

GLOBAL DEFORMATIONS OF LIE ALGEBRAS

ALICE FIALOWSKI

ABSTRACT. By considering non-trivial global deformations of the Witt (and the Virasoro) algebra given by geometric constructions it is shown that, despite their infinitesimal and formal rigidity, they are globally not rigid. This shows the need of a clear indication of the type of deformations considered. The families appearing are constructed as families of algebras of Krichever-Novikov type.

1. INTRODUCTION

This talk is based on a joint work with Martin Schlichenmayer (see [6]).

Deformations of mathematical structures are important in most part of mathematics and its applications. Considering deformations of a given object, a natural question arises: Can we equip the set of nonequivalent deformations with the structure of a topological or even geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object the deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

In this respect, a good example where the picture is completely clear is the classification of complex analytic structures on a fixed topological manifold. Also in algebraic geometry one has well-developed results in this direction. Recall for instance that for the moduli space \mathcal{M}_g of smooth projective curves of genus g over \mathbb{C} (or equivalently, compact Riemann surfaces of genus g) the tangent space $T_{[C]}\mathcal{M}_g$ can be naturally identified with $H^1(C, T_C)$, where T_C is the sheaf of holomorphic vector fields over C . This extends to higher dimension. In particular, it turns out that for compact complex manifolds M , the condition $H^1(M, T_M) = 0$ implies that M is rigid, [9, Thm. 4.4]. Rigidity means that any differentiable family $\pi : \mathcal{M} \rightarrow B \subseteq \mathbb{R}$, $0 \in B$ which contains M as the special member $M_0 := \pi^{-1}(0)$ is trivial in a neighbourhood of 0, i.e. for t small enough $M_t := \pi^{-1}(t) \cong M$. (see also [11] for definitions, results, and further references).

Such results lead to the impression that the vanishing of the relevant cohomology spaces will imply rigidity with respect to deformations also in the case of other structures.

In this talk I will consider an infinite dimensional Lie algebra example. The results will show that the theory of deformations of infinite-dimensional Lie algebras is far from being satisfactory. What I will show here, can also be done in the case of associative algebras.

Consider the complexification of the Lie algebra of polynomial vector fields on the circle with generators

$$l_n := \exp(in\varphi) \frac{d}{d\varphi}, \quad n \in \mathbb{Z}$$

where φ is the angular parameter. The bracket operation in this Lie algebra is

$$[l_n, l_m] = (m - n)l_{n+m}.$$

We call it the *Witt algebra* and denote it by \mathcal{W} . Equivalently, the Witt algebra can be described as the Lie algebra of meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which

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are holomorphic outside the points 0 and ∞ . In this presentation $l_n = z^{n+1} \frac{d}{dz}$, where z is the quasi-global coordinate on $\mathbb{P}^1(\mathbb{C})$. The Lie algebra \mathcal{W} is infinite dimensional and graded with the standard grading $\deg l_n = n$. By taking formal vector fields with the projective limit topology we get the completed topological Witt algebra $\widehat{\mathcal{W}}$. In this talk I will consider its everywhere dense subalgebra \mathcal{W} .

It is well-known that \mathcal{W} has a unique nontrivial one-dimensional central extension, the *Virasoro algebra* \mathcal{V} . It is generated by l_n ($n \in \mathbb{Z}$) and the central element c , and its bracket operation is defined by

$$[l_n, l_m] = (m - n)l_{n+m} + \frac{1}{12}(m^3 - m)\delta_{n, -m} c, \quad [l_n, c] = 0. \quad (1.1)$$

The cohomology “responsible” for deformations is $H^2(\mathcal{W}, \mathcal{W})$. It is known that $H^2(\mathcal{W}, \mathcal{W}) = \{0\}$ (see [4]). Hence, guided by the experience in the theory of deformations of complex manifolds, one might think that \mathcal{W} is rigid in the sense that all families containing \mathcal{W} as a special element will be trivial. But this is not the case as we will show. Certain natural families of Krichever-Novikov type algebras of geometric origin (see Section 3 for their definition) will appear which contain \mathcal{W} as special element. But none of the other elements will be isomorphic to \mathcal{W} . In fact, from $H^2(\mathcal{W}, \mathcal{W}) = \{0\}$ it follows that the Witt algebra is infinitesimally and formally rigid. But this condition does not imply that there are no non-trivial global deformation families. The main point to learn is that it is necessary to distinguish clearly the formal and the global deformation situation. The formal rigidity of the Witt algebra indeed follows from $H^2(\mathcal{W}, \mathcal{W}) = \{0\}$, but no statement like that about global deformations.

