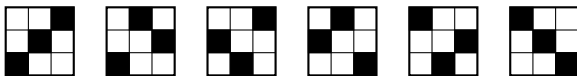


Superpermutations and Superpatterns

Miklós Bóna

University of Florida

Gainesville, FL



Rényi Institute

May 8, 2020

PATTERN CONTAINMENT

We say that a permutation $p_1p_2 \cdots p_n$ contains the shorter permutation $q_1q_2 \cdots q_k$ as a pattern if there is a subsequence of entries in p that relate to each other as the entries of q .

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That is, p contains q as a pattern if there is a subsequence of k entries $p_{i_1}p_{i_2} \cdots p_{i_k}$ so that $p_{i_a} < p_{i_b}$ if and only if $q_a < q_b$.

Example

The permutation $p = 57821346$ contains the pattern $q = 132$ as shown in the figure.

Example

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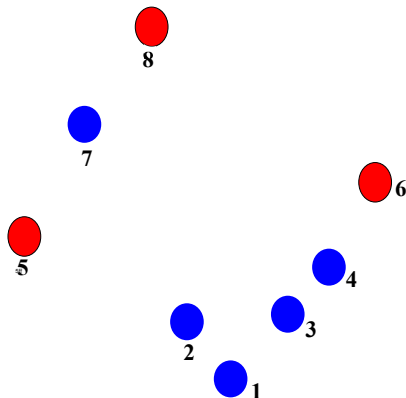


Figure: Containing the pattern 132.

k-SUPERPATTERNS

What is the shortest permutation that contains all patterns of length k ?

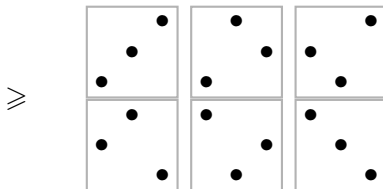
Let us call such a permutation a k -superpattern.

A 3-SUPERPATTERN

Ex: The permutation 25314 is a 3-superpattern.

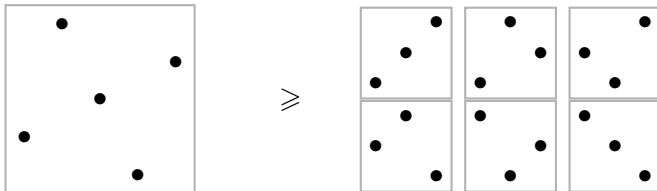
A 3-SUPERPATTERN

Ex: The permutation 25314 is a 3-superpattern.



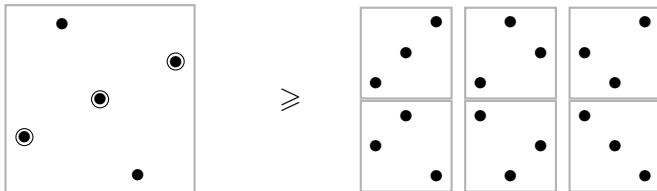
A 3-SUPERPATTERN

Ex: The permutation 25314 is a 3-superpattern.



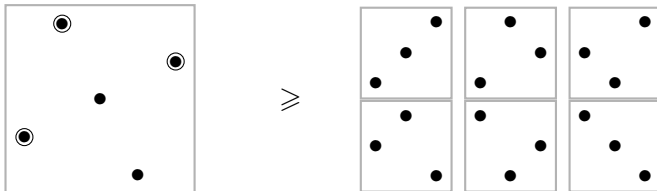
A 3-SUPERPATTERN

Ex: The permutation 25314 is a 3-superpattern.



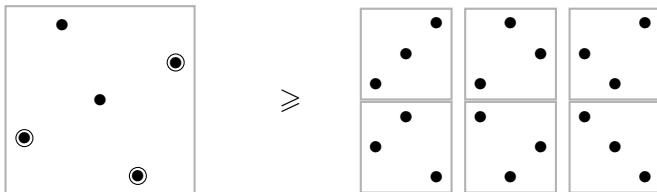
A 3-SUPERPATTERN

Ex: The permutation 25314 is a 3-superpattern.



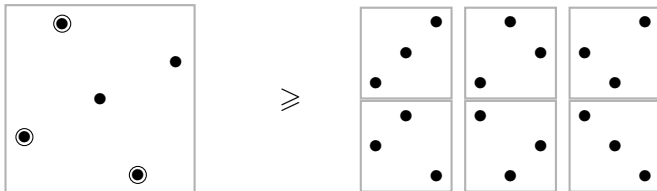
A 3-SUPERPATTERN

Ex: The permutation 25314 is a 3-superpattern.



