Algebraic Deformation Theory

(Dissertation for the D.Sc. degree of the Hungarian Academy of Sciences)

Alice Fialowski

Eötvös Loránd University
Institute of Mathematics
and
Alfréd Rényi Institute of Mathematics

EMAIL: fialowsk@cs.elte.hu
http://www.cs.elte.hu/~fialowsk/

Budapest
2008

To my family

Contents

Introduction

Part I: Versal Formal Deformations

Deforming a given mathematical structure is a tool of fundamental importance in most parts of mathematics, mathematical physics and physics. The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. The fundamental idea, which should be credited to Riemann, was to introduce an analytic structure therein.
The notion of local and infinitesimal deformations of a complex analytic manifold first appeared in the work of Kodaira and Spencer (1958). In particular, they proved that infinitesimal deformations can be parametrized by the corresponding cohomology group. The deformation theory of compact complex manifolds was devised by Kuranishi (1965) and Palamodov (1976). Shortly after the work of Kodaira and Spencer, algebro-geometric foundations were systematically developed by Artin (1960) and Schlessinger (1968). Formal deformations of arbitrary rings and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber (1964-1968). The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson (1966-68).
In this thesis I consider deformations of Lie algebras, although my general theory can be applied (and has already been applied) to other categories as well.
Deformation is one of the tools used to study a specific object, by deforming it into some families of "similar" structure objects. This way we get a richer picture about the original object itself. But there is also another question approached via deformation. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (may be up to certain equivalence) with the structure of a topological or geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object, deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$.

a) *Intuitive definition.* A deformation of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket (possibly infinite series)

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where φ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L}, \mathcal{L}) = C^2(\mathcal{L}; \mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. Two deformations, $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\widehat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + \dots$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C^1(\mathcal{L}; \mathcal{L})$ such that

$$\mu'_t(x, y) = \widehat{\psi}_t^{-1}\big(\mu_t\big(\widehat{\psi}_t(x), \widehat{\psi}_t(y)\big)\big) \quad \text{for } x, y \in \mathcal{L}.$$

The Jacobi identity for the algebras $\mathcal{L}_t$ implies that the 2-cochain φ_1 is indeed a cocycle, i.e. $d_2\varphi_1 = 0$. (Here d_i is the differential in the cochain complex.) If φ_1 vanishes identically, the first non-vanishing φ_i will be a cocycle. If μ'_t is an equivalent deformation with cochains φ'_i, then

$$\varphi'_1 - \varphi_1 = d_1\psi_1,$$

hence every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L};\mathcal{L})$. This definition was introduced by Nijenhuis and Richardson. We call a Lie algebra rigid, if it has no nontrivial deformations.

The classical one-parameter deformation theory is not satisfactory for studying the versal property of deformations.

For a more general deformation theory of Lie algebras I introduced the notion of a deformation with base and defined a versal formal deformation of a Lie algebra in [I.1] *.

b) *General definition.* Consider a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra over $\mathbb{K}$ with identity as base of a deformation. Let us fix an augmentation $\varepsilon : A \to \mathbb{K}$, $\varepsilon(1) = 1$, and set $\operatorname{Ker}\varepsilon = m$, which is a maximal ideal.
Definition A deformation λ of $\mathcal{L}$ with base (A, m) is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,\ ]_\lambda$ such that

$$\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.
Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called *equivalent* (or isomorphic) if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \mathrm{id}$.
A deformation with base A is called *local* if the algebra A is local, and it is called *infinitesimal* if, in addition to this, $m^2 = 0$. For general commutative algebra base, we call the deformation *global*.

c) *Formal deformations.* Let A be a complete local algebra (completeness means that $A = \varprojlim_{n\to\infty} (A/m^n)$, where m is the maximal ideal in A). A formal deformation of $\mathcal{L}$ with base A is a Lie A-algebra structure on the completed tensor product $A\widehat{\otimes}\mathcal{L} = \varprojlim_{n\to\infty} \big((A/m^n) \otimes \mathcal{L}\big)$ s.t.

$$\varepsilon\widehat{\otimes}\mathrm{id} : A\widehat{\otimes}\mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.
The previous notion of equivalence can be extended to formal deformations in an obvious way.

d) *Versal formal deformations.* It is known that in the category of algebraic varieties the quotient by a group action does not always exist (Hartshorne). Specifically, there is no universal deformation in general of a Lie algebra $\mathcal{L}$ with a commutative algebra base B with the property that for any other deformation of $\mathcal{L}$ with base A there exists a unique homomorphism $f : B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of $\mathcal{L}$ with base B *versal*.

Definition A formal deformation η of a Lie algebra $\mathcal{L}$ with a complete local algebra base B is called *miniversal*, if
i) for any formal deformation λ of $\mathcal{L}$ with any complete local base A there exists a homomorphism $f : B \to A$ s.t. the deformation λ is equivalent to the push-out of η by f;
ii) if A satisfies $m^2 = 0$, then f is unique.

*This notation is for paper 1 in Part I of this Thesis.

Using Schlessinger's general set-up (1968), I was able to prove, that for complete local algebra base deformations, under some minor restriction, there exists a miniversal deformation:

Theorem 1. [I.2] *Let $\mathcal{L}$ be a Lie algebra. Assume that the space $H^2(\mathcal{L};\mathcal{L})$ is finite-dimensional. Then there exists a versal formal deformation of $\mathcal{L}$, and the base of this versal deformation is formally embedded into $H^2(\mathcal{L};\mathcal{L})$, i.e. it can be described in $H^2(\mathcal{L};\mathcal{L})$ by a finite system of formal equations.*

Another question is how to construct such a deformation. I underlined a construction for the versal deformation in [I.1], using Harrison cohomology of commutative algebras. The construction is parallel to the general constructions in deformation theory, like Palamodov, Illusie, Laudal, Goldman-Milson, Kontsevich. The procedure needs a proper theory of Massey operations in the cohomology, and an algorithm for computing all the possible ways for a given infinitesimal deformation to extend to a formal deformation.

There is a confusion in the literature when one tries to describe all nonequivalent deformations of a given Lie algebra. There were several attempts to work out an appropriate theory for solving this basic problem in deformation theory, but none of them were completely adequate. In particular, the following questions remained open:

1) How many non-equivalent deformations have the same infinitesimal part?
2) Are there any singular nontrivial deformations, i.e. deformations with zero infinitesimal part?

Let $W^{\mathrm{pol}} = W_1$ be the Lie algebra of vector fields on the line with polynomial coefficients $f(x)\frac{d}{dx}$. This Lie algebra has an additive algebraic basis

$$e_i = x^{i+1}\frac{d}{dx}, \qquad i \geq -1.$$

In this basis the bracket operation is

$$[e_i, e_j] = (j-i)e_{i+j}.$$

Let us introduce the subalgebra L_i, $i \geq 0$ of W_1 which is generated by the basis elements $\{e_i, e_{i+1}, \dots\}$. In [I.2] I investigated the subalgebra $L^{\mathrm{pol}} = L_1$. It is naturally graded, the weight of e_i equals i. With this grading L_1^{pol} is a graded Lie algebra: $L_1^{\mathrm{pol}} = \bigoplus_{m=1}^{\infty} L_1^{(m)}$.

Using Feigin-Fuchs spectral sequence and some results of Feigin and Fuchs on cohomology with coefficients in tensor field modules, I was able to compute the 1- and 2-dimensional cohomology space of L_1:

Theorem 2. [I.2] *For $q > 0$, $H^q_{(m)}(L_1;L_1) \cong H^{q-1}_{(m)}(L_2;\mathbb{C})$. The cohomology space $H^q(L_1;L_1)$ has dimension $2q-1$ and is generated by elements of weight $-\frac{3q^2-q}{2}+i$, where $i = 1,2,\dots,2q-1$. In particular, $H^1(L_1;L_1)$ is of dimension 1 and has weight 0; the space $H^2(L_1;L_1)$ is three-dimensional with generators α,β,γ of weight $-2,-3$ and -4, while $\dim H^3(L_1;L_1) = 5$ with generators of weight $-7,-8,-9,-10$ and -11.*

Identifying explicit cocycles, I studied the Massey products of those. They are responsible for extending a deformation to higher order. The result is the following:

Theorem 3. [I.2] *In the case of L_1 the Massey products $\underbrace{\langle\alpha,\alpha,\dots,\alpha\rangle}_{i}$ are zero for all i, the brackets $[\beta,\beta],[\alpha,\beta]$ and $[\alpha,\gamma]$ are trivial, while $[\gamma,\gamma]$ and $[\beta,\gamma]$ are not. The only nontrivial 3-products are $\langle\beta,\beta,\beta\rangle$ and $\langle\alpha,\beta,\beta\rangle$. The higher operations are either not defined or they are trivial.*

The proof of this Theorem follows from computing all the defined Massey brackets and showing that some of them are nontrivial, while others are not. The nontrivial Massey brackets give the equations for the parameter space of the versal deformation.
I was able to give the complete description of all nonequivalent formal deformations for the Lie algebra L_1.
Let us define three real deformations of the Lie algebra L_1 with the brackets

$$[e_i,e_j]_t^1 = (j-i)(e_{i+j}+te_{i+j-1});$$
$$[e_i,e_j]_t^2 = \begin{cases}(j-i)e_{i+j} & \text{if } i,j>1,\\ (j-i)e_{i+j}+tje_j, & \text{if } i=1;\end{cases}$$
$$[e_i,e_j]_t^3 = \begin{cases}(j-i)e_{i+j} & \text{if } i,j\neq 2\\ (j-i)e_{i+j}+tje_j, & \text{if } i=2.\end{cases}$$

Denote the three Lie algebra families by $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$.

Theorem 4. [I.2] *The Lie algebra families $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$ are nontrivial and pairwise non-isomorphic.*

Based on my construction [I.1], Fuchs and I worked out a detailed straightforward recursive form of my previous construction for a versal deformation, convenient for explicit computations in [I.3]. The starting point in the construction is to explicitly give the universal infinitesimal deformation, which we then extend step by step, with the help of Massey operations. In the one-dimensional base extensions we use Harrison cohomology of commutative algebras. In [I.3] we also provide a scheme for computing the base of a miniversal deformation of a Lie algebra, convenient for practical use.

Part II: Nilpotent Lie Algebras and Cohomology

In the past decades, much attention has been payed to infinite dimensional Lie algebras, mainly because of their applications in mathematical physics. There are basically two kinds of infinite dimensional objects which are intensively studied: Lie algebras of geometric origin, like vector fields on a smooth manifold, and the so called Kac-Moody algebras, the theory of which is closely related to the theory of finite dimensional semisimple Lie algebras. Among the infinite dimensional Lie algebras - as in finite dimension - the hardest to deal with are the nilpotent ones. Any classification, cohomology or deformation result for those is really valuable.

Let me recall an old question of Kac: Which are all the graded Lie algebras $\mathfrak{g} = \oplus_{i=1}^{\infty}\mathfrak{g}_i$ over a field $\mathbb{K}$ of characteristic 0, for which $\dim\mathfrak{g}_i = 1$, with minimum possible number of generators. Obviously this number is 2.

There are three well-known Lie algebras of the above type: the Lie algebra L_1 and the algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ which are the maximal nilpotent subalgebras in Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$, respectively.

In the algebra $\mathfrak{g} = \oplus_{i=1}^{\infty}\mathfrak{g}_i$ we choose a basis of homogeneous elements $e_i \in \mathfrak{g}_i$. The generators of $\mathfrak{g}$ are e_1 and e_2. Note that $[e_1,\, e_2] \neq 0$ and $[e_1,\, [e_1,\, e_2]] \neq 0$.

In addition to these three algebras, we need two particular algebras and also a special family of algebras. These are:

- $\mathfrak{m}_0$: The algebra in which $[e_1,\, e_i] = e_{i+1}$ for $i > 1$ and $[e_i,\, e_j] = 0$ for $i, j > 1$,
- $\mathfrak{m}_2$: The algebra in which the commutator is set up as follows: $[e_i,\, e_j] = 0$ for $i, j > 2$, while $[e_1,\, e_j] = e_{j+1}$ for $j \geq 2$ and $[e_2,\, e_j] = e_{j+2}$ for $j > 2$.
- $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$: A family of Lie algebras with countably many parameters $\lambda_{4k} \in \mathbb{K}P^1$. The commutator is defined as follows: $[e_1,\, e_4] = 0$, $[e_3,\, e_4] = 0$, $[e_i,\, e_j] = 0$ if i is even but not 2 and j is any positive integer. Furthermore,

  $$[e_1,\, e_{4k-1}] = \alpha_{4k} e_{4k}, \quad \text{and} \quad [e_2,\, e_{4k-2}] = \beta_{4k} e_{4k}, \quad k = 2, 3, 4, \ldots,$$

  where the α_{4k} and β_{4k} are the homogeneous coordinates of the point $\lambda_{4k} \in \mathbb{K}P^1$. The remaining commutators can be uniquely reconstructed from the above formulas. Their structural constants are homogeneous polynomials of α_{4k} and β_{4k}.

Theorem 5. [II.1] *Let* $\mathfrak{g} = \oplus_{i=1}^{\infty}\mathfrak{g}_i$ *be an* $\mathbb{N}$*-graded Lie algebra, where* $\dim \mathfrak{g}_i = 1$*, with basis* $e_1, e_2, e_3, \ldots,$ *generated by* e_1 *and* e_2*. Then* $\mathfrak{g}$ *is one of the following.*
a). *Assume* $[e_1,\, e_4] \neq 0$ *and* $[e_2,\, e_3] \neq 0$*. If* $[e_3,\, e_4] \neq 0$*, then* $\mathfrak{g} \cong L_1$ *while if* $[e_3,\, e_4] = 0$*, then* $\mathfrak{g} \cong \mathfrak{m}_2$*.*
b). *Assume* $[e_2,\, e_3] = 0$*. If* $[e_3,\, e_4] \neq 0$*, then* $\mathfrak{g} \cong \mathfrak{n}_2$ *while if* $[e_3,\, e_4] = 0$*, then* $\mathfrak{g} \cong \mathfrak{m}_0$*.*
c). *Assume* $[e_1,\, e_4] = 0$*. If* $[e_3,\, e_4] \neq 0$*, then* $\mathfrak{g} \cong \mathfrak{n}_1$ *while if* $[e_3,\, e_4] = 0$*, then* $\mathfrak{g} \cong \mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$ *for some choice of the* $\lambda_8, \lambda_{12}, \lambda_{16}, \ldots$*.*

The proof is indirect, using the grading, the relations and the Jacobi identity. As the last choice, we obtain the infinite parameter family.

Let me recall a consequence of this classification Theorem [II.1]. We call a nilpotent Lie algebra *filiform* with the maximal possible nilindex $s(\mathfrak{g}) = \dim \mathfrak{g} - 1$ (by nilindex $s(\mathfrak{g})$ we mean the length of the descending central series $\{C^i\mathfrak{g}\}$ of $\mathfrak{g}$). $\mathbb{N}$-graded Lie algebras are closely related to nilpotent Lie algebras, for instance, a finite-dimensional $\mathbb{N}$-graded Lie algebra $\mathfrak{g}$ must be nilpotent. Infinite-dimensional ones are also called residual nilpotent Lie algebras.
In [II.1] I classified all infinite-dimensional $\mathbb{N}$-graded two-generated Lie algebras $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ with one-dimensional homogeneous components $\mathfrak{g}_i$. In particular, there are only three algebras in my list satisfying the "filiform property": $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \forall i$. They are $\mathfrak{m}_0, \mathfrak{m}_2, L_1$.
A. Shalev and E. Zelmanov defined the *coclass* (which might be infinity) of a finitely generated and residually nilpotent Lie algebra $\mathfrak{g}$, in analogy with the case of (pro-)p-groups, as $\mathrm{cc}(\mathfrak{g}) = \sum_{i \geq 1}(\dim(C^i\mathfrak{g}/C^{i+1}\mathfrak{g}) - 1)$. Obviously the coclass of a filiform algebra is equal to one and the same is true for the infinite-dimensional algebras $\mathfrak{m}_0, \mathfrak{m}_2, L_1$. Algebras of coclass 1 are also called algebras of *maximal class*. They are also *narrow* or *thin* Lie algebras (A. Shalev, A. Caranti, M. Newman, et al.). Part of my classification can be reformulated in the following way: *Up to an isomorphism there are only three* $\mathbb{N}$*-graded Lie algebras of maximal class with one-dimensional homogeneous components:* $\mathfrak{m}_0, \mathfrak{m}_2, L_1$.

This part of my Theorem [II.1] was rediscovered by Shalev and Zelmanov in 1997.

An interesting question is to consider the maximal nilpotent subalgebra of an arbitrary affine Kac Moody type Lie algebra. Two of them appeared in the previous classification. The cohomology with trivial coefficients are known. In [II.2] I computed cohomology with coefficients in the adjoint representation. By computing the one-dimensional cohomology space, we can classify the exterior derivations, while the two-dimensional cohomology space describes all infinitesimal deformations.

