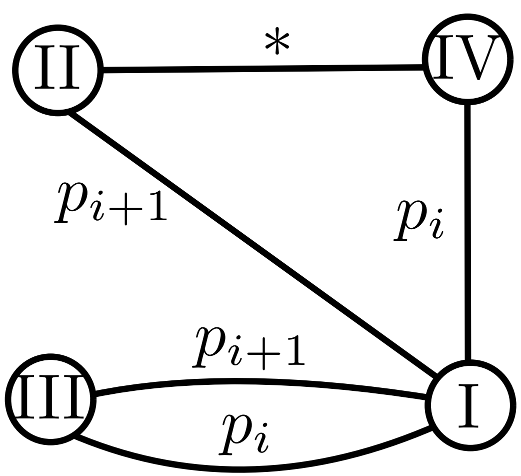
The red polygon induces an odd closed walk between cones I-IV which is a subgraph of this edge-labeled dummy graph:



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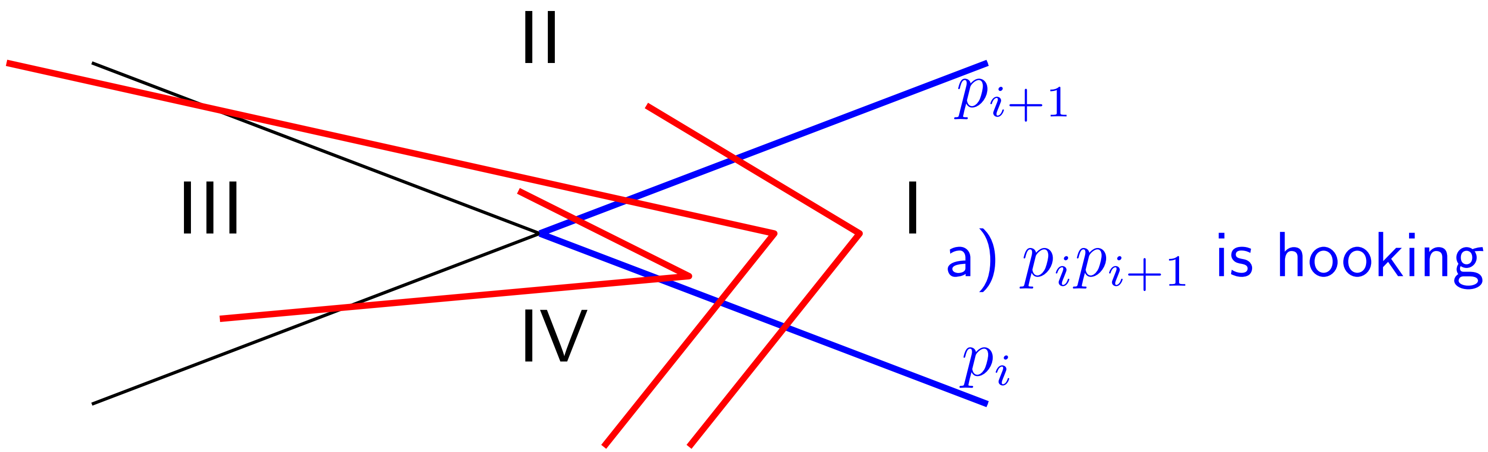

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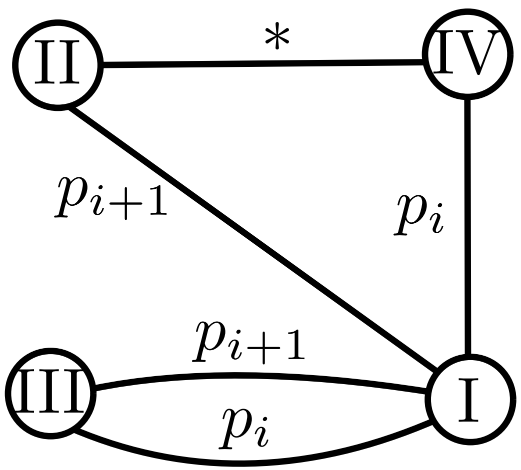

It is easy to check that there is either a
a1) IV-I-II subwalk with labels p_i and p_{i+1} or
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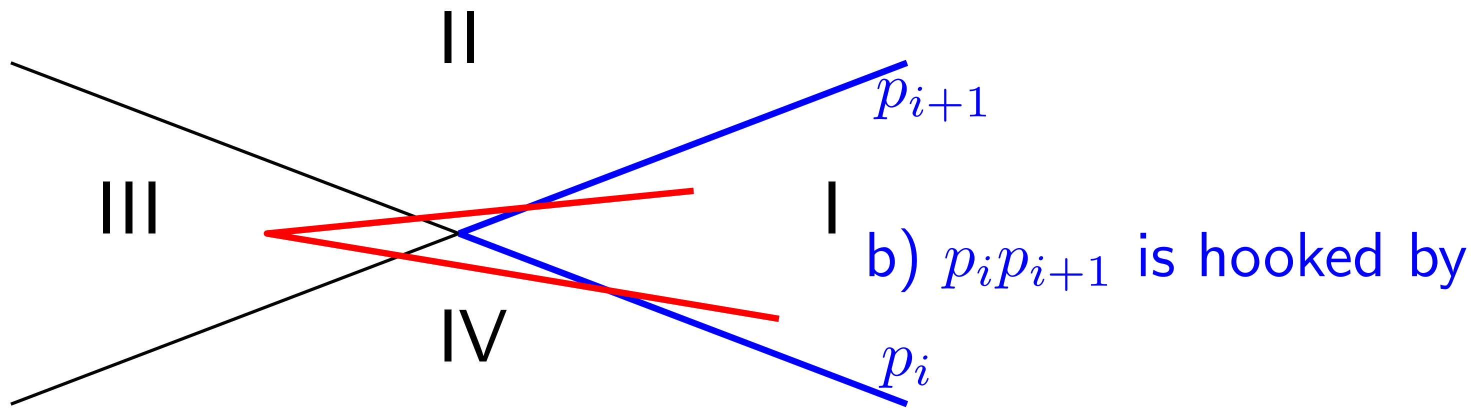