There is a lot of confusion in the literature in the notion of a deformation. Several different (inequivalent) approaches exist. One of our aims here is to clarify the difference between deformations of geometric origin and so called formal deformations. Formal deformation theory has the advantage of using cohomology. It is also complete in the sense that under some natural cohomology assumptions there exists a versal formal deformation which induces all other deformations. Formal deformations are deformations with a complete local algebra as base. A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the base of deformation – which is a commutative algebra of functions – with a smooth manifold, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations. As we will see in the case of the infinite dimensional Witt algebra, there is no tight relation between global and formal deformations.

I am going to introduce and recall the necessary properties of the Krichever-Novikov type vector field algebras. They are generalizations of the Witt algebra (in its presentation as vector fields on $\mathbb{P}^1(\mathbb{C})$) to higher genus smooth projective curves.

We construct global deformations of the Witt algebra by considering certain families of algebras for the genus one case (i.e. the elliptic curve case) and let the elliptic curve degenerate to a singular cubic. The two points, where poles are allowed, are the zero element of the elliptic curve (with respect to its additive structure) and a 2-torsion point. In this way we obtain families parameterized over the affine line with the peculiar behaviour that every family is a global deformation of the Witt algebra, i.e. \mathcal{W} is a special member, whereas all other members are mutually isomorphic but not isomorphic to \mathcal{W} , see Theorem 3.4. Globally these

families are non-trivial, but infinitesimally and formally they are trivial. The construction can be extended to the centrally extended algebras, yielding global deformations of the Virasoro algebra.

2. GLOBAL DEFORMATIONS AND FORMAL DEFORMATIONS

2.1. Intuitively. Let us start with the intuitive definition. Let \mathcal{L} be a Lie algebra with Lie bracket μ_0 over a field \mathbb{K} . A *deformation* of \mathcal{L} is a one-parameter family \mathcal{L}_t of Lie algebras with the bracket

$$\mu_t = \mu_0 + t\phi_1 + t^2\phi_2 + \dots \quad (2.1)$$

where the ϕ_i are \mathcal{L} -valued 2-cochains, i.e. elements of $\text{Hom}_{\mathbb{K}}(\wedge^2 \mathcal{L}, \mathcal{L}) = C^2(\mathcal{L}; \mathcal{L})$, and \mathcal{L}_t is a Lie algebra for each $t \in \mathbb{K}$. (see [2, 8]). Two deformations \mathcal{L}_t and \mathcal{L}'_t are equivalent if there exists a linear automorphism $\hat{\psi}_t = \text{id} + \psi_1 t + \psi_2 t^2 + \dots$ of \mathcal{L} where ψ_i are linear maps over \mathbb{K} , i.e. elements of $C^1(\mathcal{L}, \mathcal{L})$, such that

$$\mu'_t(x, y) = \hat{\psi}_t^{-1}(\mu_t(\hat{\psi}_t(x), \hat{\psi}_t(y))). \quad (2.2)$$

The Jacobi identity for the algebra \mathcal{L}_t implies that the 2-cochain ϕ_1 is indeed a cocycle, i.e. it fulfills $d_2\phi_1 = 0$ with respect to the Lie algebra cochain complex of \mathcal{L} with values in \mathcal{L} (see [7] for the definitions). If ϕ_1 vanishes identically, the first nonvanishing ϕ_i will be a cocycle. If μ'_t is an equivalent deformation (with cochains ϕ'_i) then

$$\phi'_1 - \phi_1 = d_1\psi_1. \quad (2.3)$$

Hence, every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L}, \mathcal{L})$. This class is called the *differential* of the deformation. The differential of a family which is equivalent to a trivial family will be the zero cohomology class.