Let $A = \|a_{ij}\|$ be an integer $n \times n$ matrix with $a_{11} = \cdots = a_{nn} = 2$ and $a_{ij} \leq 0$ for $i \neq j$. Suppose that A is symmetrisable, i.e. there exist positive numbers $\varrho_1, \ldots, \varrho_n$ such that the matrix $\|\varrho_i a_{ij}\| = \varrho A$ is symmetric. Define the *Kac–Moody Lie algebra* $\mathfrak{g}^A$ *with the Cartan matrix* A as a complex Lie algebra with the generators $e_1, \ldots, e_n$, $f_1, \ldots, f_n$, $h_1, \ldots, h_n$ which satisfy certain relations. Here n is called the *rank* of $\mathfrak{g}^A$.

Suppose that A is non-decomposable, i.e. it can not become of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ under any simultaneous permutation of rows and columns.

I concentrated on Lie algebras with nonnegative definite matrices ϱA of rank $n-1$. These are the so-called *affine algebras*. Their classification is well-known. They correspond to the extended Dynkin diagrams. The twisted ones are defined by means of finite order exterior automorphisms of finite-dimensional simple algebras. Let us denote the order of the automorphism by l. We call $\mathfrak{g}^A \otimes \mathbf{C}[t, t^{-1}]$ a *current algebra*. Denote the maximal nilpotent subalgebra of $\mathfrak{g}^A$ by $\mathfrak{n}_+(A)$.

In [II.2] I described some concrete deformations of $\mathfrak{n}_+(A)$.

1°. Let $\alpha \in H^1(\mathfrak{n}_+(A); \mathfrak{n}_+(A))$, $\beta \in H^1(\mathfrak{n}_+(A))$. The element α corresponds to the right extension

$$0 \to \mathfrak{n}_+(A) \to \tilde{\mathfrak{n}}_+(A) \to \mathbf{C} \to 0$$

(the elements of $H^1(\mathfrak{n}_+(A); \mathfrak{n}_+(A))$ may be interpreted not only as exterior derivations, but also as right extensions), β to a functional $\varphi\colon \mathfrak{n}_+(A) \to \mathbf{C}$. For $t \in \mathbf{C}$ denote η_t the embedding $\mathfrak{n}_+(A) \to \tilde{\mathfrak{n}}_+(A) \cong \mathfrak{n}_+(A) \oplus \mathbf{C}$ defined by $\eta_t(g) = (g, t\varphi(g))$. It may be easily checked that $\eta_t(\mathfrak{n}_+(A))$ is a subalgebra of $\tilde{\mathfrak{n}}_+(A)$, that this subalgebra is connected with $\mathfrak{n}_+(A)$ by a natural linear isomorphism, and that for $t = 0$ this isomorphism is compatible with the bracket operation. Thus we have a deformation of $\mathfrak{n}_+(A)$. The corresponding infinitesimal deformation is evidently the product

$$\alpha\beta \in H^2(\mathfrak{n}_+(A); \mathfrak{n}_+(A)).$$

(By all means, this construction may be applied to an arbitrary Lie algebra.)

2°. Let $1 \leq i \leq n$. The algebra $\mathfrak{n}_+(A)$ deforms inside $\mathfrak{g}^A$. The deformed algebra is spanned by the spaces $\mathfrak{g}^A_{(m_1, \ldots, m_n)}$ with

$$(m_1, \ldots, m_n) \neq (0, \ldots, 0, \overset{(i)}{1}, \ldots, 0)$$

and by the vector $e_i + tf_i$, where t is a parameter. (Informally speaking, e_i deforms into $e_i + tf_i$, while the other additive generators of $\mathfrak{n}_+(A)$ do not change.)

The number of such deformations is equal to the rank of $\mathfrak{g}^A$.

3°. Let $1 \leq i, j \leq n$; consider the entry $a_{ij} = -1$ and if $a_{ij} = a_{ji}$, then $i < j$. The algebra $\mathfrak{n}_+(A)$ deforms again inside $\mathfrak{g}^A$. The deformed algebra is generated by the vectors $e_i + tf_j$ and $[e_i, e_j] - th_j$.

The number of this type deformations is equal to the number of nonzero pairs (a_{ij}, a_{ji}) with $i \neq j$; this number we denote below by p.

Theorem 6. [II.2] *Suppose that $A \neq \widetilde{A}_1$. Then*
(i) *All the homogeneous infinitesimal deformations of $\mathfrak{n}_+(A)$ may be extended to real deformations.*
(ii) *The space of infinitesimal deformations $H^2(\mathfrak{n}_+(A); \mathfrak{n}_+(A))$ is spanned by deformations, corresponding to the above types 1°, 2°, 3°. In other words, the mapping*

$$\psi : \left[H^1(\mathfrak{n}_+(A); \mathfrak{n}_-(A)) \otimes H^1(\mathfrak{n}_+(A))\right] \oplus \mathbf{C}^n \oplus \mathbf{C}^p \to H^2(\mathfrak{n}_+(A); \mathfrak{n}_+(A))$$

defined by the infinitesimal deformations listed above is epimorphism.
(iii) *The kernel of the mapping ψ is contained in*

$$H^1(\mathfrak{n}_+(A); \mathfrak{n}_+(A)) \otimes H^1(\mathfrak{n}_+(A))$$

and its dimension is n.

Theorem 7. [II.2] (i) *Infinitesimal deformations, corresponding to deformations of type 1°, 2° span in $H^2\big(\mathfrak{n}_+(\widetilde{A}_1); \mathfrak{n}_+(\widetilde{A}_1)\big)$ a codimension 2 subspace. The complementary subspace is spanned by elements from $H^2_{(-1,-2)}$ and $H^2_{(-2,-1)}$ respectively. These elements can not be extended to the deformation of $\mathfrak{n}_+(\widetilde{A}_1)$.*
(ii) *The kernel of the mapping*

$$\left[H^1(\mathfrak{n}_+(\widetilde{A}_1); \mathfrak{n}_+(\widetilde{A}_1)) \otimes H^1(\mathfrak{n}_+(\widetilde{A}_1))\right] \oplus \mathbf{C}^2 \to H^2(\mathfrak{n}_+(\widetilde{A}_1); \mathfrak{n}_+(\widetilde{A}_1))$$

may be described just as the kernel of ψ in part (iii) *of the previous Theorem.*

The proof of these theorems is based on introducing a filtration in these nilpotent graded Lie algebras and considering the Feigin-Fuchs spectral sequence corresponding to this filtration. For each of the cases the terms and differentials of the spectral sequence may be explicitly determined, and this leads to the calculation of the indicated cohomology. After having explicit cocycles, one can compute the squares.

On my visit to M.I.T., B. Kostant asked the following question: What is the main difference between the cohomology of finite and infinite dimensional nilpotent Lie algebras with coefficients in the adjoint representation, at what points does the generalization of the finite dimensional situation fail?

Understanding this difference is especially important as the nilpotent Lie algebra cohomology is very hard to compute and in both finite and infinite dimensional cases only the one- and two-dimensional cohomology is known so far.
Let us recall the result on the Lie algebra cohomology $H^1(\mathfrak{n}; \mathfrak{n})$ where $\mathfrak{n}$ is the maximal nilpotent ideal of a Borel subalgebra of a finite dimensional simple Lie algebra $\mathfrak{g}$. Leger and Luks deduced the structure of $H^1(\mathfrak{n}; \mathfrak{n})$ from Kostant's general result.
Suppose that the dimension of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is l.

Theorem 8. [Leger-Luks] *Except for the Lie algebra $s\ell_2$,*

$$H^1(\mathfrak{n}; \mathfrak{n}) \cong \mathfrak{h} \oplus \mathfrak{h}.$$

For $s\ell_2$, $\dim H^1(\mathfrak{n}; \mathfrak{n}) = 1$.

On the other hand, the result in the analogous nilpotent affine Kac–Moody cases is completely different. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the Cartan decomposition of an affine algebra $\hat{\mathfrak{g}}$. The second space does not arise in infinite dimension, because $\hat{\mathfrak{n}}_+$ is not a highest weight representation. Instead, another algebra – now infinite dimensional – appears.

Theorem 9. [II.3] *For an affine Lie algebra* $\hat{\mathfrak{g}}$,

$$H^1(\hat{\mathfrak{n}}_+; \hat{\mathfrak{n}}_+) \cong \hat{\mathfrak{h}} \oplus L_0,$$

where L_0 *is a subalgebra of the Virasoro algebra, isomorphic to the Lie algebra of polynomial vector fields on the line, vanishing at the origin.*

Remark 0.1. The difference between the finite and infinite dimensional case is that in finite dimension, by the Bott–Kostant Theorem, the dimension of $H^1(\mathfrak{n}; \mathfrak{g})$ is equal to the elements of length l in the Weyl group, while here we have

$$H^1(\hat{\mathfrak{n}}_+; \hat{\mathfrak{g}}) \cong \mathbb{C}[t, t^{-1}].$$

The 2-dimensional cohomology result in the analogous nilpotent affine Kac–Moody cases is also completely different.
Using Kostant's results, Leger and Luks computed $H^2(\mathfrak{n}; \mathfrak{n})$ for finite dimensional simple Lie algebras $\mathfrak{g}$. Their main idea is the following. Consider the next exact sequences of $\mathfrak{n}$-modules:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathfrak{n} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathfrak{g} & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g}/\mathfrak{n} & \longrightarrow & \mathfrak{n}^* & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

These induce exact cohomology sequences of the appropriate cohomology spaces. By studying the cohomology diagram step by step, one gets the result

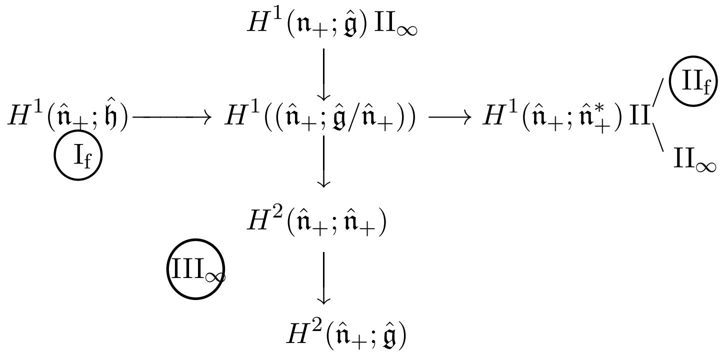
Theorem 10. (Leger-Luks) *If* $\mathfrak{g}$ *is not of type* A_1, A_2, *or* B_2 *then*

$$H^2(\mathfrak{n}; \mathfrak{n}) \approx H^2(\mathfrak{n}; \mathfrak{g}) \oplus H^1(\mathfrak{n}; \mathfrak{g}/\mathfrak{n}).$$

Here

$$\dim H^1(\mathfrak{n}; \mathfrak{g}/\mathfrak{n}) = (2l - 1) + l^2 - l,$$
$$\dim H^2(\mathfrak{n}; \mathfrak{g}) = \frac{1}{2}(l + 1)(l - 1).$$

In infinite dimension we have the following cohomology sequences:



Theorem 11. [II.3] *With the exception of $\hat{s\ell}(2,\mathbb{C})$, the space $H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ is the direct sum of three subspaces, coming from three kinds of cocycles* I–III. *The cocycles of type* $\mathrm{I_f}$ *and* $\mathrm{II_f}$ *are the same as for finite dimensional algebras. The cocycles of type* II_∞ *coming from above and from the right cancel each other. Cocycles of type* III_∞ *only appear in the affine cases. They form a space isomorphic to $H^1(\hat{\mathfrak{n}}_+)\otimes L_0$.*

Remark 0.2. Cocycles of type $\mathrm{I_f}$ and III_∞ form the space

$$H^1(\hat{\mathfrak{n}}_+)\otimes H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+).$$

The number of such deformations is $\dim(H^1(\hat{\mathfrak{n}}_+))\times\dim(H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+))$.

While nilpotent Lie algebra cohomology and deformations are usually hard to compute and a kind of spectral sequence method is needed, in some cases the grading and the structure of the algebra makes it possible to overcome the difficulties.

Algebras of maximal class are in the center of attention these days both in zero and positive characteristic. There are many open questions related to them. One natural question is their cohomology and deformations.

For the Lie algebra $\mathfrak{m}_0$, the cohomology with trivial coefficients has been studied by Millionschikov and myself. The adjoint coefficients cohomology I computed with Friedrich Wagemann [II.4].

The first and second adjoint cohomology of $\mathfrak{m}_0$ are infinite dimensional. The space $H^1(\mathfrak{m}_0;\mathfrak{m}_0)$ becomes already interesting when we split it up into homogeneous components $H^1_l(\mathfrak{m}_0;\mathfrak{m}_0)$ of weight $l\in\mathbb{Z}$, this latter space being finite dimensional for each $l\in\mathbb{Z}$. The space $H^2(\mathfrak{m}_0;\mathfrak{m}_0)$ is worse as it is infinite dimensional even in each weight separately. The interesting new feature here is that there are only finitely many generators in each negative or zero weight which give rise to *true* deformations.

Given a generator of $H^2(\mathfrak{m}_0;\mathfrak{m}_0)$, i.e. an infinitesimal deformation, corresponding to the linear term of a formal deformation, one can try to adjust higher order terms in order to have the Jacobi identity in the deformed Lie algebra up to order k. If the Jacobi identity is satisfied for all orders, we will call it a true (formal) deformation.

Theorem 12. [II.4] *The true deformations of $\mathfrak{m}_0$ are finitely generated in each weight $l\leq 1$. More precisely, the space of unobstructed cohomology classes is in degree*

- $l\leq -3$ *of dimension two,*
- $l=0$ *of dimension two,*
- $l=-2$ *of dimension three,*

while there is no true deformation in weight $l = -1$. In weight $l = 0$, these are deformations to $\mathfrak{m}_2$ and L_1. In weight $l = 1$, there are exactly two true deformations, while in weight $l \geq 2$, there are at least two.

We do not have more precise information about how many true deformations there are in positive weight, but there are always at least two. As a deformation in these weights is a true deformation if and only if all of its Massey squares are zero (as cochains !), true deformations are determined by a countable infinite system of homogeneous quadratic equations in countably infinitely many variables. We didn't succeed in determining the space of solutions of this system. In weights $l \leq -1$ we got the results using combinatorics and the graded cocycle property.

We believe that the discussion of these examples of deformations are interesting as they go beyond the usual approach where the condition that $H^2(\mathfrak{m}_0; \mathfrak{m}_0)$ should be finite dimensional is the starting point for the examination of deformations, namely the existence of a miniversal deformation.

Another attractive point of our study is the fact that in some cases the Massey squares and cubes involved are not zero because of general reasons, but because of the combinatorics of the relations. Thus the second adjoint cohomology of $\mathfrak{m}_0$ may serve as an example for studing explicitly obstruction theory.

Part III: Generalizations and Applications

There is another question approached via deformations. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (maybe up to certain equivalences) with the structure of a topological or even geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object the deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

In this respect, a clear success story is the classification of complex analytic structures on a fixed topological manifold. Also in algebraic geometry one has well-developed results in this direction. One of these results is that the local situation at a point $[C]$ of the moduli space is completely governed by the cohomological properties of the geometric object C.

As a typical example recall that for the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g over $\mathbb{C}$ (or equivalently, compact Riemann surfaces of genus g) the tangent space $T_{[C]}\mathcal{M}_g$ can be naturally identified with $\mathrm{H}^1(C; T_C)$, where T_C is the sheaf of holomorphic vector fields over C. This extends to higher dimension. In particular, it turns out that for compact complex manifolds M, the condition $\mathrm{H}^1(M; T_M) = \{0\}$ implies that M is rigid. Rigidity means that any differentiable family $\pi : \mathcal{M} \to B \subseteq \mathbb{R}$, $0 \in B$ which contains M as the special member $M_0 := \pi^{-1}(0)$ is trivial in a neighborhood of 0, i.e. for t small enough $M_t := \pi^{-1}(t) \cong M$.

Even more generally, for M a compact complex manifold and $\mathrm{H}^1(M; T_M) \neq \{0\}$ there exists a versal family which can be realized locally as a family over a certain subspace of $\mathrm{H}^1(M; T_M)$ such that every appearing deformation family is "contained" in this versal family.

These positive results lead to the impression that the vanishing of the relevant cohomology spaces will imply rigidity with respect to deformations also in the case of other structures.

There is a lot of confusion in the literature in the notion of a deformation. Several different (inequivalent) approaches exist. One of our aims with Martin

Schlichenmaier was to clarify the difference between deformations of geometric origin and so called formal deformations. Formal deformation theory has the advantage of using cohomology. It is also complete in the sense that under some natural cohomology assumptions, there exists a versal formal deformation, which induces all other deformations [I.2].

Formal deformations are deformations with a complete local algebra base. A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the base of deformation – which is a commutative algebra of functions – with a smooth manifold, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations. In infinite dimension, there is no tight relation between global and formal deformations.