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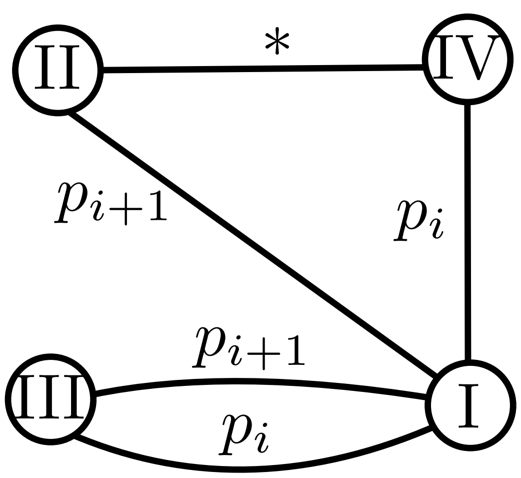

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CCp Lemma

Lemma If $CC(p_i), CC(p_{i+1}), CC(p_j)$ and $CC(p_{j+1})$ are all different then it is impossible that both of $p_i p_{i+1}$ and $p_j p_{j+1}$ are hooked or that both are hooking.

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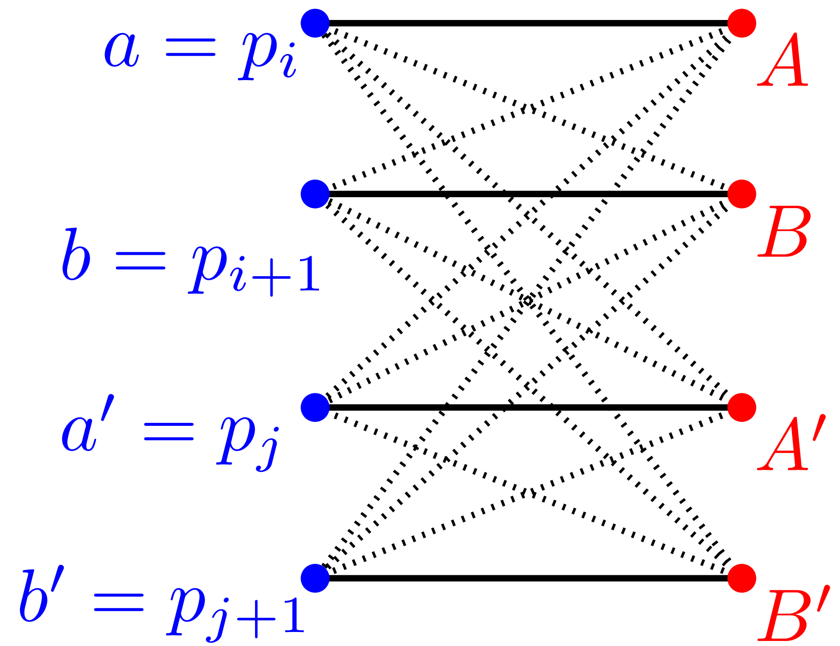