2.2. Global deformations. Consider now a deformation \mathcal{L}_t not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra A over \mathbb{K} with identity as base of a deformation.

In the following we will assume that A is a commutative algebra over the field \mathbb{K} of characteristic zero which admits an augmentation $\epsilon : A \rightarrow \mathbb{K}$. This says that ϵ is a \mathbb{K} -algebra homomorphism, e.g. $\epsilon(1_A) = 1$. The ideal $m_\epsilon := \text{Ker } \epsilon$ is a maximal ideal of A . Vice versa, given a maximal ideal m of A with $A/m \cong \mathbb{K}$, then the natural factorization map defines an augmentation.

In case that A is a finitely generated \mathbb{K} -algebra over an algebraically closed field \mathbb{K} then $A/m \cong \mathbb{K}$ is true for every maximal ideal m . Hence in this case every such A admits at least one augmentation and all maximal ideals are coming from augmentations.

Let us consider a Lie algebra \mathcal{L} over the field \mathbb{K} , ϵ a fixed augmentation of A , and $m = \text{Ker } \epsilon$ the associated maximal ideal.

Definition 2.1. A *global deformation* λ of \mathcal{L} with the base (A, m) or simply with the base A , is a Lie A -algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\cdot, \cdot]_\lambda$ such that

$$\epsilon \otimes \text{id} : A \otimes \mathcal{L} \rightarrow \mathbb{K} \otimes \mathcal{L} = \mathcal{L} \quad (2.4)$$

is a Lie algebra homomorphism (see [5]). Specifically, it means that for all $a, b \in A$ and $x, y \in \mathcal{L}$,

- (1) $[a \otimes x, b \otimes y]_\lambda = (ab \otimes \text{id})[1 \otimes x, 1 \otimes y]_\lambda$,
- (2) $[\cdot, \cdot]_\lambda$ is skew-symmetric and satisfies the Jacobi identity,
- (3) $\epsilon \otimes \text{id}([1 \otimes x, 1 \otimes y]_\lambda) = 1 \otimes [x, y]$.

By Condition (1) to describe a deformation it is enough to give the elements $[1 \otimes x, 1 \otimes y]_\lambda$ for all $x, y \in \mathcal{L}$. By condition (3) follows that for them the Lie product has the form

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y] + \sum_i a_i \otimes z_i, \quad (2.5)$$

with $a_i \in m$, $z_i \in \mathcal{L}$.

A deformation is called *trivial* if $A \otimes_{\mathbb{K}} \mathcal{L}$ carries the trivially extended Lie structure, i.e. (2.5) reads as $[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y]$. Two deformations of a Lie algebra \mathcal{L} with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\epsilon \otimes \text{id}$.

We say that a deformation is *local* if A is a local \mathbb{K} -algebra with unique maximal ideal m_A . By assumption $m_A = \text{Ker } \epsilon$ and $A/m_A \cong \mathbb{K}$. In case that in addition $m_A^2 = 0$, the deformation is called *infinitesimal*.

Example. If $A = \mathbb{K}[t]$, then this is the same as an algebraic 1-parameter deformation of \mathcal{L} . In this case we sometimes use simply the expression “deformation over the affine line.” This can be extended to the case where A is the algebra of regular functions on an affine variety X . In this way we obtain algebraic deformations over an affine variety. These deformations are non-local, and will be the objects of our study.

Let A' be another commutative algebra over \mathbb{K} with a fixed augmentation $\epsilon' : A' \rightarrow \mathbb{K}$, and let $\phi : A \rightarrow A'$ be an algebra homomorphism with $\phi(1) = 1$ and $\epsilon' \circ \phi = \epsilon$. If a deformation λ of \mathcal{L} with base $(A, \text{Ker } \epsilon = m)$ is given, then the *push-out* $\lambda' = \phi_* \lambda$ is the deformation of \mathcal{L} with base $(A', \text{Ker } \epsilon' = m')$, which is the Lie algebra structure

$$[a'_1 \otimes_A (a_1 \otimes l_1), a'_2 \otimes_A (a_2 \otimes l_2)]_{\lambda'} := a'_1 a'_2 \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda,$$

($a'_1, a'_2 \in A', a_1, a_2 \in A, l_1, l_2 \in \mathcal{L}$) on $A' \otimes \mathcal{L} = (A' \otimes_A A) \otimes \mathcal{L} = A' \otimes_A (A \otimes \mathcal{L})$. Here A' is regarded as an A -module with the structure $aa' = a'\phi(a)$.