One of my earlier results is that the Witt and Virasoro algebra are formally rigid. In our work with Martin Schlichenmaier [III.1] we constructed global deformations of the Witt algebra by considering certain families of algebras for the genus one case (i.e. the elliptic curve case) and let the elliptic curve degenerate to a singular cubic. The two points, where poles are allowed, are the zero element of the elliptic curve (with respect to its additive structure) and a 2-torsion point. In this way we obtain families parameterized over the affine line with the peculiar behavior that every family is a global deformation of the Witt algebra, i.e. $\mathcal{W}$ is a special member, whereas all other members are mutually isomorphic but not isomorphic to $\mathcal{W}$. Globally these families are non-trivial, but infinitesimally and formally they are trivial. The construction can be extended to the centrally extended algebras, yielding a global deformation of the Virasoro algebra.
The results obtained do not have only relevance to the deformation theory of algebras but also to the theory of two-dimensional conformal fields and their quantization. It is well-known that the Witt algebra, the Virasoro algebra, and their representations are of fundamental importance for the local description of conformal field theory on the Riemann sphere (i.e. for genus zero). Krichever and Novikov proposed in the case of higher genus Riemann surfaces the use of global operator fields which are given with the help of the Lie algebra of vector fields of Krichever-Novikov type, certain related algebras, and their representations.

Algebras of Krichever-Novikov types are generalizations of the Virasoro algebra and all its related algebras. Let me introduce some of them.
Let M be a compact Riemann surface of genus g, or in terms of algebraic geometry, a smooth projective curve over $\mathbb{C}$. Let $N, K \in \mathbb{N}$ with $N \geq 2$ and $1 \leq K < N$ be numbers. Fix

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_{N-K})$$

disjoint ordered tuples of distinct points ("marked points", "punctures") on the curve. In particular, we assume $P_i \neq Q_j$ for every pair (i, j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets and $A = I \cup O$ as a set.
Denote by $\mathcal{L}$ the Lie algebra consisting of those meromorphic sections of the holomorphic tangent line bundle which are holomorphic outside of A, equipped with the Lie bracket $[.,.]$ of vector fields. Its local form is

$$[e, f]_| = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] := \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz} \ .$$

To avoid cumbersome notation we use the same symbol for the section and its representing function.
For the Riemann sphere ($g = 0$) with quasi-global coordinate z, $I = \{0\}$ and $O = \{\infty\}$, the introduced vector field algebra is the Witt algebra. We denote for short this situation as the *classical situation.*
For infinite dimensional algebras and modules and their representation theory a graded structure is usually of importance to obtain structure results.
The Witt algebra is a graded Lie algebra. In our more general context the algebras will almost never be graded. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure, will be enough to develop an interesting theory of representations (Verma modules, etc.).
For the 2-point situation for M a higher genus Riemann surface and $I = \{P\}$, $O = \{Q\}$ with $P, Q \in M$, Krichever and Novikov introduced an almost-graded structure of the vector field algebras $\mathcal{L}$ by exhibiting a special basis and defining their elements to be the homogeneous elements.
We consider the genus one case, i.e. the case of one-dimensional complex tori or equivalently the elliptic curve case.
Recall that the elliptic curves can be given in the projective plane by

$$Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3, \quad g_2, g_3 \in \mathbb{C}, \quad \text{with } \Delta := {g_2}^3 - 27{g_3}^2 \neq 0.$$

The condition $\Delta \neq 0$ assures that the curve will be nonsingular.
Instead of the above elliptic curve expression we can use the description

$$Y^2Z = 4(X - e_1Z)(X - e_2Z)(X - e_3Z)$$

with

$$e_1 + e_2 + e_3 = 0, \quad \text{and} \quad \Delta = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2 \neq 0.$$

These presentations are related via

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3), \quad g_3 = 4(e_1e_2e_3).$$

We set

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0, \quad e_i \neq e_j \text{ for } i \neq j\}.$$

In the product $B \times \mathbb{P}^2$ we consider the family of elliptic curves $\mathcal{E}$ over B defined via the second expression (in product form). The family can be extended to

$$\widehat{B} := \{e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0\}.$$

The fibers above $\widehat{B} \setminus B$ are singular cubic curves. Resolving the one linear relation in $\widehat{B}$ via $e_3 = -(e_1 + e_2)$ we obtain a family over $\mathbb{C}^2$.
Consider the complex lines in $\mathbb{C}^2$

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\}, \quad s \in \mathbb{C}, \qquad D_\infty := \{(0, e_2) \in \mathbb{C}^2\}.$$

Set also

$$D_s^* = D_s \setminus \{(0, 0)\}$$

for the punctured line. Now

$$B \cong \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}).$$

We have to introduce the points where poles are allowed. For our purpose it is enough to consider two marked points. We will always put one marking to $\infty = (0 : 1 : 0)$ and the other one to the point with the affine coordinate $(e_1, 0)$. These markings define two sections of the family $\mathcal{E}$ over $\widehat{B} \cong \mathbb{C}^2$. With respect to the group structure on the elliptic curve given by ∞ as the neutral

element (the first marking) the second marking chooses a two-torsion point. All other choices of two-torsion points will yield isomorphic situations.
For any elliptic curve $E_{(e_1,e_2)}$ over $(e_1,e_2)\in\mathbb{C}^2\setminus(D_1^*\cup D_{-1/2}^*\cup D_{-2}^*)$ the Lie algebra $\mathcal{L}^{(e_1,e_2)}$ of vector fields on $E_{(e_1,e_2)}$ has a basis $\{V_n,\ n\in\mathbb{Z}\}$ such that the Lie algebra structure is given as

$$[V_n,V_m]=\begin{cases}(m-n)V_{n+m}, & n,m \text{ odd},\\ (m-n)\big(V_{n+m}+3e_1V_{n+m-2} & \\ \qquad +(e_1-e_2)(e_1-e_3)V_{n+m-4}\big), & n,m \text{ even},\\ (m-n)V_{n+m}+(m-n-1)3e_1V_{n+m-2} & \\ \quad +(m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4}, & n \text{ odd},\ m \text{ even}.\end{cases}$$

By defining $\deg(V_n):=n$, we obtain an almost-grading. We consider now the family of algebras obtained by taking as base variety the line D_s (for any s). First consider $s\neq\infty$. We calculate $(e_1-e_2)(e_1-e_3)=e_1^2(1-s)(2+s)$ and can rewrite for these curves the brackets as

$$[V_n,V_m]=\begin{cases}(m-n)V_{n+m}, & n,m \text{ odd},\\ (m-n)\big(V_{n+m}+3e_1V_{n+m-2} & \\ \qquad +e_1^2(1-s)(2+s)V_{n+m-4}\big), & n,m \text{ even},\\ (m-n)V_{n+m}+(m-n-1)3e_1V_{n+m-2} & \\ \qquad +(m-n-2)e_1^2(1-s)(2+s)V_{n+m-4}, & n \text{ odd},\ m \text{ even}.\end{cases}$$

For D_∞ we have $e_3=-e_2$ and $e_1=0$ and obtain

$$[V_n,V_m]=\begin{cases}(m-n)V_{n+m}, & n,m \text{ odd},\\ (m-n)\big(V_{n+m}-e_2^2V_{n+m-4}\big), & n,m \text{ even},\\ (m-n)V_{n+m}-(m-n-2)e_2^2V_{n+m-4}, & n \text{ odd},\ m \text{ even}.\end{cases}$$

If we take $V_n^*=(\sqrt{e_1})^{-n}V_n$ (for $s\neq\infty$) as generators, we obtain for $e_1\neq0$ always the algebra with $e_1=1$ in our structure equations. For $s=\infty$ a rescaling with $(\sqrt{e_2})^{-n}V_n$ will do the same (for $e_2\neq0$). Hence we see that for fixed s in all cases the algebras will be isomorphic above every point in D_s as long as we are not above $(0,0)$.

Theorem 13. [III.1] *For every $s\in\mathbb{C}\cup\{\infty\}$ the families of Lie algebras defined via the structure equations for $s\neq\infty$ and the brackets just above for $s=\infty$ define global deformations $\mathcal{W}_t^{(s)}$ of the Witt algebra $\mathcal{W}$ over the affine line $\mathbb{C}[t]$. Here t corresponds to the parameter e_1 and e_2 respectively. The Lie algebra above $t=0$ corresponds always to the Witt algebra, the algebras above $t\neq0$ belong (if s is fixed) to the same isomorphic type, but are not isomorphic to the Witt algebra.*

In finite dimension global deformations coincide with formal deformations, so we can use cohomology theory. Here cohomology and versal deformations make it possible to get a geometric description of the moduli space of a certain type of algebraic objects in a given dimension. This feature is completely new and underlines the importance of those invariants.

In a paper with Michael Penkava [III.2] we show that the moduli space of Lie algebras on $\mathbb{C}^4$ is essentially an orbifold given by the natural action of the symmetric group Σ_3 on the complex projective space $\mathbb{P}^2(\mathbb{C})$. In addition, there are two exceptional complex projective lines, one of which has an action of the symmetric group Σ_2. Finally, there are 6 exceptional points. The moduli space is glued together by the miniversal deformations, which determine the elements that one may deform to locally, so deformation theory determines the geometry of the space. The exceptional points play a role in refining the picture of how this space is glued together. By orbifold, we mean essentially a topological space factored out by the action of a group. In the case of $\mathbb{P}^n$, there is a natural action of Σ_{n+1} induced by the natural action of Σ_{n+1} on $\mathbb{C}^{n+1}$. An orbifold point is a point which is fixed by some element in the group. In the case of Σ_{n+1} acting on $\mathbb{P}^n$, points which have two or more coordinates with the same value are orbifold points, but there are some other ones, such as the point $(1:-1)=(-1:1)$.

In the classical theory of deformations, a deformation is called a *jump deformation* if there is a 1-parameter family of deformations of a Lie algebra structure such that every nonzero value of the parameter determines the same deformed Lie algebra, which is not the original one. There are also deformations which move along a family, meaning that the Lie algebra structure is different for each value of the parameter. There can be multiple parameter families as well.

In the picture we assembled, both of these phenomena arise. Some of the structures belong to families and their deformations simply move along the family to which they belong. If there is a jump deformation from an element to a member of a family, then there will always be deformations from that element along the family as well, although they will typically not be jump deformations. In addition, there are sometimes jump deformations either to or from the exceptional points, so these exceptional points play an interesting role in the picture of the moduli space.

In classical Lie algebra theory, the cohomology of a Lie algebra is studied by considering a differential on the dual space of the exterior algebra of the underlying vector space, considered as a cochain complex. If V is the underlying vector space on which the Lie algebra is defined, then its exterior algebra $\bigwedge V$ has a natural $\mathbb{Z}_2$-graded coalgebra structure as well. In this language, a Lie algebra is simply a quadratic odd codifferential on the exterior coalgebra of a vector space. An odd codifferential is an odd coderivation whose square is zero. The space L of coderivations has a natural $\mathbb{Z}$-grading $L=\bigoplus L_n$, where L_n is the subspace of coderivations determined by linear maps $\phi:\bigwedge^n V\to V$. A Lie algebra is a codifferential in L_2, in other words, a quadratic codifferential.

The space of coderivations has a natural structure of a $\mathbb{Z}_2$-graded Lie algebra. The condition that a coderivation d is a codifferential can be expressed in the form $[d,d]=0$. The coboundary operator $D:L\to L$ is given simply by the rule $D(\varphi)=[d,\varphi]$ for $\varphi\in L$; the fact that $D^2=0$ is a direct consequence of the fact that d is an odd codifferential. Moreover, $D(L_n)\subseteq L_{n+1}$, which means that the cohomology $H(d)=\ker D/\operatorname{Im} D$ has a natural decomposition as a $\mathbb{Z}$-graded space: $H(d)=\prod H^n(d)$, where

$$H^n(d)=\ker(D:L_n\to L_{n+1})/\operatorname{Im}(D:L_{n-1}\to L_n).$$

The Lie algebra structures are codifferentials in L_2. In order to represent a codifferential d as a matrix, we choose the following order for the increasing pairs $I=(i_1,i_2)$ of indices:

$$\{(1,2),(1,3),(2,3),(1,4),(2,4),(3,4)\},$$

and denote the ith element of this ordered set by $S(i)$. Using this order and the Einstein summation convention, we can express

$$d = a^i_j \varphi_i^{S(j)}.$$

We summarize our results and give the Lie bracket operations in standard terminology in the Table below.

Type	Brackets
$d_1(\lambda:\mu)$	$[e_2,e_3]=e_3, [e_1,e_4]=(\lambda+\mu)e_1,$ $[e_2,e_4]=\lambda e_2, [e_3,e_4]=e_2+\mu e_3$
$d_3(\lambda:\mu:\nu)$	$[e_1,e_4]=\lambda e_1, [e_2,e_4]=e_1+\mu e_2, [e_3,e_4]=e_2+\nu e_3$
$d_3(\lambda:\mu)$	$[e_1,e_4]=\lambda e_1, [e_2,e_4]=\lambda e_2, [e_3,e_4]=e_2+\mu e_3$
d_1	$[e_2,e_4]=e_1$
$d_1^\sharp$	$[e_2,e_3]=e_1, [e_1,e_4]=2e_1, [e_2,e_4]=e_2, [e_3,e_4]=e_3$
d_2^*	$[e_1,e_2]=e_1, [e_3,e_4]=e_2$
$d_2^\sharp$	$[e_1,e_2]=e_1, [e_3,e_4]=e_3$
d_3	$[e_1,e_2]=e_3, [e_1,e_3]=e_2, [e_2,e_3]=e_1$
d_3^*	$[e_1,e_4]=e_1, [e_2,e_4]=e_2, [e_3,e_4]=e_3$

TABLE 1. Table of Lie Bracket Operations

In Table 2, we give a classification of the codifferentials according to their cohomology. Note that for the most part, elements from the same family have the same cohomology. In fact, the decomposition of the codifferentials into families was strongly influenced by the desire to associate elements with the same pattern of cohomology in the same family. This is why our family $d_3(\lambda:\mu:\nu)$ was not chosen to be the diagonal matrices. Similar considerations influenced our selection of the family $d_3(\lambda:\mu)$.

Type	H^1	H^2	H^3	H^4
d_3	1	0	1	1
$d_2^\sharp$	0	0	0	0
$d_1(1:-1)$	2	2	2	1
$d_1(1:0)$	1	2	1	0
$d_1(\lambda:\mu)$	1	1	0	0
$d_1^\sharp$	3	3	0	0
$d_3(1:-1:0)$	3	5	5	2
$d_3(\lambda:\mu:\lambda+\mu)$	2	3	1	0
$d_3(\lambda:\mu:0)$	3	3	1	0
$d_3(\lambda:\mu:-\lambda-\mu)$	2	2	1	1
$d_3(\lambda:\mu:\nu)$	2	2	0	0
$d_3(1:0)$	5	7	3	0
$d_3(0:1)$	6	6	2	0
$d_3(1:2)$	4	5	1	0
$d_3(1:-2)$	4	4	1	1
$d_3(\lambda:\mu)$	4	4	0	0
d_1	8	13	10	3
d_2^*	4	6	5	2
d_3^*	8	8	0	0

TABLE 2. Table of the Cohomology

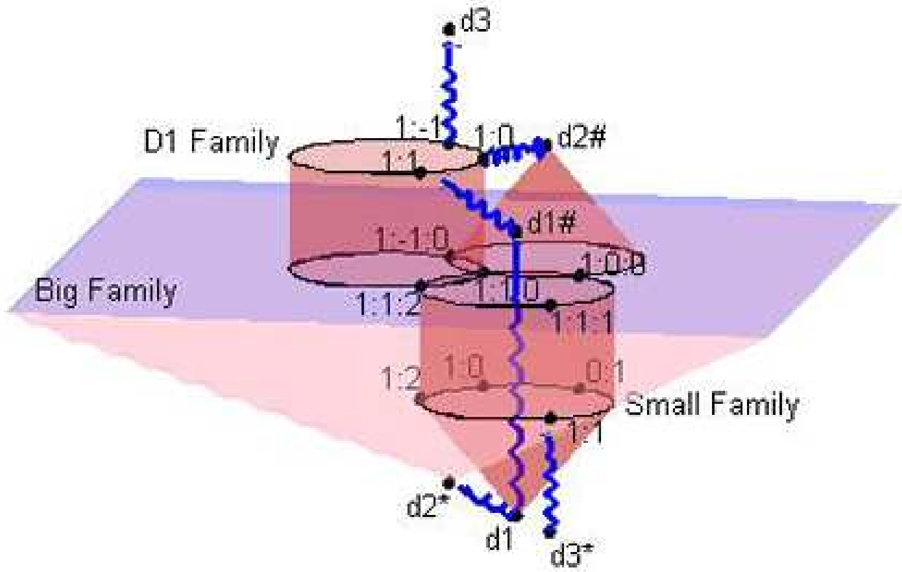


FIGURE 1. The Moduli Space of 4 dimensional Lie Algebras

In Figure 1, we give a pictorial representation of the moduli space. The big family $d_3(\lambda : \mu : \nu)$ is represented as a plane, although in reality it is $\mathbb{P}^2/\Sigma_3$. The families $d_1(\lambda : \mu)$, $d_3(\lambda : \mu)$ and the three subfamilies $d_3(\lambda : \mu : 0)$, $d_3(\lambda : \lambda : \mu)$ and $d_3(\lambda : \mu : \lambda + \mu)$ are represented by circles, mainly to reflect that the three subfamilies of the big family intersect in more than one point, because they each represent not a single $\mathbb{P}^1$, but several copies of $\mathbb{P}^1$ which are identified under the action of the symmetric group.
In the picture, jump deformations from special points are represented by curly arrows. The jump deformations from both the small family $d_3(\lambda : \mu)$ to $d_3(\lambda : \lambda : \mu)$ and from $d_3(\lambda : \mu : \lambda+\mu)$ to $d_1(\lambda : \mu)$ are represented by cylinders. The jump deformations from the family $d_3(\lambda : \mu : 0)$ to $d_2^\sharp$ and those from d_1 to the small family are represented by cones. Finally, the jump deformations from d_2^* to the big family are represented by an inverted pyramid shape. All jump deformations are either in an upwards or a horizontal direction.

