On the volume of central diagonal sections of the n-cube

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Geometry Seminar, Rényi Institute

May 15, 2020

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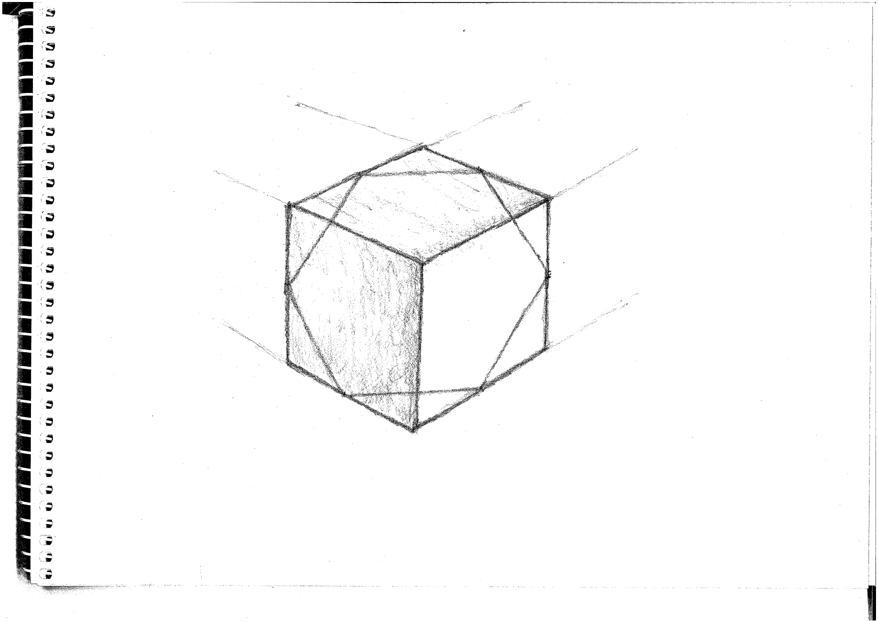
- The $n = 2$ is uninteresting, $\max \mathrm{Vol}_2(C^2 \cap H) = \sqrt{2}$.
- The $n = 3$ case is more complicated. By central symmetry, each central section of C^3 is either a hexagon or a parallelogram.



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- The maximum is attained for H orthogonal to $(1, 1, 0, \dots, 0)$.
- Hensley (1979) also described Selberg's argument to show that the volume of the central diagonal section tends to $\sqrt{6/\pi}$ as $n \to \infty$.

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$$\mathrm{Vol}_{n-1}(C^n \cap H) = \frac{\sqrt{n}}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\sin t}{t}\right)^n dt.$$

- It has been known for a long time that

$$\mathrm{Vol}_{n-1}(C^n \cap H) \to \sqrt{\frac{6}{\pi}}.$$

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$$\begin{aligned} &\mathrm{Vol}_{n-1}(C^n \cap H) \\ &\quad = \frac{\sqrt{n}}{2^{n+1}(n-1)!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-2i)^{n-1} \mathrm{sign}(n-2i). \end{aligned}$$

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- Numerical computations show that the above formula not only approaches $\sqrt{\frac{6}{\pi}}$ as $n \to \infty$, but also seems to be monotonically increasing for $n \geq 3$.

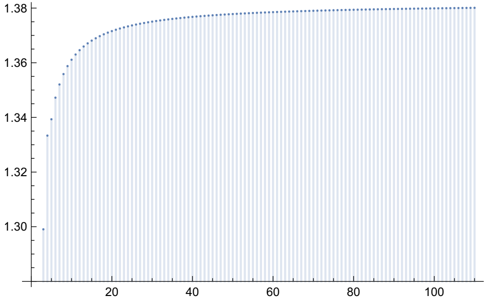

Figure: $\mathrm{Vol}_{n-1}(C^n \cap H)$ for $3 \leq n \leq 110$.

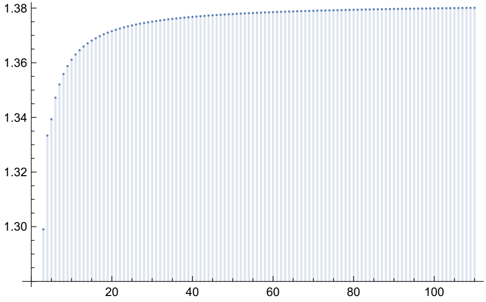

Figure: $\mathrm{Vol}_{n-1}(C^n \cap H)$ for $3 \leq n \leq 110$.

We do not know how to prove the monotonicty directly from this expression of Frank and Riede, so we will examine the integral of Ball instead.