Remark. For non-local algebras there exist more than one maximal ideal, and hence in general many different augmentations ϵ . Let \mathcal{L} be a \mathbb{K} -vector space and assume that there exists a Lie A -algebra structure $[\cdot, \cdot]_A$ on $A \otimes_{\mathbb{K}} \mathcal{L}$. Given an augmentation $\epsilon : A \rightarrow \mathbb{K}$ with associated maximal ideal $m_\epsilon = \text{Ker } \epsilon$, one obtains a Lie \mathbb{K} -algebra structure $\mathcal{L}^\epsilon = (\mathcal{L}, [\cdot, \cdot]_\epsilon)$ on the vector space \mathcal{L} by

$$\epsilon \otimes \text{id}([1 \otimes x, 1 \otimes y]_A) = 1 \otimes [x, y]_\epsilon. \quad (2.6)$$

Comparing this with Definition 2.1 we see that by construction the Lie A -algebra $A \otimes_{\mathbb{K}} \mathcal{L}$ will be a global deformation of the Lie \mathbb{K} -algebra \mathcal{L}^ϵ . On the level of structure constants the described construction corresponds simply to the effect of “reducing the structure constants of the algebra modulo m_ϵ .” In other words, for $x, y, z \in \mathcal{L}$ basis elements, let the Lie A -algebra structure be given by

$$[1 \otimes x, 1 \otimes y]_A = \sum_z C_{x,y}^z (1 \otimes z), \quad C_{x,y}^z \in A. \quad (2.7)$$

Then \mathcal{L}^ϵ is defined via

$$[x, y]_\epsilon := \sum_z (C_{x,y}^z \bmod m_\epsilon) z. \quad (2.8)$$

In general, the algebras \mathcal{L}^ϵ will be different for different ϵ . The Lie A -algebra $A \otimes_{\mathbb{K}} \mathcal{L}$ will be a deformation of different Lie \mathbb{K} -algebras L^ϵ .

Example. For a deformation of the Lie algebra $\mathcal{L} = \mathcal{L}_0$ over the affine line, the Lie structure \mathcal{L}_α in the fiber over the point $\alpha \in \mathbb{K}$ is given by considering the augmentation corresponding to the maximal ideal $m_\alpha = (t - \alpha)$. This explains the picture in the geometric interpretation of the deformation.

2.3. Formal deformations. Let A be a complete local algebra over \mathbb{K} , so $A = \varprojlim_{n \rightarrow \infty} (A/m^n)$, where m is the maximal ideal of A and we assume that $A/m \cong \mathbb{K}$.

Definition 2.2. A *formal deformation* of \mathcal{L} with base A is a Lie algebra structure on the completed tensor product $A \widehat{\otimes} \mathcal{L} = \varprojlim_{n \rightarrow \infty} ((A/m^n) \otimes \mathcal{L})$ such that

$$\epsilon \widehat{\otimes} \text{id} : A \widehat{\otimes} \mathcal{L} \rightarrow \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Example. If $A = \mathbb{K}[[t]]$, then a formal deformation of \mathcal{L} with base A is the same as a formal 1-parameter deformation of \mathcal{L} (see [8]).

There is an analogous definition for equivalence of deformations parameterized by a complete local algebra.

2.4. Infinitesimal and versal formal deformations. In the following let the base of the deformation be a local \mathbb{K} -algebra (A, m) with $A/m \cong \mathbb{K}$. In addition we assume that $\dim(m^k/m^{k+1}) < \infty$ for all k .