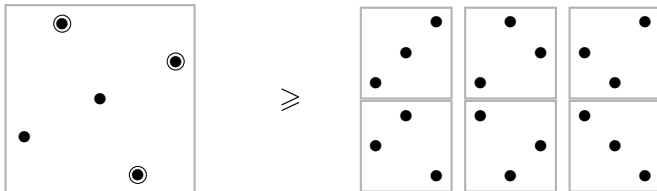
A 3-SUPERPATTERN

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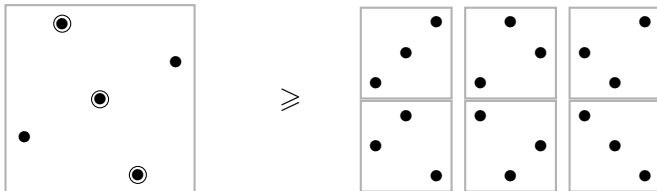
A 3-SUPERPATTERN

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A 3-SUPERPATTERN

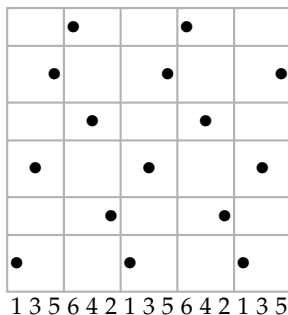
Ex: The permutation 25314 is a 3-superpattern.



An exhaustive search is feasible by computer for the next two values of n , resulting in $sp(4) = 9$ and $sp(5) = 13$.

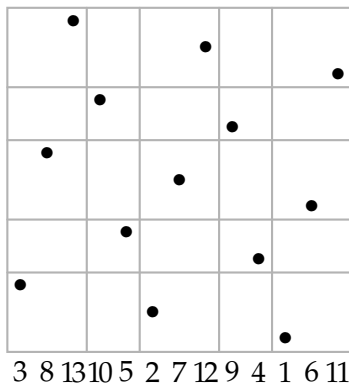
Theorem. *The inequality $\text{sp}(k) \leq \frac{k^2 + k}{2}$ holds.*

Observation. *What Miller actually proves is that there is a word of this length over the alphabet $[k + 1]$ that contains all patterns of length k as subsequences.*



ENGEN AND VATTER (2019)

Theorem. *The inequality $\text{sp}(k) \leq \left\lceil \frac{k^2 + 1}{2} \right\rceil$ holds.*



Is This It?

Is This It?

No. For $k = 6$, this bound is 19, but there is a 6-superpattern of length 17,

6 14 10 2 13 17 5 8 3 12 9 16 1 7 11 4 15,

found by Arnar Arnarson at *Permutation Patterns 2018*.

Is This It?

No. For $k = 6$, this bound is 19, but there is a 6-superpattern of length 17,

6 14 10 2 13 17 5 8 3 12 9 16 1 7 11 4 15,

found by Arnar Arnarson at *Permutation Patterns 2018*. This turns out to be optimal, after a gigantic computation.

Clearly, if there exists a k -superpattern of length n , then n has to be long enough to contain $k!$ distinct patterns of length k .

(n)

$$v^2$$

Interestingly, this trivial lower bound is quite difficult to improve, because most ideas do not lead to an improvement that is significant enough to increase the constant $1/e^2$.

SUPERPERMUTATIONS

Theorem. (Newey 1973) $\leq n^2 - 2n + 4$.

Theorem. (Adleman 1974)

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Theorem. (Newey 1973) $\leq n^2 - 2n + 4$.

Theorem. (Adleman 1974) $\leq n^2 - 2n + 4$.

Theorem. (Koutas and Hu 1975)

Theorem. (Koutas and Hu 1975) $\leq n^2 - 2n + 4$.

Theorem. (Galbiati and Preparata 1976)

Theorem. (Galbiati and Preparata 1976) $\leq n^2 - 2n + 4$.

Theorem. (Mohanty 1980) $\leq n^2 - 2n + 4$.

MOHANTY'S CONSTRUCTION

Write down n copies of 1, and between two consecutive 1s, insert any permutation of the set $\{2, 3, \dots, n\}$. This is a string of $n^2 - n + 1$ entries.