The computation of the equivalence classes of non-isomorphic Lie algebra structures in a vector space V determines the elements of the moduli space of Lie algebra structures on V, but is only the first step in the classification of these structures. When classifying the algebras, there are different ways of dividing up the structures according to families; therefore, it is desirable to have a rationale for the division. We have shown that there is a natural way to divide up the moduli space into families, using cohomology as a guide to the division, and versal deformations as a tool to refine the analysis.

In our case, there is 1 two-parameter family, 2 one-parameter families, and 6 singleton elements, giving rise to a two-dimensional orbifold, 2 one-dimensional orbifolds, and 6 one-dimensional orbifolds. The jump deformations provide maps between the families which either are smooth maps of orbifolds (or sub-orbifolds as in the case of the map $d_3(\lambda : \mu : \lambda + \mu) \to d_1(\lambda : \mu)$), or, in the case of some of the singletons, identify the element with a whole family.

The cohomology of a Lie algebra determines the tangent space to the Lie algebra, but the tangent space does not contain enough information to give a good local description of the moduli space. The relations on the base of the versal deformation determine the manner in which the moduli space contacts the tangent space. It is clear that the cohomology is not sufficient to get an accurate picture of the moduli space. Versal deformations provide important detail that characterizes the moduli space completely.

Part I: Versal Formal Deformations

APPEARED IN: *Math. USSR Sbornik*, **55** (1986), pp. 467–473.

Deformations of Lie Algebras

Alice Fialowski
Alfréd Rényi Institute of Mathematics
Budapest

Abstract The author considers general questions of deformations of Lie algebras over a field of characteristic zero, and the related problems of computing cohomology with coefficients in adjoint representations. The construction of a versal family, and the construction of obstructions to the extension of deformations are also considered.

In this paper, we consider general questions on deformations of Lie algebras over a field of characteristic zero, and related problems of computing cohomology with coefficients in adjoint representations. We consider the construction of a versal family and the nature of obstructions to the extension of deformations. Our aim is to carry over general constructions of the modern theory of deformations and related properties of the cohomology of (local) commutative algebras to Lie algebras in parallel with the papers [3], [10] and [11].

1. We shall require some information on the Harrison cohomology of commutative rings (see [12] and [8]). Harrison cohomology is the cohomology in the category of commutative rings. We shall only require the 1-dimensional and 2-dimensional cohomology, and restrict ourselves to their explicit definition. (In contrast with the traditional indexing, we consider Harrison cohomology with the indices increased by 1.)
Let A be a commutative $\mathbf{k}$-algebra, where $\mathbf{k}$ is a field of characteristic zero, and let N be an A-module. We write down a cochain complex $N \overset{d_0}{\to} K^1 \overset{d_1}{\to} K^2$, where $K^1 = \operatorname{Hom}_{\mathbf{k}}(A, N)$ and K^2 is the subspace of $\operatorname{Hom}_{\mathbf{k}}(S^2A, N)$ consisting of the maps φ for which

$$a\varphi(b, c) - \varphi(ab, c) - c\varphi(a, b) + \varphi(a, bc) = 0$$

for any three elements $a, b, c \in A$. The differentials d_0 and d_1 are arranged so that

$$d_0(n)(a) = an, \quad a \in A,\ n \in N, \quad d_1\theta(a, b) = a\theta(b) - \theta(ab) + b\theta(a), \quad a, b \in A.$$

The spaces $H^1_{\mathrm{Harr}}(A; N)$ and $H^2_{\mathrm{Harr}}(A; N)$ of 1-dimensional and 2-dimensional cohomology are by definition $\operatorname{Ker} d_1/\operatorname{Im} d_0$ and $K^2/\operatorname{Im} d_1$, respectively.
From the definition one can see that 1-cocycles are derivations. Let A be an algebra, m a maximal ideal, and $A/m \cong k$. Then $H^1_{\mathrm{Harr}}(A; \mathbf{k}) \cong (m/m^2)^*$. In other words, $H^1_{\mathrm{Harr}}(A; \mathbf{k})$ is isomorphic to the space of homomorphisms $A \to \mathbf{k}[t]/(t^2)$ for which the kernel of the composition $A \to \mathbf{k}[t]/(t^2) \to k$ is m.
The 2-dimensional cohomology is interpreted as extension (see [9]). An extension of the algebra A by a module N is an exact sequence $0 \to N \overset{i}{\to} B \overset{\pi}{\to} A \to 0$, where B is a commutative algebra and $i(N)$ is an ideal in B with trivial multiplication such that $bi(n) = \pi(b)n$ for $b \in B$ and $n \in N$. To an

1980 Mathematics Subject Classification (1985 Revision). Primary 17B56; Secondary 13D03.

extension we assign a cochain $\varphi \in K^2$ in the following way. Let $\eta : A \to B$ be a $\mathbf{k}$-linear map for which $\pi\eta = \mathrm{id}$. We put $i\varphi(a,b) = \eta(a)\eta(b) - \eta(ab)$. It can easily be verified that $\varphi \in K^2$ and that for any other choice of the map η the cochain φ is changed by a coboundary. Thus from the extension we have constructed an element of the space $H^2_{\mathrm{Harr}}(A;N)$. From the construction one can see that to every element of $H^2_{\mathrm{Harr}}(A;N)$ there corresponds an extension, and the extension is trivial (that is, B is a semidirect product) if and only if the corresponding cohomology class is zero.

An automorphism of the extension $0 \to N \xrightarrow{i} B \xrightarrow{\pi} A \to 0$ is an automorphism μ of the algebra B such that $\pi(\mu(b) - b) = 0$ for all $b \in B$ and $\mu(n) - n = 0$ if $n \in i(N)$. The map $\theta : A \to N$, $a \mapsto (\mu - 1)\eta a$ depends on the choice of η; namely, $\theta : A \to N$ is a cocycle which, as η changes, is changed by a coboundary. Thus the set of automorphisms of the extension $0 \to N \to B \to A \to 0$ can be naturally identified with the space $H^1_{\mathrm{Harr}}(A;N)$.

2. Now we move to the theory of deformations of Lie algebras. We begin with a "naive" definition of deformation of Lie algebras. Let V be a linear space, and $S(V)$ the set of all linear maps $\Lambda^2 V \to V$ satisfying the Jacobi identity; $S(V)$ is hence the set of common zeros of a certain system of second degree polynomials on the space $\Lambda^2 V' \otimes V$. This makes it possible to equip $S(V)$ with the structure of affine algebraic variety. The group $\mathrm{GL}(V)$ acts on $S(V)$. The quotient $L = S(V)/\mathrm{GL}\,(V)$ is the set of pairwise nonisomorphic Lie algebra structures on the space V.

It is well known (see [6]) that in the category of algebraic varieties the quotient by a group action does not always exist. In particular, L is not an algebraic variety. However, one can define a functor assigning to each affine algebraic variety X a would-be set of morphisms $\mathrm{Mor}\,(X,L)$. Namely, to X we assign $D(X)$, which is the quotient of the set $\mathrm{Mor}\,(X,S(V))$ by the action of the group of regular maps $X \to \mathrm{GL}\,(V)$. If the functor $D(X)$ could be represented in the category of algebraic varieties, then L would admit the structure of an algebraic variety, and $D(X) \cong \mathrm{Mor}\,(X,L)$.

The study of the quotient D is the main problem in the theory of deformations. We are mainly interested in the local theory of deformations; that is, we restrict ourselves to the subcategory Λ of the category of affine algebraic varieties which consists of the varieties of the form spec A, where A is a local algebra. We recall that a local algebra is an algebra with a unique maximal ideal m, $A/m \cong \mathbf{k}$. We now define a functor responsible for the structure of L in the neighborhood of a given point.

Let $\mathcal{L}$ be a Lie algebra, and spec A an object of the category Λ. Then $\mathrm{Def}\,(\mathcal{L}, \mathrm{spec}\,A)$ is by definition the preimage of $\mathcal{L}$ under the map $D(A) \to D(\mathbf{k})$ induced by the morphism $\mathrm{spec}\,\mathbf{k} \to \mathrm{spec}\,A$. Here we assume that $V = \mathcal{L}$ and that $\mathcal{L}$ itself is an element of the set L. The elements of the set $\mathrm{Def}\,(\mathcal{L}, \mathrm{spec}\,A)$ are called *deformations* of the Lie algebra $\mathcal{L}$ *with the base* $\mathrm{spec}\,A$.

One distinguishes especially the so-called formal 1-parameter deformations of a Lie algebra; that is, deformations over a ring of formal power series in one variable (see [13]).

Now we give a more explicit description of the set $\mathrm{Def}\,(\mathcal{L}, \mathrm{spec}\,A)$, where $\mathrm{spec}\,A \in \Lambda$. It is the set of classes by isomorphism of the following pairs: a) the Lie A-algebra $\overline{\mathcal{L}}(A)$, which is free as an A-module, and b) the isomorphism $\overline{\mathcal{L}}(A)/m\overline{\mathcal{L}}(A) \cong \mathcal{L}$, where m is a maximal ideal in A. The coincidence of this definition with that of the preceding paragraph is obvious. Indeed, we choose in the algebra $\overline{\mathcal{L}}(A)$ a basis (over A). The Lie commutator in this basis gives us a map $\operatorname{spec} A \to S(V)$ ($V \cong \mathcal{L}$). Conversely there exists a "tautological" algebra over the algebra $\mathbf{k}[S(V)]$ (that is, $\bigoplus_{s\in S(V)} V_s$, where V_s is a Lie algebra with commutator s), and by "change of base" in each morphism $\operatorname{spec} A \to S(V)$ we obtain a Lie A-algebra.

Example. Let $A \to \mathbf{k}[t]/(t^2)$. We describe $\operatorname{Def}(\mathcal{L}, \operatorname{spec} A)$.

Proposition 1. *$Def(\mathcal{L}, spec\, A) \cong H^2(\mathcal{L}; \mathcal{L})$.*

Proof. An element of the set $\operatorname{Def}(\mathcal{L}, \operatorname{spec} A)$ is a Lie algebra $\overline{\mathcal{L}}$ equipped with an endomorphism where $\overline{\mathcal{L}}/t\overline{\mathcal{L}} \cong \mathcal{L}$; $t\mathcal{L}$ is an abelian ideal. Thus $0 \to t\overline{\mathcal{L}} \to \overline{\mathcal{L}} \to \mathcal{L} \to 0$ is an extension of the algebra $\mathcal{L}$ by the adjoint representation. It is easy to show that, conversely, every such extension defines an element of $\operatorname{Def}(\mathcal{L}, \operatorname{spec} A)$. On the other hand, the extensions can be classified by the elements of the linear space $H^2(\mathcal{L}; \mathcal{L})$. The proposition is proved.

Hence $H^2(\mathcal{L}; \mathcal{L})$ is the tangent space to L at the point $\mathcal{L}$. It is natural to refer to its elements as infinitesimal deformations of the Lie algebra $\mathcal{L}$. To every deformation of $\mathcal{L}$ there corresponds a unique infinitesimal deformation, which is also called the characteristic class of this deformation.
The functor Def is, generally speaking, not representable; that is, there does not exist a universal element. However, there exists a so-called versal (more precisely, mini-versal) element, whose definition we shall now give.

Let X be an object of the category Λ and $\tau \in \operatorname{Def}(\mathcal{L}, X)$. The pair (X, τ) defines a morphism of functors $\theta : \operatorname{Mor}(Y, X) \to \operatorname{Def}(\mathcal{L}, Y)$ (both functors act from the category Λ into the category of sets). The map θ relates to the morphism $\varphi : Y \to X$ the element $\varphi^*(\tau)$. We recall that the pair (X, τ) is called universal if θ is an isomorphism for any object Y. The pair (X, τ) is called *(mini)versal* if a) the map θ is surjective for any Y, and b) θ is an isomorphism if $Y = \operatorname{spec} \mathbf{k}[t]/t^2$.
Conditions a) and b) have a simple geometric meaning. Namely, if (X, τ) is a versal deformation, then the corresponding map $X \to S(V)$ is an embedding, and the image of X intersects the $\operatorname{GL}(V)$-orbit of $\mathcal{L}$ in a unique point p, where the tangent space at p is the sum of the tangent space to the orbit and the tangent space to the image of X.
In [1] and [7] there appear general theorems which imply the existence of a versal deformation. We give an inductive construction of such a deformation. For this we require some information from the theory of obstructions.

3. Let A be a commutative algebra, $\varepsilon \in \operatorname{Def}(\mathcal{L}, \operatorname{spec} A)$, and let m be a maximal ideal in A, $A/m \cong \mathbf{k}$. We choose an element $f \in H^2_{\text{Harr}}(A; \mathbf{k})$ and let $0 \to \mathbf{k} \to B \to A \to 0$ be the corresponding extension. We denote by $\overline{m}$ a maximal ideal in B. We attempt to extend the deformation ε to a deformation with base $\operatorname{spec} B$.
The deformation ε is a Lie A-algebra structure on the space $\mathcal{L} \otimes_{\mathbf{k}} A$. We need to define a Lie B-algebra structure on the space $\mathcal{L} \otimes_{\mathbf{k}} B = \overline{\mathcal{L}}(B)$ such that the map $\chi : \mathcal{L} \otimes B \to \mathcal{L} \otimes A$ induced by the homomorphism $B \to A$ would be a Lie algebra homomorphism. We identify the kernel of χ with the algebra $\mathcal{L}$.
Let $\varphi : \overline{\mathcal{L}}(B) \to \mathcal{L}$ be the map induced by the homomorphism $B \to B/\overline{m} = \mathbf{k}$.
On $\overline{\mathcal{L}}(B)$ we can define a B-linear skew-symmetric operation $\{\ ,\ \}$ for which a) $\chi(\{l_1, l_2\}) = [\chi(l_1), \chi(l_2)]$, $l_i \in \overline{\mathcal{L}}$, and b) $\{l, l_1\} = [l, \varphi(l)]$, $l \in \ker \chi$, $l_1 \in \overline{\mathcal{L}}(B)$. The operation $\{\ ,\ \}$ "partially" satisfies the Jacobi identity; that is,

$$\{l_1, \{l_2, l_3\}\} - \{\{l_1, l_2\}, l_3\} - \{l_2, \{l_1, l_3\}\} = \Phi(l_1, l_2, l_3) \in \ker \chi.$$

Remark that the function Φ is multilinear and skew-symmetric.
Furthermore, if $l_1 \in \overline{m}\overline{\mathcal{L}}(B)$, then $\Phi(l_1, l_2, l_3) = 0$. In fact if $l_1 = nl$, $n \in \overline{m}$, then $\Phi(nl, l_2, l_3) = n\Phi(l, l_2, l_3) = 0$. This means that Φ defines a multilinear skew-symmetric form $\overline{\Phi}$ on $\mathcal{L} = \overline{\mathcal{L}}(B)/\overline{m}\overline{\mathcal{L}}(B)$ with values in $\ker\chi \cong \mathcal{L}$. We shall view $\overline{\Phi}$ as an element of the space $C^3(\mathcal{L};\mathcal{L})$. It is easy to show by direct calculation that $\delta\overline{\Phi} = 0$.
If we replace the operation $\{\ ,\ \}$ by another one which also satisfies conditions a) and b), then the cocycle $\overline{\Phi}$ is changed by a coboundary. Moreover, by changing $\{\ ,\ \}$ we can obtain any element from the cohomology class of $\overline{\Phi}$. So we have assigned to an element f of $H^2_{\mathrm{Harr}}(A;\mathbf{k})$ a cohomology class of $H^3(\mathcal{L};\mathcal{L})$. One can see from the construction that this correspondence is linear in f; that is, we have obtained a homomorphism $\mathcal{P}_2 : H^2_{\mathrm{Harr}}(A;\mathbf{k}) \to H^3(\mathcal{L};\mathcal{L})$. □

Proposition 2. *The deformation ε can be extended to a deformation with base* $\operatorname{spec} B$ *if and only if* $\mathcal{P}_2(f) = 0$.

In fact, a deformation can be extended in the case when one can choose the operation $\{\ ,\ \}$ so that the form Φ is zero. From this it follows that if $\mathcal{P}_2(f) \neq 0$, then the deformation cannot be extended. However, if $\mathcal{P}_2(f) = 0$, the operation $\{\ ,\ \}$ can be modified in such a way that the Jacobi identity holds.