Proposition 2.3. ([5]) *With the assumption $\dim H^2(\mathcal{L}; \mathcal{L}) < \infty$, there exists a universal infinitesimal deformation $\eta_{\mathcal{L}}$ of the Lie algebra \mathcal{L} with base $B = \mathbb{K} \oplus H^2(\mathcal{L}; \mathcal{L})'$, where the second summand is the dual of $H^2(\mathcal{L}; \mathcal{L})$ equipped with zero multiplication, i.e.*

$$(\alpha_1, h_1) \cdot (\alpha_2, h_2) = (\alpha_1 \alpha_2, \alpha_1 h_2 + \alpha_2 h_1).$$

This means that for any infinitesimal deformation λ of the Lie algebra \mathcal{L} with finite dimensional (local) algebra base A there exists a unique homomorphism $\phi : \mathbb{K} \oplus H^2(\mathcal{L}; \mathcal{L})' \rightarrow A$ such that λ is equivalent to the push-out $\phi_* \eta_{\mathcal{L}}$.

Although in general it is impossible to construct a universal formal deformation, there is a so-called versal element.

Definition 2.4. A formal deformation η of \mathcal{L} parameterized by a complete local algebra B is called *versal* if for any deformation λ , parameterized by a complete local algebra (A, m_A) , there is a morphism $f : B \rightarrow A$ such that

- 1) The push-out $f_* \eta$ is equivalent to λ .
- 2) If A satisfies $m_A^2 = 0$, then f is unique (see [2, 5]).

Remark. A versal formal deformation is sometimes called miniversal.

Theorem 2.5. ([3],[5, Thm. 4.6]) *Let the space $H^2(\mathcal{L}; \mathcal{L})$ be finite dimensional.*

(a) *There exists a versal formal deformation of \mathcal{L} .*

(b) *The base of the versal formal deformation is formally embedded into $H^2(\mathcal{L}; \mathcal{L})$, i.e. it can be described in $H^2(\mathcal{L}; \mathcal{L})$ by a finite system of formal equations.*

A Lie algebra \mathcal{L} is called (infinitesimally, formally, or globally) rigid if every (infinitesimal, formal, global) family is equivalent to a trivial one. Assume $H^2(\mathcal{L}, \mathcal{L}) < \infty$ in the following. By Proposition 2.3 the elements of $H^2(\mathcal{L}; \mathcal{L})$ correspond bijectively to the equivalence classes of infinitesimal deformations, as equivalent deformations up to order 1 differ from each other only in coboundary. Together with Theorem 2.5, Part (b), follows

Corollary 2.6.

(a) *\mathcal{L} is infinitesimally rigid if and only if $H^2(\mathcal{L}, \mathcal{L}) = \{0\}$.*

(b) *$H^2(\mathcal{L}; \mathcal{L}) = \{0\}$ implies that \mathcal{L} is formally rigid.*

Let us stress the fact, that $H^2(\mathcal{L}, \mathcal{L}) = \{0\}$ does not imply that every global deformation will be equivalent to a trivial one. Hence, \mathcal{L} is in this case not necessarily globally rigid. In this talk we will see plenty of nontrivial global deformations of the Witt algebra \mathcal{W} . Hence, the Witt algebra is not globally rigid.

The Lie algebras considered here are infinite dimensional. Such Lie algebras possess a topology with respect to which all algebraic operations are continuous. In this situation, in a cochain complex it is natural to distinguish the sub-complex formed by the continuous cohomology of the Lie algebra (see [1]).

It is known (see [4]) that the Witt and the Virasoro algebra are formally rigid.

3. KRICHEVER-NOVIKOV ALGEBRAS

3.1. The algebras with their almost-grading. Algebras of Krichever-Novikov types are generalizations of the Virasoro algebra and all its related algebras. We only recall the definitions and facts needed here. Let M be a compact Riemann surface of genus g , or in terms of algebraic geometry, a smooth projective curve over \mathbb{C} . Let $N, K \in \mathbb{N}$ with $N \geq 2$ and $1 \leq K < N$ be numbers. Fix

$$I = (P_1, \dots, P_K), \quad \text{and} \quad O = (Q_1, \dots, Q_{N-K})$$

disjoint ordered tuples of distinct points (“marked points”, “punctures”) on the curve. In particular, we assume $P_i \neq Q_j$ for every pair (i, j) . The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets and set $A = I \cup O$ as a set.