Remove $n - 3$ entries as follows. Let $1 < i < n - 1$. Remove an entry j from the i th segment, then in segment $i - 1$, move the entry j of that segment into the last position, and in segment $i + 1$ move the entry j of that segment into the first position.

12345 1 2345 1 2345 1 2345 1

12345 1 234 1 523 1 42351

Theorem. (Zălinescu 2011) $\leq n^2 - 2n + 3$ for $n \geq 10$.

Theorem. (Zălinescu 2011) $\leq n^2 - 2n + 3$ for $n \geq 10$.

Theorem. (Radomirović 2012) $\leq \left\lceil n^2 - \frac{7n}{3} + \frac{19}{3} \right\rceil$ for $n \geq 7$.

Theorem. (Zălinescu 2011) $\leq n^2 - 2n + 3$ for $n \geq 10$.

Theorem. (Radomirović 2012) $\leq \left\lceil n^2 - \frac{7n}{3} + \frac{19}{3} \right\rceil$ for $n \geq 7$.

Theorem. (Kleitman and Kwiatkowski 1976) $\geq n^2 - c_\epsilon n^{7/4+\epsilon}$.

CONTAINING AS FACTORS, OVER $[n]$ (SUPERFACTORS)

Theorem (Ashlock and Tillotson 1993). *There is a word of length*

$$n! + (n - 1)! + (n - 2)! + \cdots + 2! + 1$$

over the alphabet $[n]$ that contains all permutations of $[n]$ contiguously.

Conjecture (Ashlock and Tillotson 1993). *The above construction is best possible.*

Proposition (Ashlock and Tillotson 1993). *Any such word has length at least $n! + (n - 1)! + n - 2$.*

Proposition (Houston 2014). *There is a word of length*

$$6! + 5! + 4! + 3! + 2! = 872$$

over the alphabet $[6]$ that contains all permutations of $[6]$ contiguously.

The Crackpot Index

John Baez

A simple method for rating potentially revolutionary contributions to physics:

1. A -5 point starting credit.
2. 1 point for every statement that is widely agreed on to be false.
3. 2 points for every statement that is clearly vacuous.
4. 3 points for every statement that is logically inconsistent.
5. 5 points for each such statement that is adhered to despite careful correction.
6. 5 points for using a thought experiment that contradicts the results of a widely accepted real experiment.
7. 5 points for each word in all capital letters (except for those with defective keyboards).
8. 5 points for each mention of "Einstien", "Hawkins" or "Feynmann".
9. 10 points for each claim that quantum mechanics is fundamentally misguided (without good evidence).
10. 10 points for pointing out that you have gone to school, as if this were evidence of sanity.
11. 10 points for beginning the description of your theory by saying how long you have been working on it.

**John Carlos Baez**

@johncarlosbaez

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If you don't believe me, see my next tweet.

JUST WHEN YOU THOUGHT
YOU UNDERSTOOD THE PATTERN

The smallest number with the digits from 1 to 2 arranged in all possible orders has $1! + 2! = 3$ digits: 121.

The smallest number with the digits from 1 to 3 arranged in all possible orders has $1! + 2! + 3! = 9$ digits: 123121321.

The smallest number with the digits from 1 to 4 arranged in all possible orders has $1! + 2! + 3! + 4! = 33$ digits: 123412314231243121342132413214321.

The smallest number with the digits from 1 to 5 arranged in all possible orders has $1! + 2! + 3! + 4! + 5! = 153$ digits.

Does the smallest number with the digits from 1 to 6 arranged in all possible orders have $1! + 2! + 3! + 4! + 5! + 6! = 873$ digits?

No: 872 is enough. Or maybe even fewer — nobody knows yet!

3:54 PM - 26 Sep 2018

56 Retweets 121 Likes



JUST WHEN YOU THOUGHT YOU UNDERSTOOD THE PATTERN

The smallest number with the digits from 1 to 2 arranged in all possible orders has $1! + 2! = 3$ digits: 121.

The smallest number with the digits from 1 to 3 arranged in all possible orders has $1! + 2! + 3! = 9$ digits: 123121321.

The smallest number with the digits from 1 to 4 arranged in all possible orders has $1! + 2! + 3! + 4! = 33$ digits: 123412314231243121342132413214321.