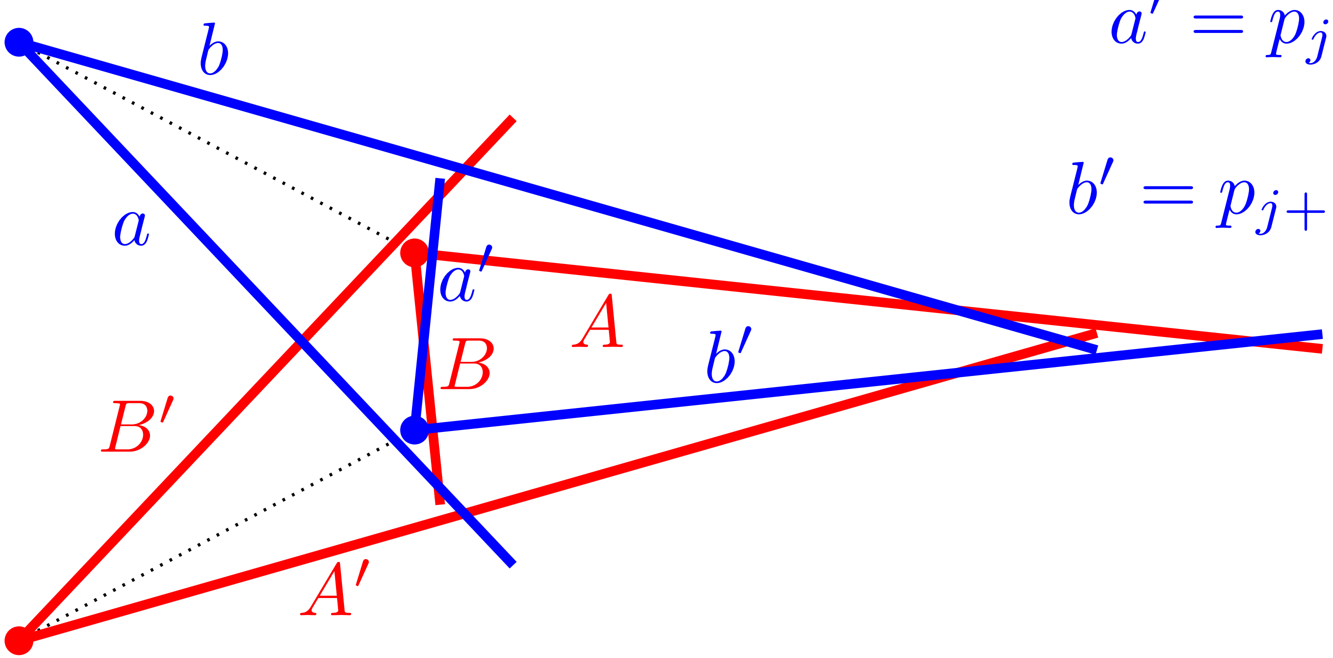

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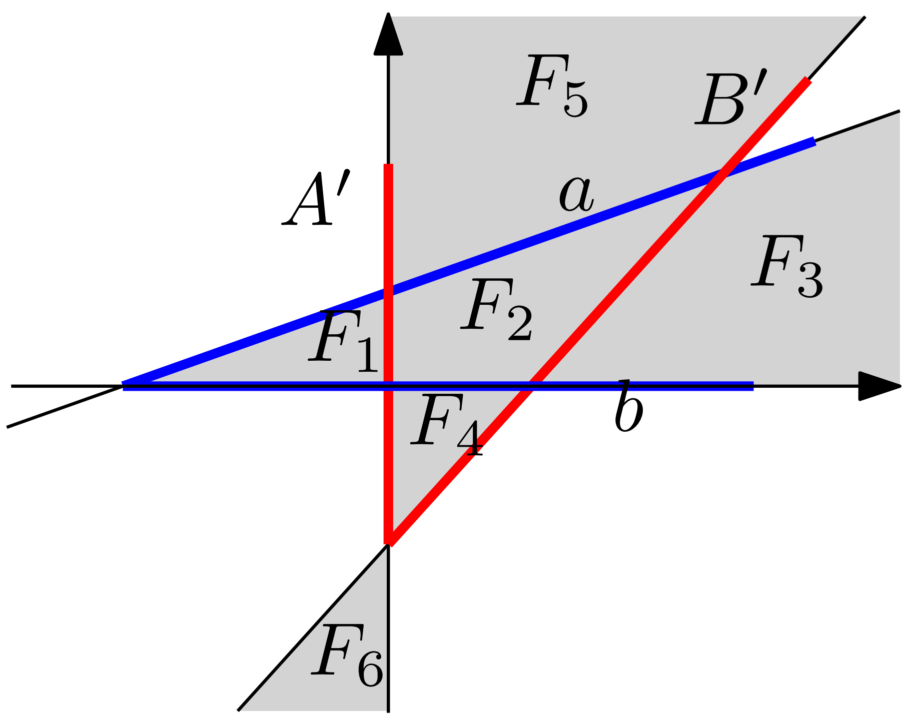
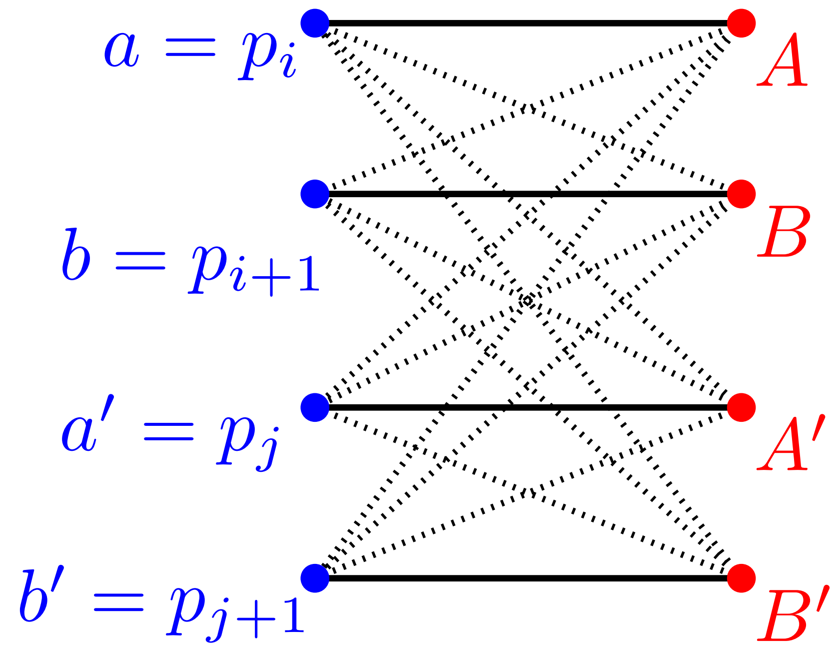
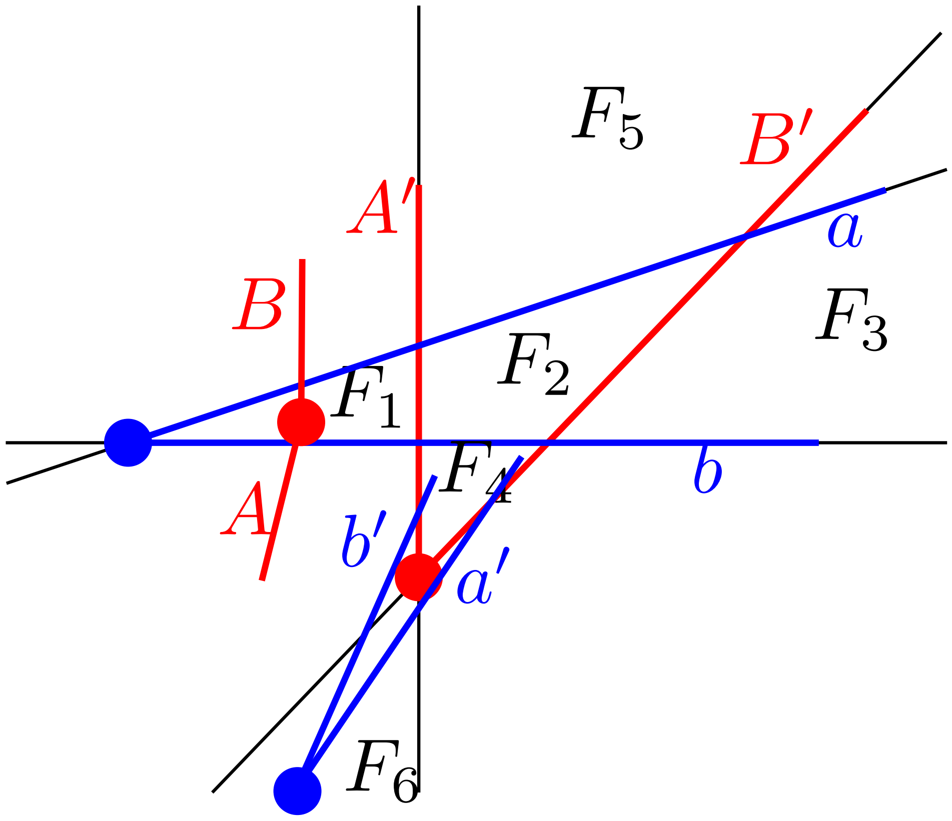




A, B meet in one of F_1, F_2, F_3 and a', b' meet in one of $F_6, F_4, F_2, F_5 \Rightarrow$ we need to check 12 cases