Denote by \mathcal{L} the Lie algebra consisting of those meromorphic sections of the holomorphic tangent line bundle which are holomorphic outside of A , equipped with the Lie bracket $[\cdot, \cdot]$ of vector fields. Its local form is

$$[e, f] = \left[e(z) \frac{d}{dz}, f(z) \frac{d}{dz} \right] := \left(e(z) \frac{df}{dz}(z) - f(z) \frac{de}{dz}(z) \right) \frac{d}{dz}. \quad (3.1)$$

To avoid cumbersome notation we will use the same symbol for the section and its representing function.

For the Riemann sphere ($g = 0$) with quasi-global coordinate z , $I = \{0\}$ and $O = \{\infty\}$, the introduced vector field algebra is the Witt algebra.

For infinite dimensional algebras and modules and their representation theory a graded structure is usually of importance to obtain structure results. The Witt algebra is a graded Lie algebra. In our more general context the algebras will almost never be graded. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure, will be enough to develop an interesting theory.

Definition 3.1. Let \mathcal{A} be an (associative or Lie) algebra admitting a direct decomposition as vector space $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$. The algebra \mathcal{A} is called an *almost-graded* algebra if (1) $\dim \mathcal{A}_n < \infty$ and (2) there are constants R and S with

$$\mathcal{A}_n \cdot \mathcal{A}_m \subseteq \bigoplus_{h=n+m+R}^{n+m+S} \mathcal{A}_h, \quad \forall n, m \in \mathbb{Z}. \quad (3.2)$$

The elements of \mathcal{A}_n are called *homogeneous elements of degree n* .

For the 2-point situation for M a higher genus Riemann surface and $I = \{P\}$, $O = \{Q\}$ with $P, Q \in M$, Krichever and Novikov [10] introduced an almost-graded structure of the vector field algebras \mathcal{L} by exhibiting a special basis and defining their elements to be the homogeneous elements.

3.2. The family of elliptic curves. We consider the genus one case, i.e. the case of one-dimensional complex tori or equivalently the elliptic curve case. We have degenerations in mind. Hence it is more convenient to use the purely algebraic picture. Recall that the elliptic curves can be given in the projective plane by

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3, \quad g_2, g_3 \in \mathbb{C}, \quad \text{with } \Delta := g_2^3 - 27g_3^2 \neq 0. \quad (3.3)$$

The condition $\Delta \neq 0$ assures that the curve will be nonsingular. Instead of (3.3) we can use the description

$$Y^2 Z = 4(X - e_1 Z)(X - e_2 Z)(X - e_3 Z) \quad (3.4)$$

with

$$e_1 + e_2 + e_3 = 0, \quad \text{and} \quad \Delta = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 \neq 0. \quad (3.5)$$

These presentations are related via

$$g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3), \quad g_3 = 4(e_1 e_2 e_3). \quad (3.6)$$

We set

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0, \quad e_i \neq e_j \text{ for } i \neq j\}. \quad (3.7)$$

In the product $B \times \mathbb{P}^2$ we consider the family of elliptic curves \mathcal{E} over B defined via (3.4). The family can be extended to

$$\widehat{B} := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0\}. \quad (3.8)$$

The fibers above $\widehat{B} \setminus B$ are singular cubic curves. Resolving the one linear relation in \widehat{B} via $e_3 = -(e_1 + e_2)$ we obtain a family over \mathbb{C}^2 .

Consider the complex lines in \mathbb{C}^2

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\}, \quad s \in \mathbb{C}, \quad D_\infty := \{(0, e_2) \in \mathbb{C}^2\}. \quad (3.9)$$

Set also

$$D_s^* = D_s \setminus \{(0, 0)\} \quad (3.10)$$

for the punctured line. Now

$$B \cong \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}). \quad (3.11)$$

Note that above D_1^* we have $e_1 = e_2 \neq e_3$, above $D_{-1/2}^*$ we have $e_2 = e_3 \neq e_1$, and above D_{-2}^* we have $e_1 = e_3 \neq e_2$. In all these cases we obtain the nodal cubic. The nodal cubic E_N can be given as

$$Y^2 Z = 4(X - eZ)^2(X + 2eZ) \quad (3.12)$$

where e denotes the value of the coinciding $e_i = e_j$ ($-2e$ is then necessarily the remaining one). The singular point is the point $(e : 0 : 1)$. It is a node. It is up to isomorphy the only singular cubic which is stable in the sense of Mumford-Deligne.