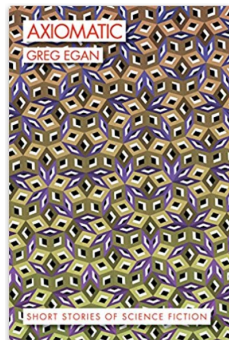
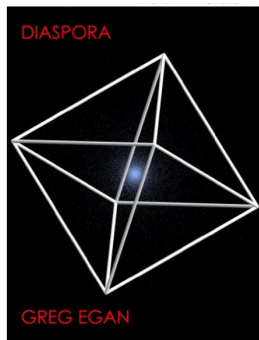
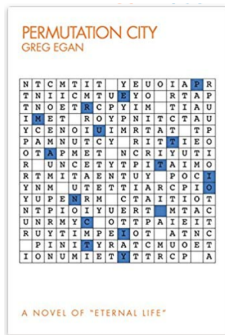
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Does the smallest number with the digits from 1 to 6 arranged in all possible orders have $1! + 2! + 3! + 4! + 5! + 6! = 873$ digits?

No: 872 is enough. Or maybe even fewer — nobody knows yet!

Theorem (Egan, 19 October 2018). *There is a word of length $n! + (n - 1)! + (n - 2)! + (n - 3)! + n - 3$ over the alphabet $[n]$ that contains all permutations of $[n]$ contiguously.*

Theorem (Egan, 19 October 2018). *There is a word of length $n! + (n - 1)! + (n - 2)! + (n - 3)! + n - 3$ over the alphabet $[n]$ that contains all permutations of $[n]$ contiguously.*



THE LOWER BOUND

Proposition (Ashlock and Tillotson 1993). $\geq n! + (n-1)! + n - 2$.

THE LOWER BOUND

Proposition (Ashlock and Tillotson 1993). $\geq n! + (n-1)! + n - 2$.

Question on 4Chan (2011).

You have an n episode tv series. You want to watch the episodes in every order possible. What is the least number of episodes that you would have to watch?

Over lapping is allowed. For example, in the case of $n = 2$, watching episode 1, then 2, then 1 again, would fit the criteria.

The orders must be continuous. For example, $(1, 2, 1, 3)$ does NOT contain the sequence $(1, 2, 3)$.

THE 4CHAN LOWER BOUND

■ **Lower bounds** Anonymous Sat Sep 17 08:35:54 2011 No.3751197

Quoted by: >>3751366 >>3751370 >>3752047_2 >>3752047_25

I think I have a proof of the lower bound $n! + (n-1)! + (n-2)! + n-3$ (for $n \geq 2$). I'll need to do this in multiple posts. Please look it over for any loopholes I might have missed.

As in other posts, let n (lowercase) = the number of symbols; there are $n!$ permutations to iterate through.

The obvious lower bound is $n! + n-1$. We can obtain this as follows:

Let

L = the running length of the string

N_0 = the number of permutations visited

$X_0 = L - N_0$

When you write down the first permutation, X_0 is already $n-1$. For each new permutation you visit, the length of the string must increase by at least 1. So X_0 can never decrease. At the end, $N_0 = n!$, giving us $L \geq n! + n-1$.

I'll use similar methods to go further, but first I'll need to explain my terminology...

THE 4CHAN LOWER BOUND

☐ **Lower bounds** Anonymous Sat Sep 17 08:35:54 2011 No.3751199

Quoted by: >>3751366

Edges:

I'm picturing the ways to get from one permutation to the next as a directed graph where the nodes correspond to permutations and the edges to ways to get from one to the next. A k-edge is an edge in which you move k symbols from the beginning of the permutation to the end; for example,

1234567 -> 4567321

would be a 3-edge. Note that I don't include edges like

12345 -> 34512

in which you pass through through a permutation in the middle (in this case 23451). This example would be considered two edges:

12345 -> 23451

23451 -> 34512

From every node there is exactly one 1-edge, e.g.:

12345 -> 23451

THE 4CHAN LOWER BOUND

☐ **Lower bounds** Anonymous Sat Sep 17 08:37:54 2011 No.3751204

Quoted by: >>3751366 >>3751541

1-loops:

I call the set of n permutations connected by a cyclic path of 1-edges a 1-loop. There are $(n-1)!$ 1-loops.

The concept of 1-loops is enough to get the next easiest lower bound of $n! + (n-1)! + n-2$. That's because to pass from one 1-loop to another, it is necessary to take a 2-edge or higher. Let us define:

N_1 = the number of 1-cycles completed or that we are currently in

$X_1 = L - N_1 - N_2$

The definition of N_1 is a bit more complicated than we need for this proof, but we'll need it later. You might ask, isn't N_1 just one more than the number of completed 1-cycles? No! When we have just completed a 1-cycle, it is equal to the number of completed 1-cycles.