One of the cases: $A \cap B \in F_1$ and $a' \cap b' \in F_6$



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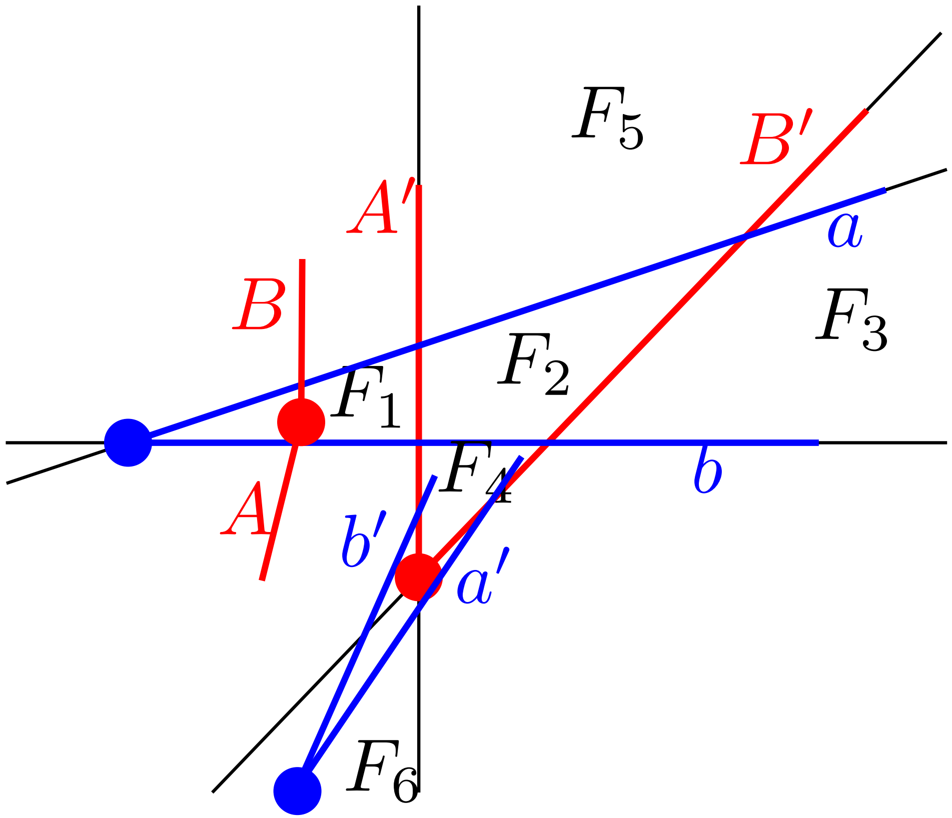

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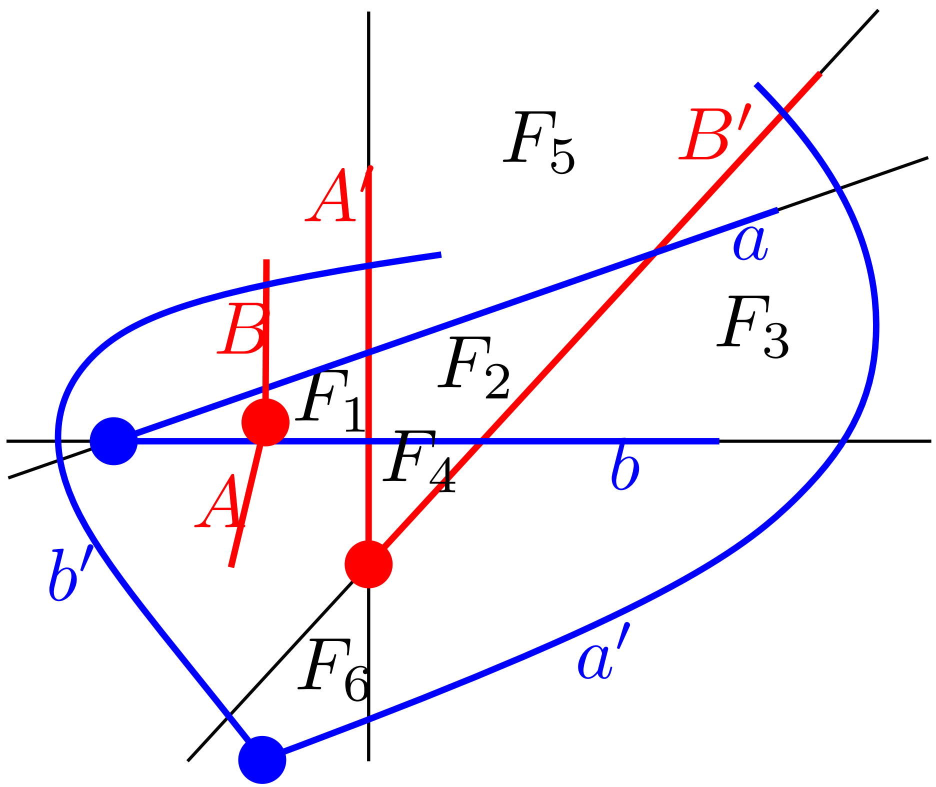

b) Both sides a' and b' have an endpoint in $F_2 \cup F_5$
$\Rightarrow$ A and B meet inside the cone of a', b'
$\Rightarrow$ none of A, B can intersect both of a', b'

Showing that there are at most $\approx n/2$ CC's

Lemma If $CC(p_i), CC(p_{i+1}), CC(p_j)$ and $CC(p_{j+1})$ are all different then it is impossible that both of $p_i p_{i+1}$ and $p_j p_{j+1}$ are hooked or that both are hooking.