- König and Koldobky (2018) proved that, in fact,

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- Very recently, Aliev (2020) proved that

$$\frac{\sqrt{n}}{\sqrt{n+1}} \le \frac{\mathrm{Vol}_n(C^{n+1} \cap H)}{\mathrm{Vol}_{n-1}(C^n \cap H)},$$

  which is slightly less than monotonicity.

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- If one has a bound for n_0, then one can verify monotonicity for all n by directly calculating the section volumes for $n < n_0$ with the formula of Frank and Riede.
- Such a verification also yields, as a corollary, the upper bound of König and Koldobsky.

Proof.

- We need to examine the behaviour of

$$I(n) = \frac{2\sqrt{n}}{\pi} \int_0^{+\infty} \left(\frac{\sin t}{t} \right)^n dt, \quad \text{for } n \geq 3.$$

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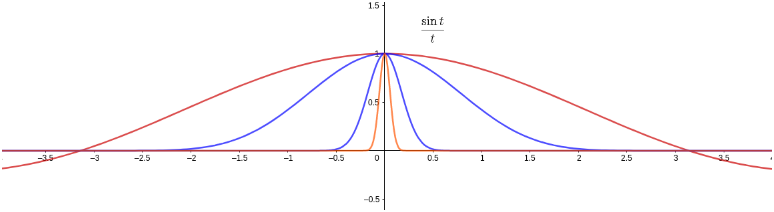

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- Therefore, $x(t)$ has an inverse $t = t(x) : [0, x(a)] \to [0, a]$.
- Since $x'(t) \neq 0$ everywhere in $[0, a]$, the inverse function $t(x)$ is analytic in $[0, x(a)]$ by the Lagrange Inversion Theorem.

- We can get the first few terms of the Taylor series of $t(x)$ around $x = 0$ by inverting the Taylor series of $x(t)$ at $t = 0$ as follows:

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- Since $t^{(7)}$ is C^∞ in $[0, x(a)]$, it attains its maximum, so $|t^{(7}(x)| \leq R$ for some $R > 0$ and every $x \in [0, x(a)]$.

Therefore, after the change of variables, we get

$$\begin{aligned}
I_a(n) &= \frac{2\sqrt{n}}{\pi} \int_0^{x(a)} e^{-nx^2/6} t'(x) dx \\
&= \frac{2\sqrt{n}}{\pi} \int_0^{x(a)} e^{-nx^2/6} \left(1 - \frac{x^2}{20} - \frac{13x^4}{30240} + R_6(x)\right) dx \\
&= \frac{2\sqrt{n}}{\pi} \int_0^{x(a)} e^{-nx^2/6} \left(1 - \frac{x^2}{20} - \frac{13x^4}{30240}\right) dx \\
&+ \frac{2\sqrt{n}}{\pi} \int_0^{x(a)} e^{-nx^2/6} R_6(x) dx.
\end{aligned}$$

In order to evaluate the above integrals we will use the central moments of the normal distribution: If $y = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, then for an integer $p \geq 0$ it holds that

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$$\mathbb{E}[y^p] = \begin{cases} 0, & \text{if } p \text{ is odd,} \\ \sigma^p (p-1)!!, & \text{if } p \text{ is even.} \end{cases}$$

In our case $\mu = 0$ and $\sigma^2 = 3/n$. Thus, we get that

$$\begin{aligned}
\frac{2\sqrt{n}}{\pi} \int_0^{x(a)} e^{-nx^2/6} |R_6(x)| dx &\leq \frac{2R\sqrt{n}}{\pi 6!} \int_0^{x(a)} e^{-nx^2/6} x^6 dx \\
&< \frac{2R\sqrt{n}}{\pi 6!} \int_0^{+\infty} e^{-nx^2/6} x^6 dx \\
&= \frac{2R\sqrt{n}}{\pi 6!} \frac{3^3}{n^3} 5!! \\
&= \frac{9R}{8\pi} \frac{1}{n^{5/2}} \\
&< \frac{R}{2} \frac{1}{n^{5/2}} =: e_2(n).
\end{aligned}$$

Notice also that

$$\frac{2\sqrt{n}}{\pi}\int_0^{+\infty} e^{-nx^2/6}\left(1-\frac{x^2}{20}-\frac{13x^4}{30240}\right)dx$$
$$=\sqrt{\frac{3\pi}{2}}\frac{2\sqrt{n}}{\pi}\left(\frac{1}{n^{1/2}}-\frac{3}{20n^{3/2}}-\frac{13}{1120n^{5/2}}\right)$$
$$=\sqrt{\frac{6}{\pi}}\left(1-\frac{3}{20n}-\frac{13}{1120n^2}\right).$$

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$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-\tau^2} d\tau.$$