Example. Let $A = \mathbf{k}[t]/t^2$ and $B = \mathbf{k}[t]/t^3$, and let f be a class of extension $0 \to \mathbf{k} \to B \to A \to 0$. A deformation ε with base $\operatorname{spec} A$ is nothing more than an infinitesimal deformation; that is, an element of $H^2(\mathcal{L};\mathcal{L})$. The obstruction $\mathcal{P}_2(f)$ to the extension of ε to a deformation with base $\operatorname{spec} B$ is directly computed. This is equivalent to the cohomology class of the cocycle

$$(l_1, l_2, l_3) \to e(e(l_1, l_2), l_3) + e(e(l_2, l_3), l_1) + e(e(l_3, l_1), l_2),$$

where e is the cocycle representing ε. This class is called the Lie square of the class ε and is denoted by $[\varepsilon, \varepsilon]$. Thus the infinitesimal deformation ε can be extended to a deformation with base $\operatorname{spec} \mathbf{k}[t]/t^3$ if and only if the class $[\varepsilon, \varepsilon] \in H^3(\mathcal{L};\mathcal{L})$ is 0.

Obstructions to the further extension of a deformation onto $\operatorname{spec} \mathbf{k}[t]/t^q$, $q = 4, 5, \dots$, which also lie in $H^3(\mathcal{L};\mathcal{L})$ can be described using Massey Lie operations (see Section 5 below, and also [13]).
The map $\mathcal{P}_2$ is characteristic of the deformation ε. We require one more homomorphism $\mathcal{P}_1 : H^1_{\mathrm{Harr}}(A;\mathbf{k}) \to H^2(\mathcal{L};\mathcal{L})$, also characteristic of the deformation. This homomorphism relates to an element of $H^1_{\mathrm{Harr}}(A;\mathbf{k})$; that is, to a homomorphism $r : A \to \mathbf{k}[t]/t^2$, a characteristic class of the deformation $r^*\varepsilon$ over $\mathbf{k}[t]/t^2$.
Suppose that $\mathcal{P}_2(f) = 0$. Then the deformation ε can be extended to a deformation with base $\operatorname{spec} B$, and in many ways. Let $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ be two brackets on $\overline{\mathcal{L}}(B)$, and let us consider the difference $\{\ ,\ \}_1 - \{\ ,\ \}_2$. This is a skew-symmetric bilinear function ρ which relates to a pair of elements $l_1, l_2 \in \overline{\mathcal{L}}(B)$ an element of $\ker\chi$ and, moreover, $\rho(l_1, l_2) = 0$ if $l_1 \in \overline{m}\overline{\mathcal{L}}(B)$. From this it follows that ρ defines a 2-form

$$\overline{\rho} : \Lambda^2(\overline{\mathcal{L}}(B)/\overline{m}\overline{\mathcal{L}}(B)) \cong \Lambda^2\mathcal{L} \to \ker\chi \cong \mathcal{L}.$$

It can be verified directly that $\overline{\rho}$ is a closed form. The cohomology class of the form $\overline{\rho}$ is the "distinguisher" for the two brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$. The endomorphisms of the extension $0 \to \mathbf{k} \to B \to A \to 0$ act on the set of brackets on $\overline{\mathcal{L}}(B)$. We now study this action.

Proposition 3. *Let a be an automorphism of the extension $0 \to \mathbf{k} \to B \to A \to 0$, and let $\overline{a}$ be the corresponding element of the space $H^1_{\mathrm{Harr}}(A;\mathbf{k})$. Furthermore, let $\{\ ,\ \}$ be a bracket on $\overline{\mathcal{L}}(B)$. Then the difference between the bracket $a\{\ ,\ \}$ and $\{\ ,\ \}$ is $\mathcal{P}_1(\overline{a})$.*

The proof of this Proposition is obvious.

Corollary. *Let $\varepsilon \in Def(\mathcal{L}, A)$ be a deformation for which the homomorphism $\mathcal{P}_1$ is a surjection, and let $0 \to \mathbf{k} \to B \to A \to 0$ be an extension for which ε can be extended to a deformation over $spec\, B$. Then the automorphism group of the extension acts transitively on the set of all extensions of ε.*

4. We now move to the construction of a versal deformation. Let Σ be the subcategory of the category of local algebras which consists of algebras A for which $m^2 = 0$, m is a maximal ideal, and $A/m \cong \mathbf{k}$. Then the functor $\mathrm{Def}\,(\mathcal{L}, \mathrm{spec}\, A)$ can be represented on the category Σ; that is, it admits a universal pair (X, ε) $\varepsilon \in \mathrm{Def}\,(\mathcal{L}, X)$ (see [7]). We construct such a pair. We put $X = \mathrm{spec}\, A$, where $A = \mathbf{k} \otimes H_2(\mathcal{L};\mathcal{L})$ (here $H_2(\mathcal{L};\mathcal{L})$ is an ideal in A with zero multiplication). We now remark that the space

$$H^2(\mathcal{L}; H_2(\mathcal{L};\mathcal{L}) \cdot \mathcal{L}) = H^2(\mathcal{L};\mathcal{L}) \otimes H_2(\mathcal{L};\mathcal{L}),$$

like every tensor product of dual spaces, has a distinguished element. Let v be a cochain representing this element. We now define a Lie A-algebra structure on the space

$$\mathcal{L} \otimes A = \mathcal{L} \otimes 1 \oplus \mathcal{L} \otimes H^2(\mathcal{L};\mathcal{L}),$$

by putting

$$[l_1 \otimes 1, l_2 \otimes 1] = [l_1, l_2] \otimes 1 + v(l_1, l_2), \qquad l_1, l_2 \in \mathcal{L}.$$

This is the Lie A-algebra that corresponds to the deformation ε.

If $C = \mathbf{k} \oplus m$, $m^2 = 0$, is an object of the category Σ, then to each element of the set $\mathrm{Def}\,(\mathcal{L}, \mathrm{spec}\, C)$ there corresponds naturally (as in Proposition 1) an element of $H^2(\mathcal{L}; m \otimes \mathcal{L})$. This correspondence is one-to-one. We note that

$$H^2(\mathcal{L}; m \otimes \mathcal{L}) \cong \mathrm{Hom}\,(H_2(\mathcal{L};\mathcal{L}), m) \cong \mathrm{Hom}\,(A, C) \cong \mathrm{Def}\,(\mathcal{L}, \mathrm{spec}\, C).$$

But this implies that the pair (X, ε) is universal.

Let A be a local algebra, m a maximal ideal, and let $N = (H^2(A; A/m))^* \otimes_{\mathbf{k}} A/m$ be an A-module. We identify the latter with the space $H^2_{\mathrm{Harr}}(A;\mathbf{k})^*$. The space

$$H^2_{\mathrm{Harr}}(A; N) = H^2_{\mathrm{Harr}}(A;\mathbf{k}) \otimes (H^2_{\mathrm{Harr}}(A,\mathbf{k}))^*$$

contains a canonical element u. We construct the corresponding extension $0 \to N \to F(A) \to A \to 0$. Let $A = \mathbf{k}[x_1, \ldots, x_n]/m^2$, $m = (x_1, \ldots, x_n)$. Then the projective limit of the system $A \leftarrow F(A) \leftarrow F(F(A)) \leftarrow \ldots$ is an algebra of formal series in n variables.

Let (X, ε) be a universal pair in the category Σ, and let $A = \mathbf{k}[x]$. The deformation ε gives us the homomorphism

$$P_2 : H^2_{\mathrm{Harr}}(A;\mathbf{k}) \to H^3(\mathcal{L};\mathcal{L}).$$

Consider the dual map

$$P_2^* : H_3(\mathcal{L};\mathcal{L}) \to \big(H^2_{\mathrm{Harr}}(A,\mathbf{k})\big)^*,$$

and let $\overline{u}$ be the image of the class $u \in H^2\big(A; (H^2_{\mathrm{Harr}}(A,\mathbf{k}))^*\big)$ under the homomorphism

$$H^2\big(A; (H^2_{\mathrm{Harr}}(A;\mathbf{k}))^*\big) \to H^2\big(A; (H^2_{\mathrm{Harr}}(A,\mathbf{k}))^*/\mathrm{Im}\,\mathcal{P}_2^*\big).$$

We construct the extension corresponding to $\overline{u}$:

$$0 \to H^2_{\text{Harr}}(A; \mathbf{k})^*/\operatorname{Im} P_2^* \to \overline{F}(A) \to A \to 0.$$

From the constructions of Section 3 it follows that the family of ε can be extended to a family with base $\operatorname{spec} \overline{F}(A)$. From the corollary of Proposition 3 it follows that the automorphism group of the extension acts transitively on the set of extensions. This means that the Lie algebra $\overline{\mathcal{L}}(\overline{F}(A))$ is uniquely defined up to isomorphism. We apply the same construction to the algebra $\overline{F}(A)$ again, then once more, etc. We obtain a projective system of algebras:

$$\cdots \to \overline{\overline{F}}(A) \to \overline{F}(A) \to A.$$

By $v(\mathcal{L})$ we denote the projective limit of this system of algebras. From the above it follows that $v(\mathcal{L})$ is the quotient of the algebra $\mathbf{k}[[H_2(\mathcal{L};\mathcal{L})]]$ by a certain ideal. So, in the obvious way, there is a deformation of $\overline{\mathcal{L}}(v(\mathcal{L}))$ with base $\operatorname{spec} v(\mathcal{L})$.

Proposition 4. *The deformation with base* $\operatorname{spec} v(\mathcal{L})$ *just constructed is versal.*

This proposition is proved by standard means. The proof of a similar result for local commutative algebras can be found in a paper by Schlessinger [7].
The algebra $v(\mathcal{L})$ is the quotient of $\mathbf{k}[[H_2(\mathcal{L};\mathcal{L})]]$ by an ideal J. We assume that $\dim H_2(\mathcal{L};\mathcal{L}) < \infty$. The algebra of formal power series in a finite number of variables is Noetherian, and hence the ideal J has finitely many generators. The space of generators for J can be identified (see [7]) with the space $(H^2_{\text{Harr}}(v(\mathcal{L}), \mathbf{k}))^*$. From the construction of $v(\mathcal{L})$ it follows that the map $H_3(\mathcal{L};\mathcal{L}) \to (H^2_{\text{Harr}}(v(\mathcal{L}); \mathbf{k}))^*$ is surjective. Thus the coordinate ring of the base of the versal deformation is the quotient of $\mathbf{k}[[H_2(\mathcal{L};\mathcal{L})]]$ by the ideal generated by the relations corresponding to the elements of $H_3(\mathcal{L};\mathcal{L})$.

5. In this section, following [4], we introduce certain cohomology operations which serve as the main means for computing the versal deformation. For this we require the standard homology complex of a Lie superalgebra. We do not quote its definition (see [2] or [5]).
Let $A = C^*(\mathcal{L})$ be the standard cochain complex consisting of cochains of the Lie algebra $\mathcal{L}$. Let A be a differential $\mathbf{Z}$-graded algebra. By $\operatorname{Der} A$ we denote the set of superderivations of the algebra A. The space $\operatorname{Der} A$ is equipped in the usual way with a Lie superalgebra structure. We note that $DerA \cong C^*(\mathcal{L};\mathcal{L}) \cong \Lambda^*(\mathcal{L}^*) \otimes \mathcal{L}$ (an element $\omega_1 \otimes l \in \Lambda^*(\mathcal{L}^*) \otimes \mathcal{L}$ is assigned the derivation $\omega_1 \partial/\partial l$, $l \in \mathcal{L}$, $\omega_1 \in \Lambda^*(\mathcal{L}^*)$). The space $\operatorname{Der} A$ has a distinguished element δ to which there corresponds a commutation operator $\Lambda^2\mathcal{L} \to \mathcal{L}$. The differential in the complex $\operatorname{Der} A$ is given by the formula $u \to [u\delta]$; so $\operatorname{Der} A$ is turned into a differential graded Lie superalgebra. The space $H^*(\mathcal{L};\mathcal{L})$ inherits the Lie superalgebra structure.
Let K be the standard complex of the superalgebra $\operatorname{Der} A$; that is, $K = \{\operatorname{Der} A \leftarrow \Lambda^2 \operatorname{Der} A \leftarrow \dots\}$; we recall that the exterior power is here understood in the super sense. We specify on this complex the filtration

$$K_i = \operatorname{Der} A \oplus \Lambda^2 \operatorname{Der} A \oplus \cdots \oplus \Lambda^i \operatorname{Der} A.$$

The first term of the associated spectral sequence, called the Quillen spectral sequence for superalgebras $\operatorname{Der} A$ (see [5]), is isomorphic to the standard complex of the superalgebra $H^*(\mathcal{L};\mathcal{L})$. According to [5], if a Massey operation is defined on the elements $\alpha_1, \dots \alpha_n \in H^*(\mathcal{L};\mathcal{L})$, then a boundary differential is also defined in the spectral sequence under consideration. Hence the images

of the boundary differentials in this spectral sequence can naturally be called generalized Massey Lie operations.
Let us choose in $H^2(\mathcal{L};\mathcal{L})$ an element α. This element is even, and hence in the standard complex $H^*(\mathcal{L};\mathcal{L})$ the elements α^n are defined. As computations show, the first differential is $d_1\alpha^2 = \langle\alpha,\alpha\rangle \in H^2(\mathcal{L};\mathcal{L})$. If $\langle\alpha,\alpha\rangle = 0$, then the differential

$$d_2\alpha^3 \in H^3(\mathcal{L};\mathcal{L})/d_1(S^2H^2(\mathcal{L};\mathcal{L}))$$

is defined (we assume that $d_2\alpha^3 = \langle\alpha,\alpha,\alpha\rangle$), and so on.
The equation $\langle\alpha,\alpha\rangle = 0$ defines a quadratic cone in $H^2(\mathcal{L};\mathcal{L})$. If $\langle\alpha,\alpha\rangle = 0$, then the condition $\langle\alpha,\alpha,\alpha\rangle \ni 0$ defines a subset of this quadratic cone. Iterating this procedure, we obtain a certain homogeneous subvariety V in $H^2(\mathcal{L};\mathcal{L})$. Let $v(\mathcal{L})$ be the algebra constructed in the preceding section, let m be a maximal ideal in $v(\mathcal{L})$, and let $\overline{v}(\mathcal{L})$ be the graded algebra $\mathbf{k} \oplus m/m^2 \oplus m^2/m^3 \oplus \dots$.

Proposition 5 (see [4]). *The spectrum of the algebra $\overline{v}(\mathcal{L})$ is V.*

Corollary. *If for every $\alpha \in H^2(\mathcal{L};\mathcal{L})$*

$$\langle\alpha,\alpha\rangle = 0, \quad \langle\alpha,\alpha,\alpha\rangle \ni 0, \dots,$$

then $\operatorname{spec} v(\mathcal{L}) = H^2(\mathcal{L};\mathcal{L})$; that is, the base of a versal deformation of the algebra $\mathcal{L}$ is $H^2(\mathcal{L};\mathcal{L})$.

References

[1] M. Artin, *Algebraization of formal moduli. I, Global Analysis* (Papers in Honor of K. Kodaira), Univ. of Tokyo Press, Tokyo, and Princeton Univ. Press, Princeton. N. J., 1969, pp. 21–71.
[2] D. A. Leites, Cohomology of Lie superalgebras, *Funktsional. Anal. i Prilozhen.* **9** (1975), no. 4, 75–76; English transl. in Functional Anal. Appl. **9** (1975).
[3] V. P. Palamodov, Deformations of complex spaces, *Uspekhi Mat. Nauk* **31** (1976), no. 3(189), 129–194; English transl. in *Russian Math. Surveys* **31** (1976).
[4] V. S. Retakh, The Massey operations in Lie superalgebras, and deformations of complex-analytic algebras, *Funktsional. Anal. i Prilozhen.* **11** (1977), no. 4. 88–89; English transl. in *Functional Anal. Appl.* **11** (1977).
[5] V. S. Retakh, The Massey operations in Lie superalgebras, and differentials of the Quillen spectral sequence, *Funktsional. Anal. i Prilozhen.* **12** (1978), no. 4. 91–92: English transl. in *Functional Anal. Appl.* **12** (1978).
[6] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag. 1977.
[7] Michael Schlessinger, Functors of Artin rings, *Trans. Amer. Math. Soc.* **130** (1968), 208–222.
[8] Michael Barr, Harrison homology, Hochschild homology and triples, *J. Algebra* **8** (1968), 314–323.
[9] D. K. Harrison, Commutative algebras and cohomology, *Trans. Amer. Math. Soc.* **104** (1962), 191–204.
[10] Luc Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Math., vol. 239, Springer-Verlag, 1971.
[11] Olav A. Laudal, *Formal moduli of algebraic structures*, Lecture Notes in Math., vol. 754, Springer-Verlag, 1979.
[12] Daniel Quillen, On the (co-) homology of commutative rings, *Applications of Categorical Algebra*, Proc. Sympos. Pure Math., vol. 17, Amer. Math. Soc., Providence, R. I., 1970, pp. 65–87.
[13] Murray Gerstenhaber, On the deformation of rings and algebras. I, III, *Ann. of Math.* (2) **79** (1964), 59–103: (2) **88** (1968), 1–34.

APPEARED IN: *Deformation Theory of Algebras and Structures and Applications*, Kluwer Acad. Publishers 1988, 375–401.