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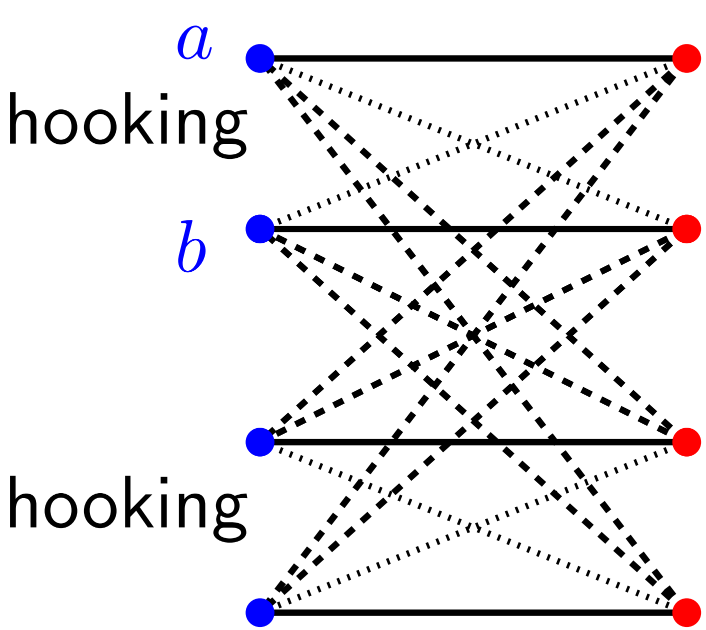
$\Rightarrow$ There can be a pair of edges $CC(p_i) \neq CC(p_{i+1})$ which is hooking ($= ab$), another that is hooked ($= a'b'$), the rest of such pairs must share a CC with one of ab and $a'b'$.

one of the dashed edges must be in G:



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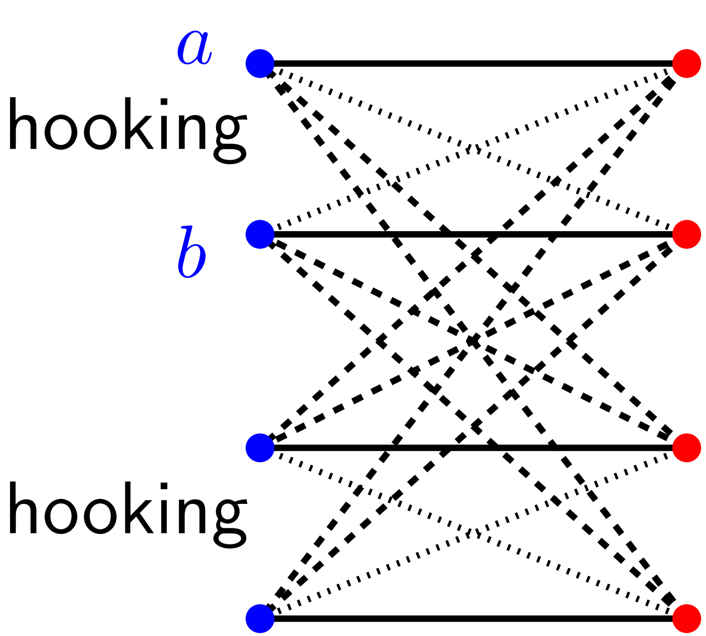

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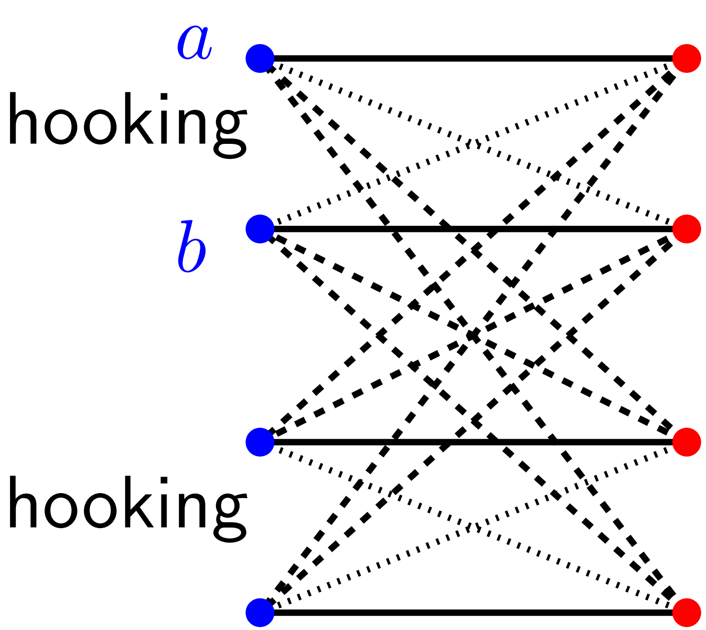

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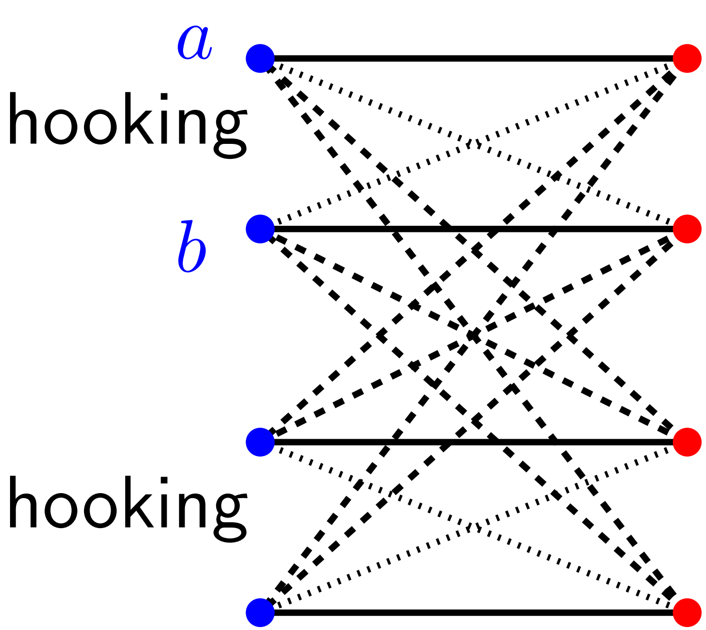