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$$\begin{aligned}
&\frac{2\sqrt{n}}{\pi} \left| \int_{x(a)}^{+\infty} e^{-nx^2/6} \left(1 - \frac{x^2}{20} - \frac{13x^4}{30240}\right) dx \right| \\
&\le \frac{2\sqrt{n}}{\pi} \int_{x(a)}^{+\infty} e^{-nx^2/6} \left|1 - \frac{x^2}{20} - \frac{13x^4}{30240}\right| dx \\
&\qquad < \frac{2\sqrt{n}}{\pi} \int_1^{+\infty} e^{-nx^2/6} \left(1 + \frac{x^2}{20} + \frac{13x^4}{30240}\right) dx \\
&\qquad = \sqrt{6\pi} \operatorname{erfc}(\sqrt{n/6}) \left(\frac{13 + 168n + 1120n^2}{2240n^{5/2}}\right) \\
&\qquad + e^{-n/6}\sqrt{n}\frac{117 + 1525n}{10080n^{5/2}} < 5e^{-n/6} =: e_3(n).
\end{aligned}$$

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Furthermore,

$$I_a(n{+}1) \geq \sqrt{\frac{6}{\pi}}\left(1 - \frac{3}{20(n+1)} - \frac{13}{1120(n+1)^2}\right) - e_2(n{+}1) - e_3(n{+}1),$$

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$$I_a(n) \leq \sqrt{\frac{6}{\pi}} \left(1 - \frac{3}{20n} - \frac{13}{1120n^2}\right) + e_2(n) + e_3(n).$$

Therefore

$$\begin{aligned} I(n+1) - I(n) \\ &\geq \sqrt{\frac{6}{\pi}}\left(\frac{3}{20n(n+1)} + \frac{13(2n+1)}{1120n^2(n+1)^2}\right) \\ &- 4a^{-n} - (e_2(n) + e_2(n+1) + e_3(n) + e_3(n+1)) \\ &\geq \sqrt{\frac{6}{\pi}}\left(\frac{3}{20n(n+1)}\right) - 4 \cdot 1.1^{-n} - \frac{R}{n^{5/2}} - 10e^{-n/6}. \end{aligned}$$

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&\qquad \geq \sqrt{\frac{6}{\pi}} \left(\frac{3}{20n(n+1)} + \frac{13(2n+1)}{1120n^2(n+1)^2} \right) \\
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Clearly, there exists an n_0, such that for all $n \geq n_0$ the above expression is strictly positive. Thus, $\mathrm{Vol}_{n-1}(C^n \cap H)$ is strictly monotonically increasing for $n \geq n_0$.

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- We note that with the same method (but more terms in the Taylor expansion) we can prove that $\mathrm{Vol}_{n-1}(C_n \cap H)$ is a concave sequence, i.e., $2I(n+1) \geq I(n) + I(n+2)$ for sufficiently large n.

Numerical bounds

- The following rigorous upper bound

$$|t^{(7)}(x)| \leq 1.25 \quad \text{for all } x \in [0, x(a)]$$

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  can be obtained using interval arithmetic and automatic differentiation combined with general analytic methods.
- Comparing this with the above estimate, we obtain that $n_0 = 145$ is sufficient.
- We can check $I(n)$ for $3 \leq n \leq 145$ by calculating the exact value by the Frank-Riede formula:

Numerical bounds

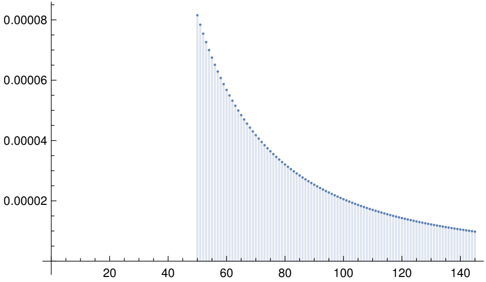

Figure: $I(n+1) - I(n)$ for $50 \leq n \leq 145$ plotted by Mathematica

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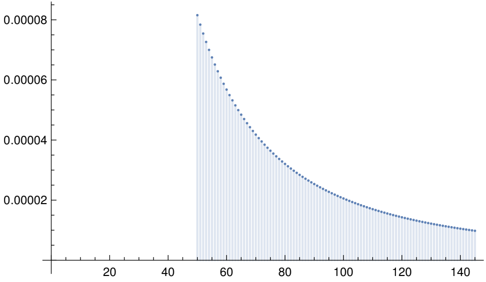

Figure: $I(n+1) - I(n)$ for $50 \leq n \leq 145$ plotted by Mathematica

Thus, we get the following theorem:

Theorem (F. Bartha, F.F., B. Gonzalez)

$Vol_{n-1}(C^n \cap H)$ is a strictly monotonically increasing function of n for all $n \geq 3$.

Thank you for your attention.