Above the unique common intersection point $(0, 0)$ of all D_s there is the cuspidal cubic E_C

$$Y^2 Z = 4X^3. \quad (3.13)$$

The singular point is $(0 : 0 : 1)$. The curve is not stable in the sense of Mumford-Deligne. In both cases the complex projective line is the desingularisation,

In all cases (non-singular or singular) the point $\infty = (0 : 1 : 0)$ lies on the curves. It is the only intersection with the line at infinity, and is a non-singular point. In passing to an affine chart in the following we will loose nothing.

For the curves above the points in D_s^* we calculate $e_2 = se_1$ and $e_3 = -(1+s)e_1$ (resp. $e_3 = -e_2$ if $s = \infty$). Due to the homogeneity, the modular parameter j for the curves above D_s^* will be constant along the line. In particular, the curves in the family lying above D_s^* will be isomorphic.

3.3. The family of vector field algebras. We have to introduce the points where poles are allowed. For our purpose it is enough to consider two marked points. One marking we will always put to $\infty = (0 : 1 : 0)$ and the other one to the point with the affine coordinate $(e_1, 0)$. This marking defines two section of the family \mathcal{E} over $\widehat{B} \cong \mathbb{C}^2$. With respect to the group structure on the elliptic curves given by ∞ as the neutral element (the first marking) the second marking chooses a two-torsion point. All other choices of two-torsion points will yield isomorphic situations.

In [12] for this situation (and for a three-point situation) a basis of the Krichever-Novikov type vector field algebras were given:

Theorem 3.2. *For any elliptic curve $E_{(e_1, e_2)}$ over $(e_1, e_2) \in \mathbb{C}^2 \setminus (D_1^* \cup D_{-1/2}^* \cup D_{-2}^*)$ the Lie algebra $\mathcal{L}^{(e_1, e_2)}$ of vector fields on $E_{(e_1, e_2)}$ has a basis $\{V_n, n \in \mathbb{Z}\}$ such that the Lie algebra structure is given as*

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd,} \\ (m-n)(V_{n+m} + 3e_1 V_{n+m-2} \\ \quad + (e_1 - e_2)(e_1 - e_3)V_{n+m-4}), & n, m \text{ even,} \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \quad + (m-n-2)(e_1 - e_2)(e_1 - e_3)V_{n+m-4}, & n \text{ odd, } m \text{ even.} \end{cases} \quad (3.14)$$

By defining $\deg(V_n) := n$, we obtain an almost-grading.

The algebras of Theorem 3.2 defined with the structure (3.14) make sense also for the points $(e_1, e_2) \in D_1 \cup D_{-1/2} \cup D_{-2}$. Altogether this defines a two-dimensional family of Lie algebras parameterized over \mathbb{C}^2 . In particular, note that we obtain for $(e_1, e_2) = 0$ the Witt algebra.

We consider now the family of algebras obtained by taking as base variety the line D_s (for any s). First consider $s \neq \infty$. We calculate $(e_1 - e_2)(e_1 - e_3) = e_1^2(1 - s)(2 + s)$ and can rewrite for these curves (3.14) as

$$[V_n, V_m] = \begin{cases} (m - n)V_{n+m}, & n, m \text{ odd,} \\ (m - n)(V_{n+m} + 3e_1V_{n+m-2} \\ \quad + e_1^2(1 - s)(2 + s)V_{n+m-4}), & n, m \text{ even,} \\ (m - n)V_{n+m} + (m - n - 1)3e_1V_{n+m-2} \\ \quad + (m - n - 2)e_1^2(1 - s)(2 + s)V_{n+m-4}, & n \text{ odd, } m \text{ even.} \end{cases} \quad (3.15)$$

For D_∞ we have $e_3 = -e_2$ and $e_1 = 0$ and obtain

$$[V_n, V_m] = \begin{cases} (m - n)V_{n+m}, & n, m \text{ odd,} \\ (m - n)(V_{n+m} - e_2^2V_{n+m-4}), & n, m \text{ even,} \\ (m - n)V_{n+m} - (m - n - 2)e_2^2V_{n+m-4}, & n \text{ odd, } m \text{ even.} \end{cases} \quad (3.16)$$

If we take $V_n^* = (\sqrt{e_1})^{-n}V_n$ (for $s \neq \infty$) as generators we obtain for $e_1 \neq 0$ always the algebra with $e_1 = 1$ in our structure equations. For $s = \infty$ a rescaling with $(\sqrt{e_2})^{-n}V_n$ will do the same (for $e_2 \neq 0$).