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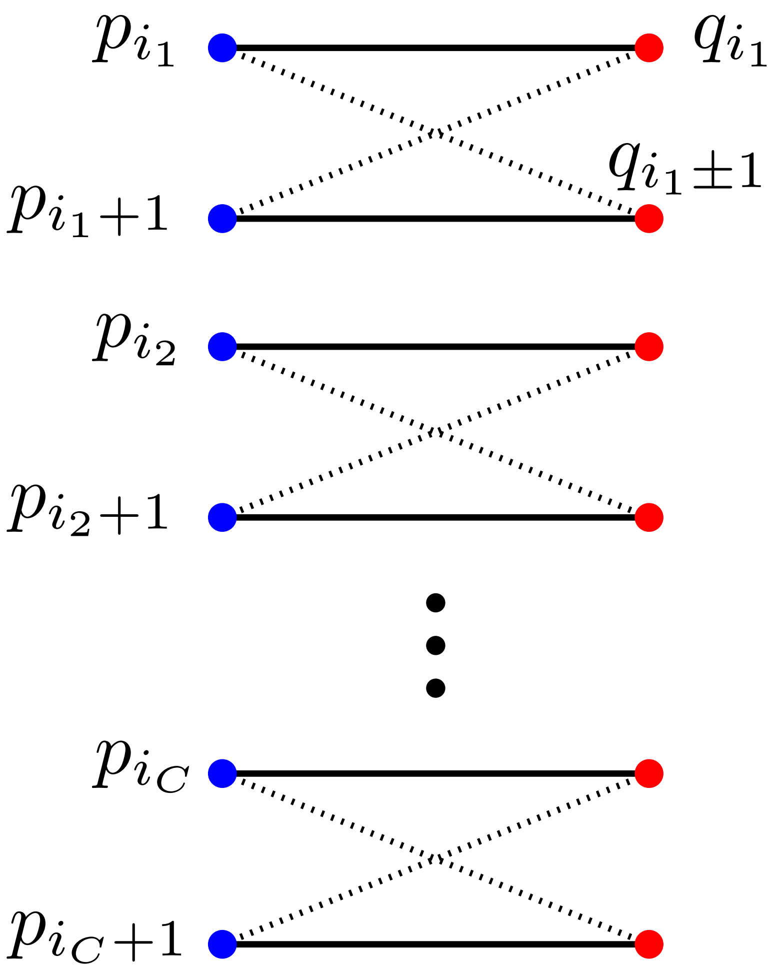

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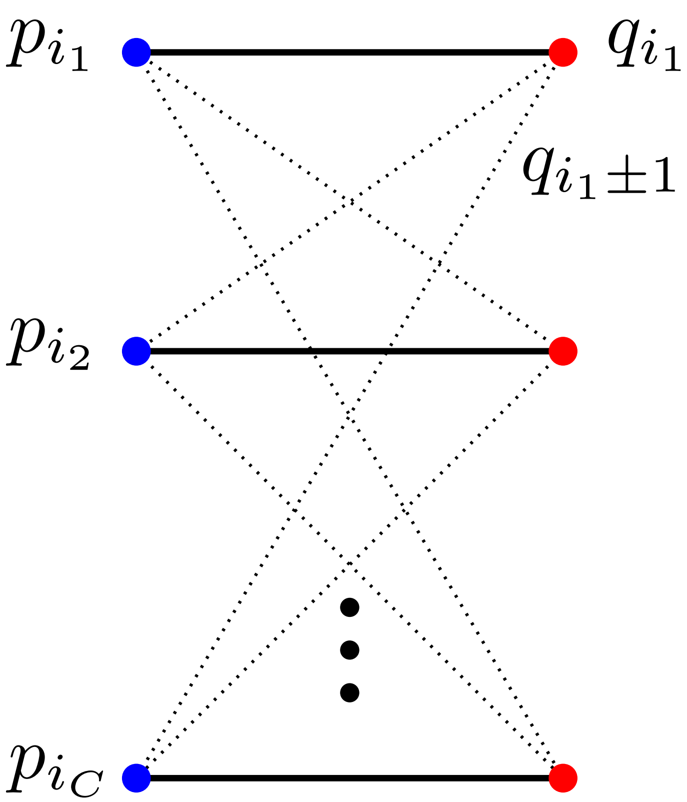

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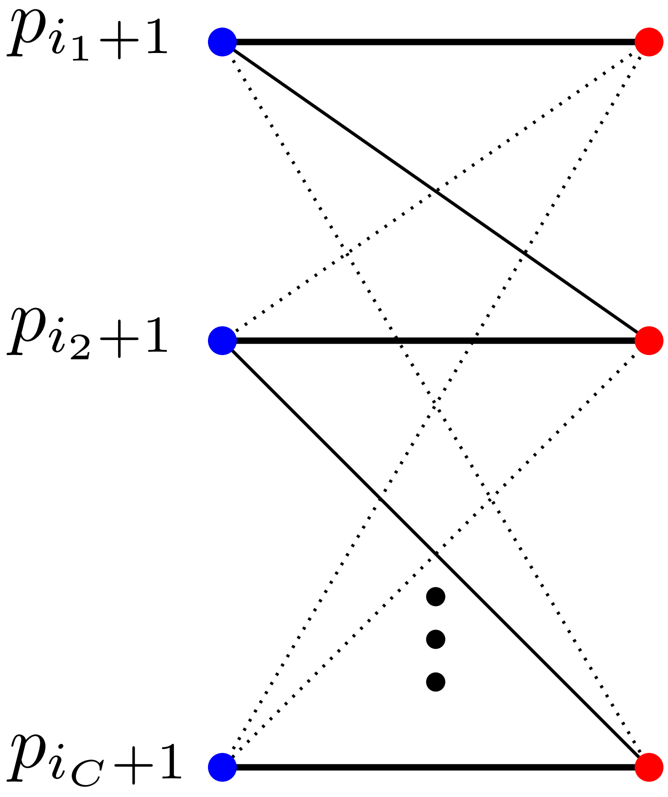

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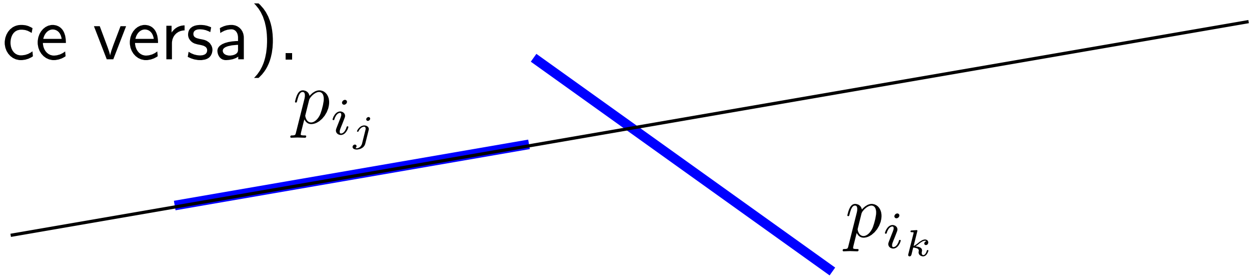
$CC(p_{i_j+1})$ are all the same.