An Example of Formal Deformations of Lie Algebras

Alice Fialowski
Institute of Mathematics
Eötvös Loránd University, Budapest

Introduction

In this work we are going to investigate the formal deformations of an infinite dimensional Lie algebra of vector fields on the line with polynomial coefficients. This Lie algebra L_1 consists of the fields which vanish with their first derivative at the origin. For finding the deformations, we have to consider the cohomology with coefficients in the adjoint representation.

In Section 1 we recall – following Nijenhuis and Richardson [6] – the construction of the differential Lie superalgebra structure in the cochain complex of an arbitrary Lie algebra with coefficients in the adjoint representation. In Section 2 we apply the general theory of Schlessinger [8] to the formal deformations of a Lie algebra. In Section 3 we compute the cohomology $H^\bullet(L_1; L_1)$ with the help of the Feigin–Fuchs spectral sequences [1]. In Section 4 we deal with the obstruction theory of Lie algebras and give concrete computations in the case of L_1. In Section 5 we give examples of deformations of this infinite dimensional Lie algebra.

This work was supported by the Swiss National Foundation.
I would like to thank Professor A. Haefliger for the help during the preparation of this lecture and paper.

1. The differential Lie superalgebra $C^\bullet(L; L)$

Let L be a Lie algebra. For a positive integer q denote by $C^q(L; L)$ the space of q-linear, antisymmetric, L-valued functions on L. This is the space of q-dimensional cochains of L with coefficients in the adjoint representation. For $q < 0$ put $C^q(L; L) = 0$. Let $d_q = d$ denote the differential or coboundary operator $d_q = d : C^q(L; L) \to C^{q+1}(L; L)$ which acts as follows.

For $q \geqq 0$, $\varphi \in C^q(L; L)$

$$d\varphi(g_1, \dots, g_{q+1}) := \sum_{1 \leqq s < t \leqq q+1} (-1)^{s+t-1} \varphi([g_s, g_t], g_1, \overset{\hat{s}\,\hat{t}}{\dots}, g_{q+1}) + \\ + \sum_{1 \leqq s \leqq q+1} (-1)^s \big[g_s, \varphi(g_1, \overset{\hat{s}}{\dots}, g_{q+1})\big]$$

where ˆ means that the element with the indicated index is missing. For $q < 0$ let $d_q = 0$. From the definition it follows that $d_{q+1} \circ d_q = 0$, so we get a complex $C^\bullet(L; L)$. By a differential Lie superalgebra we mean a complex $C = (X_n, d)_{n=0}^\infty$ with an operation $[\ ,\ ]$ such that for $x \in X_p$, $y \in X_q$

$$[x, y] = -(-1)^{p \cdot q}[y, x] \tag{1.1}$$

where p is the degree of x. The super-Jacobi identity is satisfied for $x \in X_p$, $y \in X_q$, $z \in X_r$:

$$(-1)^{p \cdot q}[[x, y], z] + (-1)^{q \cdot r}[[y, z], x] + (-1)^{r \cdot p}[[z, x], y] = 0 \tag{1.2}$$

and the differential d of degree $+1$ is such that

$$d([x, y]) = [dx, y] - (-1)^p [x, dy]. \tag{1.3}$$

Proposition 1.1. *The complex $C^\bullet(L; L)$ is a differential Lie superalgebra, the degree of $a \in C^p(L; L)$ being $p - 1$.*

Proof. For $a \in C^p(L;L)$ and $b \in C^q(L;L)$ define the cochain $ab \in C^{p+q+1}(L;L)$ by

$$ab(g_1,\dots,g_{p+q-1}) := \sum_\sigma \operatorname{sgn}(\sigma) a\big(b(g_{i_1},\dots,g_{i_q}), g_{j_1},\dots,g_{j_{p-1}}\big)$$

where the sum runs over the shuffles

$$\{1,\dots,p+q+1\} = \{i_1,\dots,i_q\} \cup \{j_1,\dots,j_{p-1}\}$$

$(i_1 < \dots < i_q, j_1 < \dots < j_{p-1})$.
Put $[a,b] = ab - (-1)^{(p-1)(q-1)} ba$.
It is easy to verify that for this superbracket operation the identities (1.1)–(1.3) are satisfied with $c \in C^r(L;L)$.
From (1.3) it follows that if a, b are cocycles then the superbracket $[a,b]$ is also a cocycle, and the cohomology class of $[a,b]$ depends only on the class of a and b. That means that a multiplication can be defined in the cohomology space

$$H^p(L;L) \otimes H^q(L;L) \longrightarrow H^{p+q-1}(L;L)$$

which satisfies (1.1) and (1.2) with $a \in H^p(L;L)$, $b \in H^q(L;L)$, $c \in H^r(L;L)$. □

Corollary. *The Lie superalgebra structure on $C^\bullet(L;L)$ induces a structure of Lie superalgebra on the cohomology space, in which the usual grading is reduced by one. In this way we get an analogy with the Kodaira–Spencer theory* (see [6]).

2. Formal deformations of Lie algebras. General theory

In this section we explain, how the general theory of Schessinger applies to formal deformations of Lie algebras.

Let L be a Lie algebra over a field K. Let $\mathcal{C}$ be the category of local finite dimensional algebras A over K. For such an A there exists a unique maximal ideal m_A such that $A/m_A = K$ and $\dim_K A$ is finite. Let us denote by ε the canonical map $A \longrightarrow A/m_A = K$. If $t_1,\dots,t_n$ are elements of m_A such that their images in m_A/m_A^2 form a basis, then $A = K[[t_1,\dots,t_n]]/I$ where I contains a power of the maximal ideal of $K[[t_1,\dots,t_n]]$. The morphisms in $\mathcal{C}$ are the homomorphisms of local algebras (so commuting with ε).
A deformation L_A of L parametrized by $A \in \mathcal{C}$ is a Lie algebra structure over A on $L \otimes_K A$ such that the Lie algebra structure on

$$L = (L_A) \otimes_A K = (L \otimes A) \otimes_A K$$

is the given one on L (obtained from L_A by the extension of the scalars given by ε).
If $f : A \to B$ is a morphism in $\mathcal{C}$ then the Lie algebra $L_B = (L_A) \otimes_A B$ is the deformation of L parametrized by B induced by f from L_A.

Two deformations L_A and L'_A of L parametrized by A are equivalent, if there exists a Lie algebra isomorphism over A of L_A on L'_A inducing the identity of $L_A \otimes_A K = L$ on $L'_A \otimes_A K = L$.
The functor $F : \mathcal{C} \to \text{Sets}$ associates to $A \in \mathcal{C}$ the set $F(A)$ of equivalence classes of deformations of L parametrized by A.
The algebra A is of order less or equal to k if $m^{k+1} = 0$. (For instance if $k = 1$, $K = \mathbb{C}$, $t_1,\dots,t_n$ form a basis of m/m^2, then $A = \mathbb{C}\cdot 1 \oplus \mathbb{C}t_1 \oplus \dots \oplus \mathbb{C}t_n$ and $t_i t_j = 0$ for all $1 \leqq i,j \leqq n$.)
A deformation L_A is of order k if A is of order $\leqq k$. An infinitesimal deformation is a deformation of order 1, parametrized by $K[t]/(t^2)$.

More generally, let $\hat{\mathcal{C}}$ be the category of complete local algebras A over K such that $A/m_A^n \in \mathcal{C}$ for all n (complete means that $A = \varprojlim A/m_A^n$).
A deformation of L parametrized by A is the projective limit $\varprojlim L_A/m_A^n$ of deformations of L parametrized by A/m_A^n. In other words, a deformation of L parametrized by $A \in \hat{\mathcal{C}}$ is a Lie algebra structure over A on $L\hat{\otimes}A = \varprojlim L \otimes A/m_A^n$, inducing the given structure on L.

There is an analogous definition for isomorphisms of deformations parametrized by $A \in \hat{\mathcal{C}}$. The functor F can be extended as a functor $\hat{F} : \hat{\mathcal{C}} \to \text{Sets}$.
A deformation L_R of L parametrized by $R \in \hat{\mathcal{C}}$ is called *formally versal* if for any deformation L_A of L parametrized by $A \in \hat{\mathcal{C}}$ there is a morphism $f : R \to A$ such that $L_R \otimes {}_R A$ is equivalent to L_A and if the map $m_R/m_R^2 \to m_A/m_A^2$ induced by f is unique. (In particular, all the other deformations of L can be induced from the versal one.)
A versal deformation up to order k is defined similarly, where $\hat{\mathcal{C}}$ is replaced by the subcategory of algebras of order $\leqq k$.

Theorem (Schlessinger)**.** *If the space $H^2(L; L)$ is finite, then there exists a formal versal deformation of L.*

Proof. This follows from the general Theorem of Schlessinger ([8, Theorem 2.11]), provided we check the following properties of the functor F.
Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$. Consider the map

$$\tau : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

associating to the equivalence class of a deformation of L parametrized by $A' \times_A A''$ the equivalence classes of the deformations of L parametrized by A' and A'' respectively, induced by the morphism $A' \times_A A'' \to A'$ and $A' \times_A A'' \to A''$. Then
a) τ is surjective whenever $A'' \to A$ is surjective;
b) τ is bijective when $A = K$.
To check a) consider two deformations $L_{A'}$ and $L_{A''}$ of L parametrized by A' and A'' respectively, and such that there exists an equivalence ξ of $L'_A = L_{A'} \otimes_{A'} A$ on $L''_A = L_{A''} \otimes_{A''} A$ of the associated deformations of L parametrized by A. The equivalence classes of $L_{A'}$ and $L_{A''}$ give an element of $F_{A'} \times_{F_A} F_{A''}$. As the morphism $A'' \to A$ is surjective and as $L_{A''}$ is a free A''-module, after changing $L_{A'}$ in its equivalence class, we can assume that the equivalence ξ is the identity of $L_A = L \otimes A$, using the canonical A-module isomorphism of L'_A and L''_A with $L \otimes A$. Then the image by τ of the equivalence class of the deformation $L_{A'} \times_{L_A} L_{A''}$ of L parametrized by $A' \times_A A''$ will be the given element of $F_{A'} \times_{F_A} F_{A''}$.
The condition b) can be easily verified because in that case $B = A' \times_A A''$ is equivalent to $K \cdot 1 \oplus m_{A'} \oplus m_{A''}$ with $m_{A'} \cdot m_{A''} = 0$ and a deformation of L parametrized by $A' \times_A A''$ is characterized by the induced deformation parametrized by A' and A''. □

To link this section to the preceding one we can give a more concrete description of the Lie algebra L_A.
Suppose $A \in \hat{\mathcal{C}}$, then $A = K[[t_1, \ldots, t_n]]/I$. We can express $L \hat{\otimes} A$ in the form $L \hat{\otimes} K[[t_1, \ldots, t_n]]/I$. The Lie algebra structure on L_A will be described by a bilinear alternate map

$$\mu_t : L \times L \to L \hat{\otimes} K[[t_1, \ldots, t_n]] \text{ such that}$$

$$\mu_t(x, y) = [x, yl \otimes 1 + \sum_{|\alpha| \geqq 1} \varphi_\alpha(x, y) \otimes t^\alpha$$

where $\varphi_\alpha \in C^2(L; L)$ and $\mu_t(x, y)$ composed with the projection on $L \hat{\otimes} A$ is equal to $[x \otimes 1, y \otimes 1]_{L_A}$. This lifting is unique mod I.
The map μ_t will define a bracket in $L \hat{\otimes} A$ verifying the Jacobi identity iff

$$2d\varphi + [\varphi, \varphi] \equiv 0 \text{ mod } I$$

where d and [,] were defined in Section 2. This means that the coefficients of the formal power series obtained from

$$2 \sum_{|\alpha| \geqq 1} (d\varphi_\alpha) t^\alpha + \sum_{|\alpha| \geqq 1} \sum_{\beta + \gamma = \alpha} [\varphi_\beta, \varphi_\gamma] t^\beta t^\gamma$$

by applying to each coefficient an arbitrary linear form on $C^2(L; L)$ belong to I.
In case $|\alpha| = 1$ we get $d\varphi_\alpha = 0$ for each α which means that φ_α is a cocycle.

The elements of $H^2(L;L)$ correspond bijectively to the equivalent classes of infinitesimal deformations. Suppose that $\dim H^2(L;L)$ is finite. Let $\varphi_1, \varphi_2, \dots, \varphi_n$ be cocycles whose cohomology classes form a basis of $H^2(L;L)$. Then a versal deformation of order 1 of L is parametrized by $K[t_1,\dots,t_n]/(m^2)$ and is given by the bilinear alternate map

$$\mu_t = \mu_0 + \varphi_1 t_1 + \cdots + \varphi_n t_n$$

where μ_0 is the original bracket in L.
Let us try to extend this deformation to a versal deformation of order 2 parametrized by $K[t_1,\dots,t_n]/I$ where I contains m^3. The bracket should be of the form

$$\mu_t = \mu_0 + \sum_{i=1}^{n} \varphi_i t_i + \sum \varphi_{ij} t_i t_j$$

with the conditions that

$$-2\Sigma d\varphi_{ij} t_i t_j \equiv \Sigma[\varphi_i, \varphi_j] t_i t_j \bmod I.$$

This means that the right-hand side (which is always a three-cocycle) must be coboundary. So the ideal I is generated by the polynomials, obtained by composing the cohomology class of the right-hand side with linear forms on $H^3(L;L)$, and m^3. For φ_{ij} one can choose any 2-cochain satisfying the above condition.

3. Computation of $H^\bullet(L_1;L_1)$

Let $W^{\text{pol}} = W_1$ be the Lie algebra of vector fields on the line with polynomial coefficients $f(x)\frac{d}{dx}$. This Lie algebra has an additive algebraic basis

$$e_i = x^{i+1}\frac{d}{dx}, \qquad i \geqq -1.$$

In this basis the bracket operation is

$$[e_i, e_j] = (j-i)e_{i+j}.$$

Let us introduce the subalgebra L_i, $i \geqq 0$ of W_1 which is generated by the basis elements $\{e_i, e_{i+1}, \dots\}$.
We shall investigate the subalgebra L_1. The Lie algebra L_1 is naturally graded, the weight of e_i equals i. With this grading L_1^{pol} is a graded Lie algebra: $L_1^{\text{pol}} = \bigoplus_{m=1}^{\infty} L_1^{(m)}$.
We consider the cohomology of L_1 with coefficients in the adjoint representation. The cochain complex is defined in the graded sense:

$$C^q(L_1;L_1) = \oplus C^q_{(m)}(L_1;L_1),$$

where for the cochain $\varphi \in C^q_{(m)}(L_1;L_1)$ the weight of $\varphi(e_{i_1},\dots,e_{i_q})$ is $m+i_1+\cdots+i_q$. The grading is inherited by the cohomology spaces $H^q(L_1;L_1)$.

Theorem (see [2]). *For $q>0$, $H^q_{(m)}(L_1;L_1) \cong H^{q-1}_{(m)}(L_2;\mathbb{C})$. The cohomology space $H^q(L_1;L_1)$ has dimension $2q-1$ and is generated by elements of weight $-\frac{3q^2-q}{2}+i$ where $i=1,2,\dots,2q-1$.*
In particular, $H^1(L_1;L_1)$ is of dimension 1 and has weight 0; the space $H^2(L_1;L_1)$ is three-dimensional with generators of weight $-2,-3$ and -4, while $\dim H^3(L_1;L_1)=5$ with generators of weight $-7,-8,-9,-10$ and -11.

Proof. Define the module F_λ over W_1, where $\lambda \in \mathbb{C}$ is arbitrary, as the space of expressions $f(x)dx^{-\lambda}$ where $f(x)$ is a formal power series of x (see [1]). Then the formula

$$\left(g\frac{d}{dx}\right) f dx^{-\lambda} = (gf' - \lambda f g')dx^{-\lambda}$$

gives the action of W_1 in F_λ. (For λ an integer they are modules of formal tensor fields; formal power series for $\lambda=0$, formal differential 1-forms for $\lambda=-1$ and formal vector fields for $\lambda = 1$.) The module F_λ has an additive basis $\{f_j \mid j = 0, 1, \dots\}$ where $f_j = x^j dx^{-\lambda}$ and the action on the basis elements is

$$e_i f_j = (j - (i+1)\lambda) f_{i+j}.$$

Denote by $\mathcal{F}_\lambda$ the W_1-module which is defined in the same way, only the index j runs over all integers. The adjoint modules F'_λ, $\mathcal{F}'_\lambda$ are defined as modules of linear functionals $F_\lambda \to \mathbb{C}$, $\mathcal{F}_\lambda \to \mathbb{C}$ which are finite in the sense that they take nonzero value only on a finite number of f_j-s. That means F'_λ and $\mathcal{F}'_\lambda$ are generated by elements f'_j, and W_1 acts on them by the formula

$$e_i f'_j = \begin{cases} (-(j-i) + (i+1)\lambda) f'_{j-i} & \text{if } f'_j \in \mathcal{F}'_\lambda \text{ or } j \geqq i, \\ 0 & \text{if } f'_j \in F'_\lambda \text{ and } j < 0. \end{cases}$$

The correspondence $f'_j \leftrightarrow f_{-1-j}$ defines for any λ an isomorphism $\mathcal{F}'_\lambda = \mathcal{F}_{-1-\lambda}$ and as $F_{-1-\lambda} = \operatorname{ann} F_\lambda$, it follows that $F'_\lambda = \mathcal{F}_{-1-\lambda}/F_{-1-\lambda}$.