Hence we see that in all cases the algebras will be isomorphic above every point in D_s as long as we are not above $(0, 0)$.

Proposition 3.3. *For $(e_1, e_2) \neq (0, 0)$ the algebras $\mathcal{L}^{(e_1, e_2)}$ are not isomorphic to the Witt algebra.*

Proof. Assume that we have an Lie isomorphism $\Phi : \mathcal{W} = \mathcal{L}^{(0,0)} \rightarrow \mathcal{L}^{(e_1, e_2)}$. Denote the generators of the Witt algebra by $\{l_n, n \in \mathbb{Z}\}$. In particular, we have $[l_0, l_n] = nl_n$ for every n . We assign to every l_n numbers $m(n) \leq M(n)$ such that $\Phi(l_n) = \sum_{k=m(n)}^{M(n)} \alpha_k(n)V_k$ with $\alpha_{m(n)}(n), \alpha_{M(n)}(n) \neq 0$. From the relation in the Witt algebra we obtain

$$[\Phi(l_0), \Phi(l_n)] = \sum_{k=m(0)}^{M(0)} \sum_{l=m(n)}^{M(n)} \alpha_k(0)\alpha_l(n)[V_k, V_l] = n \cdot \sum_{l=m(n)}^{M(n)} \alpha_l(n)V_l.$$

We can choose n in such a way that the structure constants in the expression of $[V_k, V_l]$ at the boundary terms will not vanish. Using the almost-graded structure we obtain $M(0) + M(n) = M(n)$ which implies $M(0) = 0$, and $m(0) + m(n) - 4 = m(n)$ or $m(0) + m(n) - 2 = m(n)$ (for $s = 1$ or $s = -2$) which implies $2 \leq m(0) \leq M(0) = 0$ which is a contradiction. \square

It is necessary to stress the fact, that in our approach the elements of the algebras are only finite linear combinations of the basis elements V_n .

In particular, we obtain a family of algebras over the base D_s , which is always the affine line. In this family the algebra over the point $(0, 0)$ is the Witt algebra and the isomorphy type above all other points will be the same but different from the special element, the Witt algebra. This is a phenomena also appearing in algebraic geometry. There it is related to

non-stable singular curves (which is for genus one only the cuspidal cubic). Note that it is necessary to consider the two-dimensional family introduced above to “see the full behaviour” of the cuspidal cubic E_C .

Let us collect the facts:

Theorem 3.4. *For every $s \in \mathbb{C} \cup \{\infty\}$ the families of Lie algebras defined via the structure equations (3.15) for $s \neq \infty$ and (3.16) for $s = \infty$ define global deformations $\mathcal{W}_t^{(s)}$ of the Witt algebra \mathcal{W} over the affine line $\mathbb{C}[t]$. Here t corresponds to the parameter e_1 and e_2 respectively. The Lie algebra above $t = 0$ corresponds always to the Witt algebra, the algebras above $t \neq 0$ belong (if s is fixed) to the same isomorphism type, but are not isomorphic to the Witt algebra.*

If we denote by $g(s) := (1 - s)(2 + s)$ the polynomial appearing in the structure equations (3.15), we see that the algebras over D_s will be isomorphic to the algebras over D_t if $g(s) = g(t)$. This is the case if and only if $t = -1 - s$. Under this map the lines D_∞ and $D_{-1/2}$ remain fixed. Geometrically this corresponds to interchanging the role of e_2 and e_3 .

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DEPARTMENT OF APPLIED ANALYSIS, EÖTVÖS LORÁND UNIVERSITY, PÁZMÁNY PÉTER SÉTÁNY 1, H-1117 BUDAPEST, HUNGARY

E-mail address: fialowsk@cs.elte.hu