Ramsey-type arguments that improve the structure

Hooking Lemma for avoiding pairs $+$ CCp Lemma $\Rightarrow$ For $> C' - 5$ pairs all p_{i_j} and p_{i_k} are non-avoiding: p_{i_j} ***stabs*** p_{i_k} (or vice versa).



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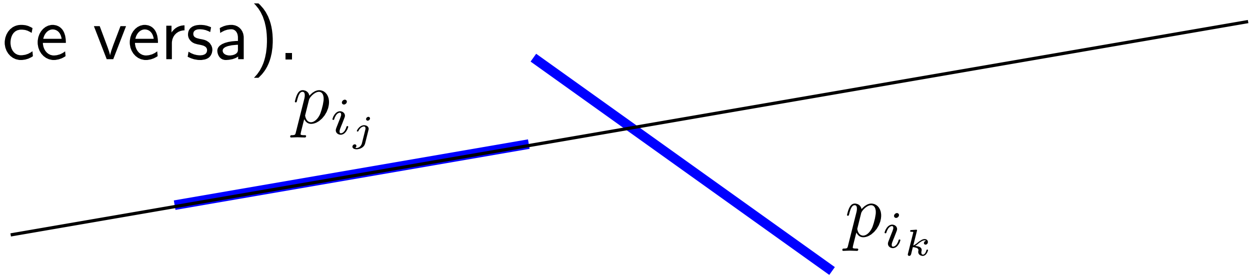

Stabbings define a tournament on the p_{i_j}'s. It has a transitive subtournament of size $C'' \approx \log(C')$.

Ramsey-type arguments that improve the structure

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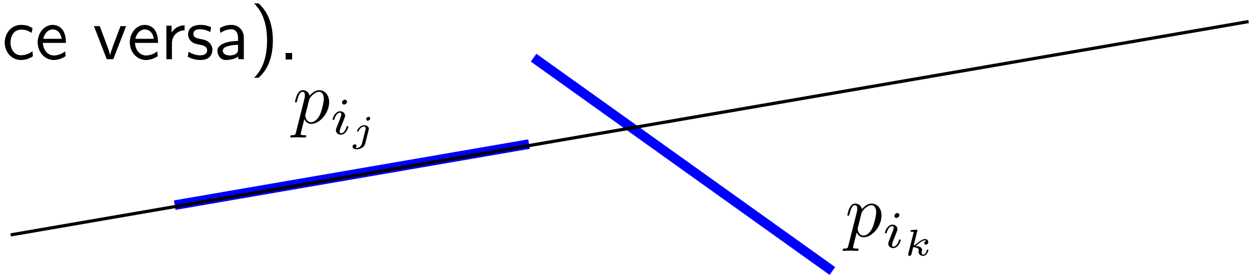

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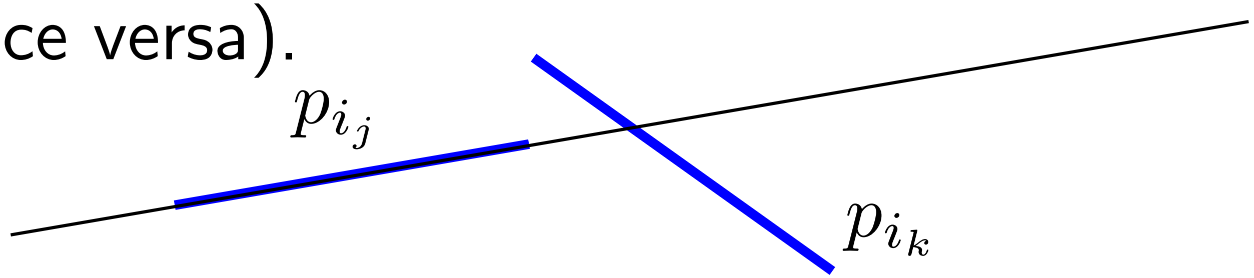

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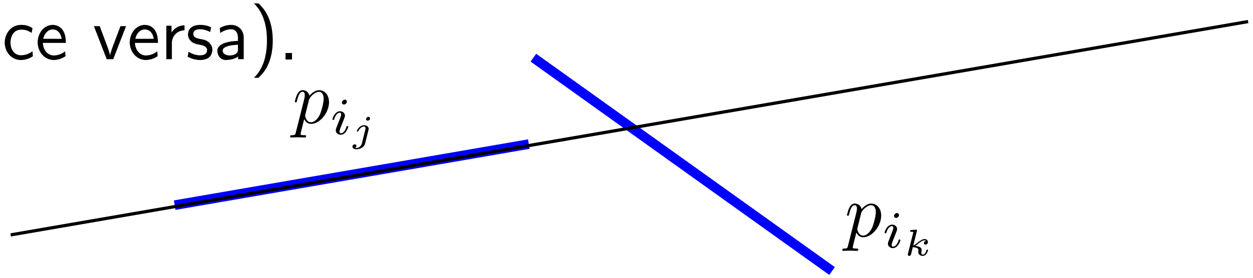