In order to increment N_1 , we have to take a 2-edge, which increases L by 2 instead of 1. Therefore X_1 can never decrease. Since X_1 starts out with the value $n-2$,

THE 4CHAN LOWER BOUND

☐ **Lower bounds** Anonymous Sat Sep 17 08:37:54 2011 No.3751205

Quoted by: >>3751366

2-loops:

Suppose we enter a 1-loop, iterate through all n nodes (as is done in the greedy palindrome algorithm), and then take a 2-edge out. The edge we exit by is determined by the entry point. The permutation that the 2-edge takes us to is determined by taking the entry point and rotating the first $n-1$ characters, e.g.:

12345

is taken by $n-1$ 1-edges to

51234

which is taken by a 2-edge to


23415

If we repeat this process, it takes us around in a larger loop passing through $n(n-1)$ permutations. I call this greater loop a 2-loop.

The greedy palindrome algorithm uses ever-larger loops; it connects $(n-k+1)$ k -loops via $(k+1)$ -edges to make $(k+1)$ -loops. But I haven't been able to prove anything about these larger loops yet.

The tricky thing about 2-loops is that which 2-loop you're in depends on the point at which you entered the current 1-loop. Each of the n possible entry points to a 1-loop

THE 4CHAN LOWER BOUND

 **Anonymous** Sat Sep 17 08:38:54 2011 No.3751207

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y: >>3751366

And now for the proof of the $n! + (n-1)! + (n-2)!$ lower bound...

To review:

n = alphabet length

L = running string length

N_0 = number of permutations visited

$X_0 = L - N_0$

N_1 = number of 1-cycles completed or that we are currently in

$X_1 = L - N_1 - N_2$

In order to increase N_1 , you must jump to a new 1-cycle -- having completed the one you are leaving. That means the next permutation P' in the 1-cycle (following your exit point P) is one you have already visited. Either you have at some point entered the 1-cycle at P' , or this is the second or greater time you've visited P . If you have ever entered the 1-cycle at P' , leaving at P by a 2-edge will not take you to a new 2-cycle; you will be in the same 2-cycle you were in when you entered at P' .

So these are the available ways to enter a 2-cycle you've never been in before:

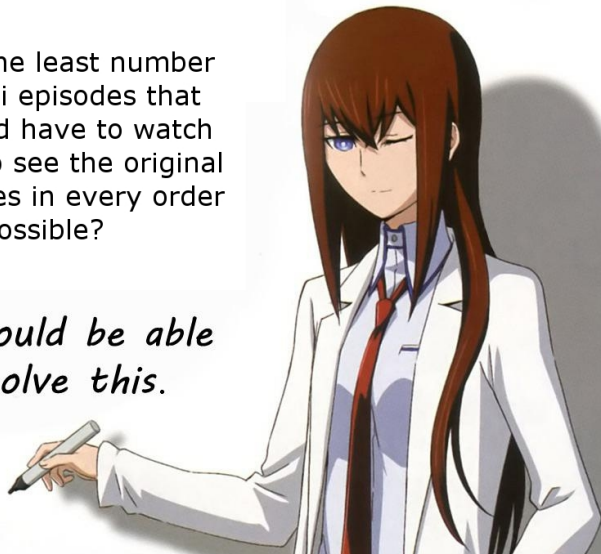
- * take a 3-edge or higher

- * take a 2-edge but don't increase N_1

- * take a 2-edge from a permutation P that you were visiting for the second or greater

What is the least number
of Haruhi episodes that
you would have to watch
in order to see the original
14 episodes in every order
possible?

*You should be able
to solve this.*



**Robin Houston**

@robinhouston

Following



A curious situation. The best known lower bound for the minimal length of superpermutations was proved by an anonymous user of a wiki mainly devoted to anime.