For $\lambda \neq 0$ the W_1-module F_λ is irreducible. But if we consider it as an L_0-module, it is reducible. For obtaining the L_0-submodules of the module F_λ it is sufficient to take the subspace generated by those f_j-s with $j \geqq \mu$ where μ is some positive integer. Denote the L_0-module we get by $F_{\lambda,\mu}$. We can define it directly as the subspace, generated – like F_λ – by the elements f_j, $j = 0, 1, \dots$, on which L_0 acts by

$$e_i f_j = (j + \mu - (i+1)\lambda) f_{i+j}.$$

In this definition μ can be an arbitrary complex number. (For positive integer μ the embedding $F_{\lambda,\mu} \to F_\lambda$ is defined by the formula $f_j \mapsto f_{j+\mu}$.) Let $F'_{\lambda,\mu}$ denote the module, conjugate to $F_{\lambda,\mu}$. At last define the modules $\mathcal{F}_{\lambda,\mu}$ over W_1 as $F_{\lambda,\mu}$ above, without requiring the positivity of j. Obviously $\mathcal{F}'_{\lambda,\mu} = \mathcal{F}_{-1-\lambda,-\mu}$ and $F'_{\lambda,\mu} = \mathcal{F}_{-1-\lambda,-\mu}/F_{-1-\lambda,-\mu}$.

All these modules are graded. Their basis elements f_j are homogeneous and the grading is defined by $\deg f_j = j$, $\deg f'_j = -j$. (Mention that in $F_{\lambda,\mu}$ and $\mathcal{F}_{\lambda,\mu}$ the grading is independent of λ and μ.)

Our aim is to calculate the homology of the Lie algebra L_1 with coefficients in $\mathcal{F}_{\lambda,\mu}$ and $F_{\lambda,\mu}$. (The calculations for the modules $\mathcal{F}_\lambda$ and F_λ see in [1] or [3].)

The space of chains $C_q^{(m)}(L_1; \mathcal{F}_{\lambda,\mu})$ is generated by "monomials", i.e. by the chains

$$f_j \otimes e_{i_1} \wedge \cdots \wedge e_{i_q} \quad \text{with} \quad j + i_1 + \cdots + i_q = m.$$

Denote by $G_p C_q^{(m)}(L_1; \mathcal{F}_{\lambda,\mu})$ the subspace of the space $C_q^{(m)}(L_1; \mathcal{F}_{\lambda,\mu})$, generated by monomials with $i_1 + \cdots + i_q \leqq p$. Evidently, $\{G_p C_q^{(m)}(L_1; F_{\lambda,\mu})\}_p$ is a decreasing filtration in $C_\bullet^{(m)}(L_1; \mathcal{F}_{\lambda,\mu})$.

Denote the spectral sequence corresponding to this filtration by $E(\lambda, m-\mu)$. In this spectral sequence $E^0_{p,q} = C^{(p)}_{p+q}\big(L_1; (\mathcal{F}_{\lambda,\mu})_{m-p}\big)$ where $(\mathcal{F}_{\lambda,\mu})_{m-p}$ is considered as a trivial L_1-module, and $d^0_{p,q}$ is the differential

$$d_{p+q} : C^{(p)}_{p+q}(L_1; \mathbb{C}) \to C^{(p)}_{p+q-1}(L_1; \mathbb{C}).$$

Hence

$$E^1_{p,q} = H^{(p)}_{p+q}(L_1; \mathbb{C}).$$

From [5] it follows that $E^1_{p,q} = \mathbb{C}$ for $p = \frac{3r^2 \pm r}{2}$, $p + q = r$ and $E^1_{p,q} = 0$ for other p and q. We set $E^r_p = \oplus_q E^r_{p,q}$; $d^r_p = \oplus_q d^r_{p,q}$, then obviously

$$H_q^{(m)}(L_1; \mathbb{C}) \cong E^\infty_{\frac{3q^2+q}{2}} \oplus E^\infty_{\frac{3q^2-q}{2}}.$$

If the coefficients are taken not in $\mathcal{F}_{\lambda,\mu}$ but in $F_{\lambda,\mu}$ then the filtration is the same. The new spectral sequence can be mapped into the old one. On E^1_p with $p \leqq m$ this map is an isomorphism, and for $p > m$ we have in the new spectral sequence $E^1_p = 0$. If we "truncate" the spectral sequence $E(\lambda, m-\mu)$ from the other side, leaving in it the part corresponding to $p \geqq m$, then obviously we get a spectral sequence converging to $H_\bullet^{(m)}(L_1; F'_{-1-\lambda,-\mu})$. (We recall that $F'_{-1-\lambda,-\mu} = \mathcal{F}_{\lambda,\mu}/F_{\lambda,\mu}$.)

Let us set $e(t) = (3t^2 + t)/2$ (Euler polynomial) and define the k-th parabola ($k = 0, 1, 2, \dots$) as a curve on the complex plane with the parametric equations

$$\lambda = e(t) - 1$$
$$m - k = e(t) + e(t + k) - 1.$$

If in the second equation we take a negative integer k then we get another parametric equation for the $|k|$-th parabola.
For $k_1, k_2 \in \mathbb{Z}$ we set

$$P(k_1, k_2) = (e(k_1) - 1, e(k_1) + e(k_2) - 1)$$

(the points $P(k_1, k_2)$ are pairwise distinct) and let

$$\mathbb{P} = \{P(k_1, k_2) \mid k_1, k_2 \in \mathbb{Z}\}.$$

Lemma. (i) *If the point $(\lambda, m - \mu)$ does not lie on any parabola then in the spectral sequence $E(\lambda, m - \mu)$ all the first differentials are different from zero.*
(ii) *If the point $(\lambda, m - \mu)$ lies on the k-th parabola but does not lie on parabolas with smaller indices, and is not contained in $\mathbb{P}$, then the differentials $d_1(r)$ with $r \leqq k$ and $d_2(k + 2s, 1)$, $d_2(k + 2s - 1, 2)$ with $s > 0$ are nontrivial in the spectral sequence $E(\lambda, m - \mu)$.*

For the proof of this Lemma see [1, Lemma 3.1.(A) and (B)].
From the Lemma the generated version of Theorem 4.1(A) and 4.2(A) in [1] follows easily. □

Theorem a). *If $(\lambda, m - \mu) \notin \mathbb{P}$ then*

$$H_\bullet^{(m)}(L_1; F_{\lambda,\mu}) = 0;$$

if the point $(\lambda, m - \mu)$ does not lie on any of the parabolas of the Lemma then

$$H_q^{(m)}(L_1; F_{\lambda,\mu}) = H_q^{(m)}(L_2; \mathbb{C}).$$

Another theorem we need is the generalization of Theorem 3.1 in [1] for $F'_{\lambda,\mu}$-modules.

Theorem b). *For those $(\lambda, m - \mu)$ considered in Theorem a),*

$$H_q^{(m)}(L_1; \mathcal{F}_{\lambda,\mu}/F_{\lambda,\mu}) = H_{q-1}^{(m)}(L_1; F_{\lambda,\mu}) \oplus H_q^{(m)}(L_1; \mathcal{F}_{\lambda,\mu}).$$

For the proof let us consider the short exact sequence

$$0 \to F_{\lambda,\mu} \to \mathcal{F}_{\lambda,\mu} \to \mathcal{F}_{\lambda,\mu}/F_{\lambda,\mu} \to 0.$$

One should check that the homomorphism $H_q(L_1; F_{\lambda,\mu}) \to H_q(L_1; \mathcal{F}_{\lambda,\mu})$ is always trivial, which is evident from Theorem a) for dimensional reasons.

The adjoint representation is $F_{1,1}$. Remark that $H^q(L_1; F_{1,1})$ is dual to $H_q(L_1; F'_{1,1})$ in the graded sense, i.e.

$$H^q_{(-m)}(L_1; L_1) = H_q^{(m)}(L_1; L'_1) = H_q^{(m)}(L_1; F'_{1,1}) = H_q^{(m)}\big(L_1; \mathcal{F}_{-2,-1}/F_{-2,-1}\big).$$

In our case $\lambda = -2$, $\mu = -1$. The line $\lambda = -2$ does not intersect any of the parabolas of the Lemma. From Theorem a) and b) we obtain that

$$\dim H^q_{(-m)}(L_1; L_1) = \dim H^q_{(-m)}(L_2; \mathbb{C}).$$

The spaces $H^\bullet(L_q; \mathbb{C})$ are calculated in [5]. By comparing the results of that calculations with the above one we get the required Theorem.

Remark. *A cocycle φ, representing a generator of $H^1(L_1; L_1)$ has the form $\varphi(e_i) = ie_i$. We know that each element of $H^1(L_1; L_1)$ defines a Lie algebra, containing L_1 as an ideal of codimension 1. In the present case we get L_0.*

Let us denote by α, β and λ the three homogeneous nonzero elements of weights $-2, -3$ and -4 in $H^2(L_1; L_1)$. It is not difficult to find cocycles in those cohomology classes.

Proposition 3.1. *Representative cocycles are, for instance,* $\overline{\alpha} \in C^2_{(-2)}(L_1; L_1)$, $\overline{\beta} \in C^2_{(-3)}(L_1; L_1)$, $\overline{\gamma} \in C^2_{(-4)}(L_1; L_1)$, *defined as follows:*

$$\overline{\alpha}(e_2, e_3) = 4e_3,$$
$$\overline{\alpha}(e_2, e_j) = je_j, \quad \overline{\alpha}(e_3, e_j) = -(j-1)e_{j+1} \ \text{ for } \ j \geqq 4,$$
$$\overline{\alpha}(e_i, e_j) = 0 \ \text{ for other } \ i, j;$$
$$\overline{\beta}(e_2, e_3) = 8e_2, \quad \overline{\beta}(e_2, e_4) = 4e_3, \quad \overline{\beta}(e_3, e_4) = -10e_4,$$
$$\overline{\beta}(e_2, e_j) = (j+1)e_{j-1}, \ \overline{\beta}(e_3, e_j) = -2je_j, \ \overline{\beta}(e_4, e_j) = (j-1)e_{j+1} \text{ for } j \geqq 5,$$
$$\overline{\beta}(e_i, e_j) = 0 \ \text{ for other } i, j;$$
$$\overline{\gamma}(e_2, e_3) = 14e_1, \quad \overline{\gamma}(e_2, e_5) = 8e_3, \quad \overline{\gamma}(e_3, e_4) = -24e_3,$$
$$\overline{\gamma}(e_3, e_5) = -16e_4, \quad \overline{\gamma}(e_4, e_5) = 18e_5$$
$$\left.\begin{array}{ll} \overline{\gamma}(e_2, e_j) = (j+2)e_{j-2}, & \overline{\gamma}(e_3, e_j) = -3(j+1)e_{j-1} \\ \overline{\gamma}(e_4, e_j) = 3je_j, & \overline{\gamma}(e_5, e_j) = -(j-1)e_{j+1} \end{array}\right\} \text{ for } j \geqq 6,$$
$$\overline{\gamma}(e_i, e_j) = 0 \ \text{ for other } i, j.$$

Proof. The fact that $\overline{\alpha}, \overline{\beta}$ and $\overline{\gamma}$ are cocycles, follows from direct computations, and the fact that they are not cohomological to zero follows from the next

Lemma. *Each class of* $H^2_{(-m)}(L_1; L_1)$ *with* $m \geqq 2$ *is represented by a unique cocycle* ω*, which vanishes on* $e_1 : \omega(e_1, e_j) = 0$ *for all* j. □

Remark. *Using this lemma we can give an elementary proof of the fact that* α, β *and* γ *generate* $H^2(L_1; L_1)$.

Remark. *By constructing the cocycles* $\overline{\alpha}, \overline{\beta}$ *and* $\overline{\lambda}$ *we can give all the nonequivalent infinitesimal deformations of the Lie algebra* L_1.

4. Obstructions

A natural question is when is it possible to extend an infinitesimal deformation to a deformation of higher order. To extend an infinitesimal deformation represented by a cocycle φ_1 to a second order deformation, parametrized by $\mathbb{C}[t]/(t^3)$ it is necessary and sufficient that $[\varphi_1, \varphi_1]$ is cohomological to zero. If φ_2 is a cochain such that

$$-2d\varphi_2 = [\varphi_1, \varphi_1]$$

then we can define a 2-order deformation with the bracket

$$\mu_t(x, y) = [x, y] + \varphi_1(x, y)t + \varphi_2(x, y)t^2.$$

Here φ_2 is well-defined up to a 2-cocycle. The cohomology class of $[\varphi_1, \varphi_1]$ is the first obstruction to forming a one-parameter family of deformations whose first term is cohomological to φ_1.

To extend now a second order deformation to a third-order one, parametrized by $\mathbb{C}[t]/(t^4)$, it is necessary and sufficient that $[\varphi_1, \varphi_2]$ is also cohomological to zero. If φ_3 is a cochain such that

$$-2d\varphi_3 = [\varphi_1, \varphi_2]$$

then we can define a 3-order deformation with the bracket

$$\mu_t = \mu_0 + \varphi_1 t + \varphi_2 t^2 + \varphi_3 t^3.$$

Here φ_3 is also well-defined up to a 2-cocycle. The cohomology class of $[\varphi_1, \varphi_2]$ is the second obstruction to forming a one-parameter family of deformations whose first term is cohomological to φ_1. If we take another cocycle $\varphi_2' = \varphi_2 + c$ then the obstruction is the class of $[\varphi_1, \varphi_2] + [\varphi_2, c]$ which is in the factorspace $H^3(L_1; L_1)/H^2(L_1; L_1)$. We can extend a given deformation step by step, until the first obstruction appears. As we see, each further step depends on the choice of the earlier ones.

In general, let us define in $H^\bullet(L; L)$ higher order operations, called Massey operations. These n-order operations are partially defined and they are well-defined mod the $(n-1)$-order ones. The second-order operation is the superbracket. Suppose that

$y_1 \in H^p(L;L)$, $y_2 \in H^q(L;L)$ and $y_3 \in H^r(L;L)$ are such that $[y_i, y_j] = 0$, i.e. for cocycles x_i representing y_i, $[x_i, x_j] = dx_{ij}$. Then the third-order $\langle y_1, y_2, y_3 \rangle$ Massey operation (cube) takes value in the factorspace

$$H^{p+q+r-3}(L;L)/[y_1, H^{q+r-2}(L;L) + [y_2, H^{p+r-2}(L;L)] + [y_3, H^{p+q-2}(L;L)]$$

and is equal to the image of the cohomology class of the cocycle

$$[x_{12}, x_3] + [x_1, x_{23}] + (-1)^{q \cdot r}[x_{13}, x_2].$$

The cohomology class of this cocycle depends on the choice of x_{ij}, but its image in the factorspace is well-defined. The next Massey operation $\langle y_1, y_2, y_3, y_4 \rangle$ is defined on four elements $y_1 \in H^p(L;L)$, $y_2 \in H^q(L;L)$, $y_3 \in H^r(L;L)$, $y_4 \in H^s(L;L)$ when all the operations of lower order are defined and are cohomological to zero. It takes value in the factorspace

$$\begin{aligned} H^{p+q+r+s-5}(L;L)/[y_1, H^{q+r+s-3}] &+ [y_2, H^{p+r+s-3}] \\ &+ [y_3, H^{p+q+s-3}] + [y_4, H^{p+q+r-3}] \end{aligned}$$

etc. For the general definition see Retakh [7].
If $y_i \in H^2(L;L)$ then these operations take value in a factorspace of $H^3(L;L)$. The Massey operations are closely connected with the "obstructions" to "extending" an infinitesimal deformation of the Lie algebra.