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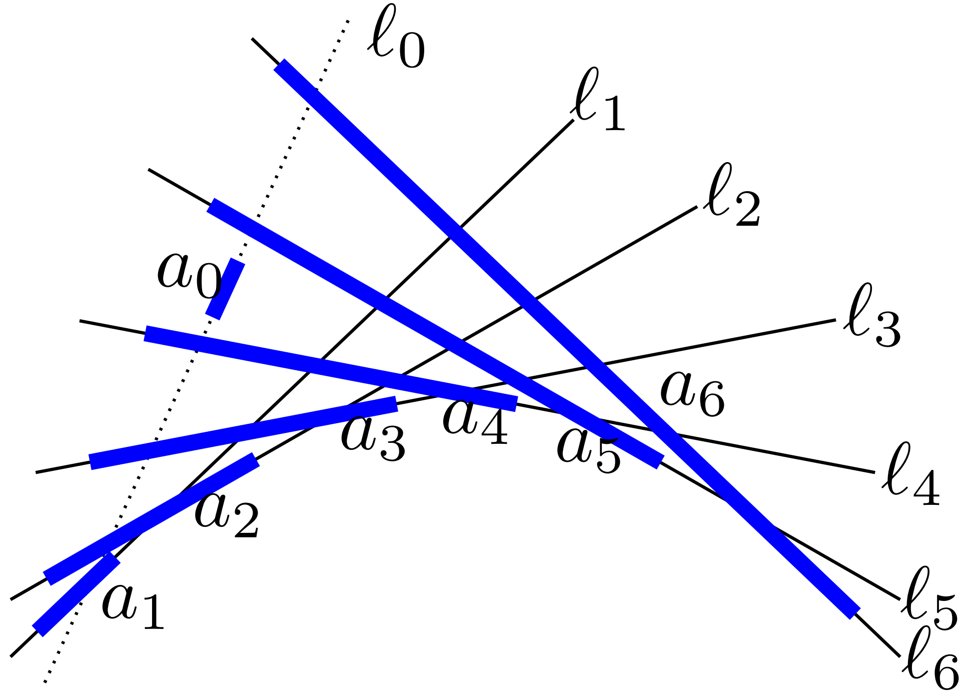
By Erdős-Szekeres Theorem on monotone subsequences there is $> C'''' = 7$ p_{i_j}'s such that their slopes are monotone increasing or decreasing.

Concluding the proof

The remaining ≥ 7 sides must have this structure:



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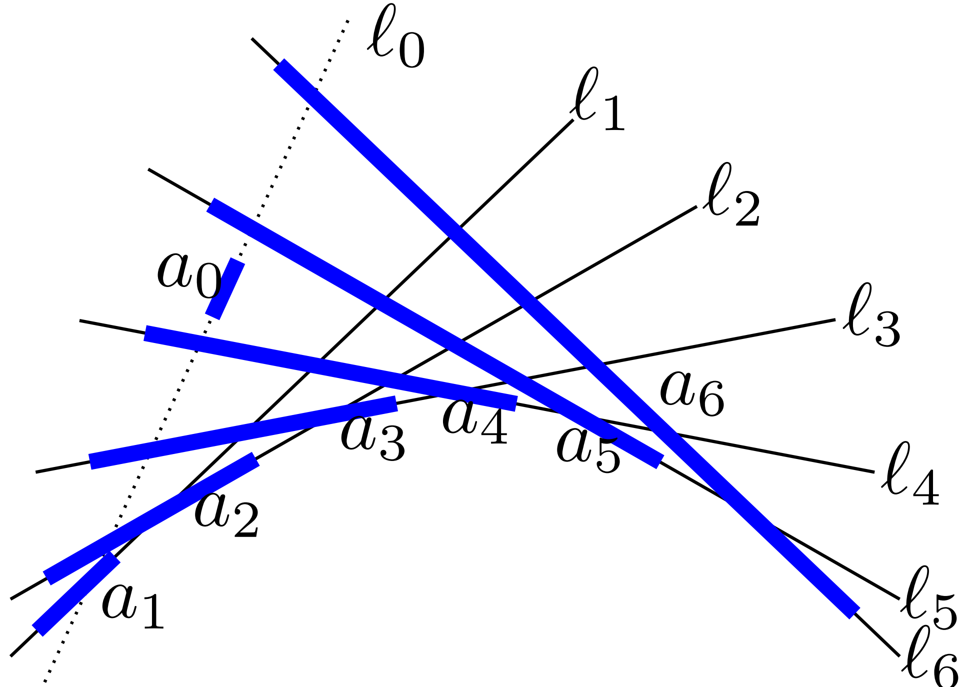
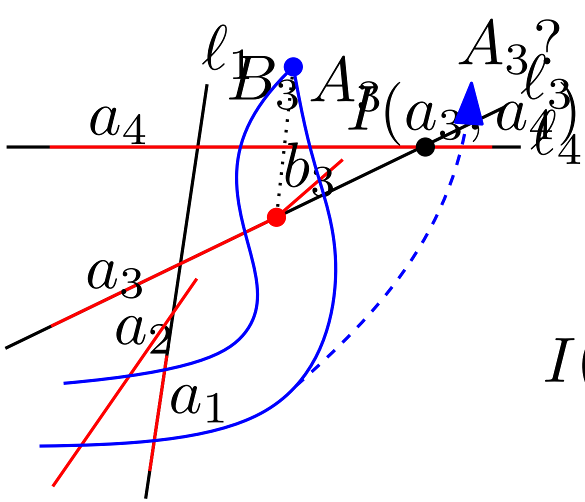
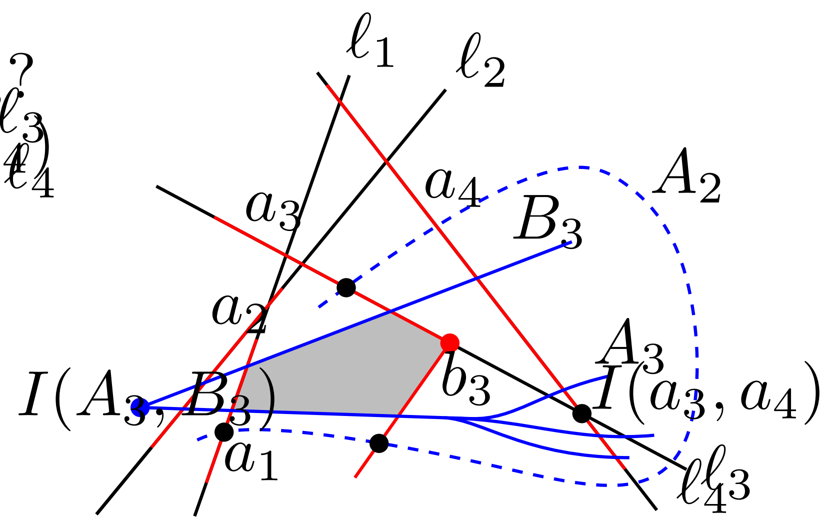


Again a case analysis of the hooking pairs leads to contradiction.





etc.

Thank you for your attention