**The Haruhi Problem | /sci/ - Math & Science Wiki | ...**

More formally, "what is the shortest string containing all permutations of a set of n elements?" Started by:二ア愛!pQsULI4sXc

mathsci.wikia.com

12:37 AM - 23 Oct 2018

4,530 Retweets 9,378 Likes

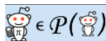


107

4.5K

9.4K





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∞ An anonymous user on 4chan solved an interesting problem, but no one knows how to cite them. (twitter.com)
 3339 submitted 9 days ago by Araraguy
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∞ [-] [TheCatcherOfThePie](#) Undergraduate 356 points 9 days ago*

For anyone wondering why the proof appeared on an anime wiki of all places: "The Melancholy of Haruhi Suzumiya" is an anime which has multiple orders in which it can be watched (the main two are broadcast order and chronological order). This question is a joke asking how many episodes you would need to watch in order to see every possible watch order.

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∞ [-] [Aenonimos](#) 102 points 9 days ago

Yeah that was the joke. In 2011 there was 4chan meme with the format "difficult problems: "you should be able to

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An anonymous 4chan post could help solve a 25-year-old math mystery

An anime math problem

By [Mary Beth Griggs](#) | Oct 24, 2018, 4:33pm EDT

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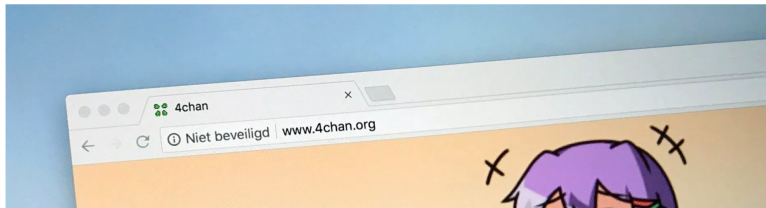
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Anonymous 4chan user solves 25 year old math problem

 **Pranav** on October 25, 2018

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Anonymous anime fan might just have solved a question that's stumped maths experts for 25 years

Posted 2 days ago by [Greg Evans](#) in [offbeat](#)

UPVOTE

Robin Houston

@robinhouston

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A curious situation. The best known lower bound for the minimal length of superpermutations was proved by an anonymous user of a wiki mainly devoted to anime.

Find out if a

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Haruhi Helps Crack a 25-Year-Old Mathematical Conundrum

posted on **2018-10-25 15:45 EDT** by Lynzee Loveridge





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Infamous Math Problem Finally Gets Solved by Fan in Anime Chatroom

By NICK VALDEZ - October 27, 2018



Anime fans surprise all the time with the variety of fans there are, and one fan just might have used anime to help solve a tricky problem mathematicians have been trying to solve for the past 25 years.

Although the fan just used the equation to figure out the best way to watch every order of *The Melancholy of Haruhi Suzumiya* in the shortest time possible, now mathematicians are wondering how to progress with this solved equation from an anonymous fan.

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4chan solved part of math problem puzzling scientists for decades

Published time: 26 Oct, 2018 17:54

Edited time: 27 Oct, 2018 10:00

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COMBINATORICS

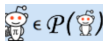
A new proof from the Australian science fiction writer Greg Egan and a 2011 proof anonymously posted online are now being hailed as significant advances on a puzzle mathematicians have been studying for at least 25 years.

 |

— By ERICA KLARREICH

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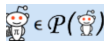
An anonymous user on 4chan solved an interesting problem, but no one knows how to cite them. (twitter.com)

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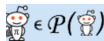
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A lower bound on the length of the shortest superpermutation

Anonymous 4chan Poster, Michael Albert, Robin Houston, Jay Pantone, and Vince Vatter



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A lower bound on the length of the shortest superpermutation

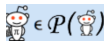
Anonymous 4chan Poster, Michael Albert, Robin Houston, Jay Pantone, and Vince Vatter

Upper bound (as of now). $\leq n! + (n-1)! + (n-2)! + (n-3)! + n - 3$.

Lower bound (as of now). $\geq n! + (n-1)! + (n-2)! + n - 3$.

Conjecture. $n! + (n-1)! + (n-2)! + (n-3)! + n - 4$.

This conjecture was disproved by Robin Houston for $n = 7$.



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A lower bound on the length of the shortest superpermutation

Anonymous 4chan Poster, Michael Albert, Robin Houston, Jay Pantone, and Vince Vatter

Upper bound (as of now). $\leq n! + (n-1)! + (n-2)! + (n-3)! + n - 3$.

Lower bound (as of now). $\geq n! + (n-1)! + (n-2)! + n - 3$.

Conjecture. $n! + (n-1)! + (n-2)! + (n-3)! + n - 4$.

This conjecture was disproved by Robin Houston for $n = 7$.

Thank you.