Theorem (Retakh, [7]). *Given an element α belonging to $H^2(L;L)$ there exists a formal deformation of the Lie algebra L parametrized by $K[[t]]$ with infinitesimal deformation α if and only if all the Massey products $\langle \alpha, \dots, \alpha, \alpha \rangle$ are zero.*

Remark. *The question of convergence of this formal power series remains open in general.*

Theorem 14. *In the case of L_1 the Massey products $\underbrace{\langle \alpha, \alpha, \dots, \alpha \rangle}_{i}$ are zero for all i, the brackets $[\beta, \beta], [\alpha, \beta]$ and $[\alpha, \gamma]$ are trivial, while $[\gamma, \gamma]$ and $[\beta, \gamma]$ are not. The only nontrivial 3-products are $\langle \beta, \beta, \beta \rangle$ and $\langle \alpha, \beta, \beta \rangle$. The higher operations are either not defined or they are trivial.*

Proof. The superbrackets $[\alpha, \alpha], [\beta, \beta]$ are trivial, because the weight of $[\alpha, \alpha]$ and $[\beta, \beta]$ equals -4 and -6 and any such three-dimensional cohomology class is zero. Similarly, by dimensional considerations we have $[\alpha, \beta] = [\alpha, \gamma] = 0$.
The triviality of the class $\underbrace{\langle \alpha, \alpha, \dots, \alpha \rangle}_{i}$ for any i follows from the fact that there exists a deformation with infinitesimal deformation, equal to $-\frac{1}{3}\alpha$. Namely, the deformation is

$$[e_i, e_j]_t = (j-i)e_{i+j} + (j-i)te_{i+j-2}.$$

A geometric realization of this deformation is the following. Denote by $L_1(t) \subset W_1$ the algebra of vector fields $(x^2+t)\varphi(x)\frac{d}{dx}$. Define a linear isomorphism $\varepsilon_t : L_1 \to L_1(t)$ by the formula

$$\varepsilon_t(e_i) = (x^2+t)x^{i-1}\frac{d}{dx} = e_i + te_{i-2}.$$

Then

$$[e_i, e_j]_t = \varepsilon_t^{-1}[\varepsilon_t(e_i), \varepsilon_t(e_j)]$$

gives the above deformation of the Lie algebra L_1.
To prove $[\gamma, \gamma] \neq 0$ substitute $\overline{\gamma}$ into the superbracket formula. The three-dimensional cocycle we obtain is not cohomological to zero, as its value on a homology class of weight -8 is different from zero. Direct computation shows that $[\overline{\gamma}, \overline{\gamma}]$ has nonzero value on the class of weight -8.
Let us verify that $[\beta, \gamma] \neq 0$. Here the triviality would imply that L_1 has a deformation over $K[t_1, t_2]/(t_1^2, t_2^2)$ such that in the deformed algebra

$$[e_i, e_j]_{t_1,t_2} = (j-i)e_{i+j} + \overline{\beta}(e_i, e_j)t_1e_{i+j-3} + \overline{\gamma}(e_i, e_j)t_2e_{i+j-4} + \kappa(e_i, e_j)t_1t_2e_{i+j-7}.$$

Straight calculation shows that such an algebra cannot exist. Namely, the numbers $\kappa_{i,j} = \kappa(e_i, e_j)$ can be defined step by step. At the 12-th step we get a system of equations for κ_{ij} from the Jacobi identities, which has no solution.
The nontriviality of the class $\langle\beta, \beta, \beta\rangle$ is equivalent to the fact that for any Lie algebra over $K[t]/(t^4)$ with the basis $\{e_i, i = 1, 2, \dots\}$ the bracket cannot be of the following form:

$$[e_i, e_j]_t = (j - i)e_{i+j} + t\overline{\beta}(e_i, e_j)e_{i+j-3} + t^2\kappa_1(e_i, e_j)e_{i+j-6} + t^3\kappa_2(e_i, e_j)e_{i+j-9}.$$

Here $\kappa_1(e_i, e_j)$ and $\kappa_2(e_i, e_j)$ can be defined step by step ($i + j = 1, 2, \dots$) from the system of equations, following from the Jacobi identity. For $i + j = 12$ we get a contradiction.
The three-bracket $\langle\alpha, \alpha, \beta\rangle$ is also defined, because the two-brackets are zero. It is defined mod $[\beta, H^2(L_1; L_1)] + [\alpha, H^2(L_1; L_1)]$. As the weight of $[\beta, \gamma]$ equals to -7, the three-bracket $\langle\alpha, \alpha, \beta\rangle$ is trivial. The last three-bracket which can be defined is $\langle\alpha, \beta, \beta\rangle$. For computing it we have to choose a coboundary for $[\alpha, \beta], [\alpha, \alpha]$ and $[\beta, \beta]$. It turns out that $\langle\alpha, \beta, \beta\rangle$ is not containing zero.
The only four-bracket which can be defined is $\langle\alpha, \alpha, \alpha, \beta\rangle$ which occurs to be the trivial cohomology class. □

A versal deformation of the Lie algebra L_1 of order 1 is given by the bilinear map

$$\mu_{t_1,t_2,t_3} = \mu_0 + \overline{\alpha}t_1 + \overline{\beta}t_2 + \overline{\gamma}t_3$$

and is parametrized by $\mathbb{C}[t_1, t_2, t_3]/(t^2)$.
The nontrivial superbrackets give the equation for the parameter space of the versal deformation of order 2:

$$[\beta, \gamma]t_2t_3 + [\gamma, \gamma]t_3^2 = 0.$$

Since $[\beta, \gamma]$ and $[\gamma, \gamma]$ have different grading, we conclude that the parameter space is $\mathbb{C}[t_1, t_2, t_3]/I$, where I is generated by t_2t_3, t_3^2 and m^3.

Theorem 15. *A versal deformation of order two of the Lie algebra L_1 is parametrized by $\mathbb{C}[t_1, t_2, t_3]/I$ and is of the form*

$$\mu_{t_1,t_2,t_3} = \mu_0 + \overline{\alpha}t_1 + \overline{\beta}t_2 + \overline{\gamma}t_3 + \sum_{i,j=1}^{3} \varphi_{ij}t_it_j$$

where the coefficients φ_{ij} satisfy the identity

$$-2\sum_{i,j} d\varphi_{ij}t_it_j = \sum_{i,j}[\varphi_i, \varphi_j]t_it_j \mod I$$

(here $\varphi_1 = \overline{\alpha}$, $\varphi_2 = \overline{\beta}$ and $\varphi_3 = \overline{\gamma}$).

5. Examples of deformations

Let us now define three real deformations of the Lie algebra L_1 with the brackets

$$[e_i, e_j]_t^1 = (j - i)(e_{i+j} + te_{i+j-1});$$

$$[e_i, e_j]_t^2 = \begin{cases} (j - i)e_{i+j} & \text{if } i, j > 1, \\ (j - i)e_{i+j} + tje_j, & \text{if } i = 1; \end{cases}$$

$$[e_i, e_j]_t^3 = \begin{cases} (j - i)e_{i+j} & \text{if } i, j \neq 2 \\ (j - i)e_{i+j} + tje_j, & \text{if } i = 2. \end{cases}$$

These deformations have infinitesimal deformations of weight $-1, -1$ and -2. Denote the three Lie algebra families by $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$. They can be realized as families of subalgebras of L_0. In the first deformation e_i deforms into $e_i + te_{i-1}$, $i > 0$. In other words, $L_1^{(1)}$ consists of the vector fields on the line which vanish at 0 and t. In the second one the e_i-s, $i > 1$ remain and e_1 deforms into $e_1 + te_0$, while in the third one e_2 turns into $e_2 + te_0$ and the rest elements remain.

Theorem. *The Lie algebra families $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$ are nontrivial and pairwise nonisomorphic.*

Proof. The commutant $[L_1^{(1)}, L_1^{(1)}]$ consists of vector fields, vanishing at 0 and t together with their first derivative. From this it follows that $\operatorname{codim}[L_1^{(1)}, L_1^{(1)}] = 2$. At the same time $\operatorname{codim}[L_1^{(2)}, L_1^{(2)}] = \operatorname{codim}[L_1^{(3)}, L_1^{(3)}] = 1$ since $[L_1^{(2)}, L_1^{(2)}]$ is isomorphic to L_2 and $[L_1^{(3)}, L_1^{(3)}]$ is isomorphic to $\mathbb{C}e_1 \oplus e_3$. Finally $[L_1^{(2)}, L_1^{(2)}]$ is not isomorphic to $[L_1^{(3)}, L_1^{(3)}]$, because the first Lie algebra has three generators: e_2, e_3, e_4, while the second one has only two: e_1, e_3. So the three families are pairwise nonisomorphic and the last two ones are nontrivial. The nontriviality of the first family is obvious. □

Remark 1. *In the third family the infinitesimal deformation is $\overline{\alpha}$. More exactly, the cocycle $\alpha_3(e_2, e_j) = je_j$, $\alpha_3(e_i, e_j) = 0$ for $i \neq 2$ is cohomological to $\overline{\alpha}$, as $\overline{\alpha} - \alpha_3$ is the coboundary of the one-cochain $\kappa(e_3) = e_1$, $\kappa(e_i) = 0$ for $i \neq 3$.*

Remark 2. *The first two families have trivial infinitesimal deformations (the coefficient of t is a coboundary). If we change the bracket with adding a trivial cocycle, we can get an equivalent deformation with vanishing first term (see* [4]*). Let us investigate the coefficient of the t^2-term.*

For the first family the cocycle $\omega_1 : (e_i, e_j) \to (j - i)e_{i+j-1}$ is trivial, as there exists a κ_1 1-cochain for which $d\kappa_1 = \omega_1$: $\kappa_1(e_{2i+1}) = ie_{2i}$, $i \geqq 0$; $\kappa_1(e_{2i}) = \frac{2i-1}{2}e_{2i-1}$, $i \geqq 1$. With the transformation $\phi_t(k) = x + t\kappa_1(x)$ we get an equivalent deformation $[\widetilde{e_i, e_j}]_t^1 = \phi_t^{-1}([\phi_t(e_i), \phi_t(e_j)])$ without t-term, where the coefficient φ_2 of the t^2-term is a nontrivial cocycle. A straightforward calculation shows that $\overline{\alpha} - \frac{1}{12}\varphi_2$ is a trivial cocycle.

In the second family for the cocycle $\omega_2 : (e_1, e_j) \to je_j$, $(e_i, e_j) \to 0$ if $i \neq 1$ there exists a κ_2 1-cochain for which $d\kappa_2 = \omega_2 : \kappa_2(e_1) = 0$, $\kappa_2(e_{2i+1}) = (i+1)e_{2i}$, $i \geqq 1$; $\kappa_2(e_{2i}) = \frac{2i+1}{2}e_{2i+1}$, $i \geqq 1$. With the $x \to x + t\kappa_2(x)$ transformation we get an equivalent deformation without a t-term, where the coefficient φ_2' of the t^2-term is a nontrivial cocycle and $\overline{\alpha}$ is cohomological to $\frac{12}{13}\varphi_2'$.

References

[1] Feigin, B. L., Fuchs, D. B.: Homology of the Lie algebra of vector fields on the line. Funct. Anal. and Appl. Vol. 14 (1980), N3, 201–212.

[2] Fialowski, A.: Deformations of a Lie algebra of vector fields on the line. Russian Math. Surveys Vol. 38 (1983), N1, 201–202.

[3] Fuchs, D. B.: Cohomology of infinite dimensional Lie algebras. Moscow, Nauka, 1984 (in Russian). Contemporary Soviet Math. Consultants Bureau, New York, 1986.

[4] Gerstenhaber, M.: On the deformation of rings and algebras. Ann. of Math. 79 (1964), 59–104.

[5] Goncharova, L. V.: The cohomologies of Lie algebras of formal vector fields on the line. Funct. Anal. and Appl. Vol. 7 (1973), N2, 91–97.

[6] Nijenhuis, A., Richardson, R. W.: Deformations of Lie algebra structures. J. Math. Mech. 17 (1967), 89–105.

[7] Retakh, V. S.: Massey operations in the superalgebras and deformations of complexly analytic algebras. Funct. Anal. and Appl. Vol. 11 (1977), N4, 319–321.

[8] Schlessinger, M.: Functors of Artin Rings. Trans. Amer. Math. Soc. 130 (1968), 208–222.

APPEARED IN: *Journal of Functional Analysis*, **161** (1999), pp. 76–110

Construction of Miniversal Deformations of Lie Algebras

Alice Fialowski
Institute of Mathematics
Eötvös Loránd University, Budapest

Dmitry Fuchs
Department of Mathematics
University of California, Davis

Introduction

In this paper we consider deformations of finite or infinite dimensional Lie algebras over a field of characteristic 0. By "deformations of a Lie algebra" we mean the (affine algebraic) manifold of all Lie brackets. Consider the quotient of this variety by the action of the group GL. It is well-known (see [Hart]) that in the category of algebraic varieties the quotient by a group action does not always exist. Specifically, there is in general no universal deformation of a Lie algebra L with a commutative algebra base A with the property that for any other deformation of L with base B there exists a unique homomorphism $f\colon B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of L with base A *versal*.

Classical deformation theory of associative and Lie algebras began with the works of Gerstenhaber [G] and Nijenhuis-Richardson [NR] in the 1960's. They studied one-parameter deformations and established the connection between Lie algebra cohomology and infinitesimal deformations. They did not study the versal property of deformations.

A more general deformation theory for Lie algebras follows from Schlessinger's work [Sch]. If we consider deformations with base spec A, where A is a local algebra, this set-up is adequate to study the problem of "universality" among formal deformations. This was worked out for Lie algebras in [Fi1], [Fi3]; it turns out that in this case under some minor restrictions there exists a so-called *miniversal* element. The problem is to construct this element.

There is confusion in the literature when one tries to describe *all* the nonequivalent deformations of a given Lie algebra. There were several attempts to work out an appropriate theory for solving this basic problem in deformation theory, but none of them were completely adequate.

The construction below is parallel to the general constructions in deformation theory, as in [P], [I], [La], [GoM], [K]. The general theory, which can provide a construction of a local miniversal deformation, is outlined in [Fi1]. The procedure however needs a proper theory of Massey operations in the cohomology, and an algorithm for computing all the possible ways for a given infinitesimal deformation to extend to a formal deformation. The proper theory of Massey operations is developed in [FuL]. Our understanding of the construction arose from the study of the infinite dimensional Lie algebra L_1 of polynomial vector fields in $\mathbb{C}$ with trivial 1-jet at 0, in which case we completely described a miniversal deformation. In [FiFu] we proved that the base of the miniversal deformation of this Lie algebra is the union of three algebraic curves: two smooth curves and another curve with a cusp at 0, with the tangent lines to all three curves coinciding at 0.

The structure of the paper is as follows: In Section 1 we give the necessary definitions and some facts on infinitesimal deformations. In Section 2 we recall Harrison cohomology, in Section 3 discuss obstruction theory. Section 4 gives the theoretical construction of a miniversal deformation, and some preliminary computations. Section 5 recalls the proper Massey product definition and describes its properties (see

[FuL]). In Section 6 we calculate obstructions. Section 7 provides a scheme for computing the base of a miniversal deformation of a Lie algebra convenient for practical use. In Section 8 we apply the construction to the Lie algebra L_1.

Acknowledgments

We thank the Erwin Schrödinger Institute for Mathematical Physics, Vienna, where the work began, the first author thanks the Max-Planck-Institute für Mathematik, Bonn, for hospitality and partial support. Both authors thank Michael Penkava for reading the manuscript, correcting a number of misprints and making valuable suggestions.

1. Lie algebra deformations

1.1. Let L be a Lie algebra over a characteristic 0 field $\mathbb{K}$, and let A be a commutative algebra with identity over $\mathbb{K}$ with a fixed augmentation $\varepsilon\colon A \to \mathbb{K}$, $\varepsilon(1) = 1$; we set $\operatorname{Ker}\varepsilon = \mathfrak{m}$. To avoid transfinite induction, we will assume that $\dim(\mathfrak{m}^k/\mathfrak{m}^{k+1}) < \infty$ for all k.

Definition 1.1. A *deformation* λ of L with base $(A, \mathfrak{m})$, or simply with base A, is a Lie A-algebra structure on the tensor product $A \otimes_{\mathbb{K}} L$ with the bracket $[\,,\,]_\lambda$, such that

$$\varepsilon \otimes \mathrm{id}\colon A \otimes L \to \mathbb{K} \otimes L = L$$

is a Lie algebra homomorphism. (We usually abbreviate $\otimes_{\mathbb{K}}$ to $\otimes$.) See [Fi1], [Fi3].

Example 1.2. If $A = \mathbb{K}[t]$, then a deformation of L with base A is the same as an algebraic 1-parameter deformation of L. More generally, if A is the algebra of regular functions on an affine algebraic manifold X, then a deformation of L with base A is the same as an algebraic deformation of L with base X.

Two deformations of a Lie algebra L with the same base A are called *equivalent* (or isomorphic) if there exists a Lie algebra isomorphism between the two copies of $A \otimes L$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \mathrm{id}$. A deformation with base A is called *local* if the algebra A is local, and it is called *infinitesimal* if, in addition to this, $\mathfrak{m}^2 = 0$. Let A be a complete local algebra (*completeness* means that $A = \varprojlim_{n\to\infty} (A/\mathfrak{m}^n)$, where $\mathfrak{m}$ is the maximal ideal in A).

Definition 1.3. A *formal deformation* of L with base A is a Lie A-algebra structure on the completed tensor product $A \hat{\otimes} L = \varprojlim_{n\to\infty} ((A/\mathfrak{m}^n) \otimes L)$ such that

$$\varepsilon \hat{\otimes} \mathrm{id}\colon A \hat{\otimes} L \to \mathbb{K} \otimes L = L$$

is a Lie algebra homomorphism (see [Fi3]) .

The above notion of equivalence is extended to formal deformations in an obvious way.

Example 1.4. If $A = \mathbb{K}[[t]]$ then a formal deformation of L with base A is the same as a formal 1-parameter deformation of L. See [G], [NR].

Let A' be another commutative algebra with identity over $\mathbb{K}$ with a fixed augmentation $\varepsilon'\colon A' \to \mathbb{K}$, and let $\varphi\colon A \to A'$ be an algebra homomorphism with $\varphi(1) = 1$ and $\varepsilon' \circ \varphi = \varepsilon$.

Definition 1.5. If a deformation λ of L with base $(A, \mathfrak{m})$ is given, then the *push-out* $\varphi_*\lambda$ is the deformation of L with base $(A', \mathfrak{m}' = \operatorname{Ker}\varepsilon')$, which is the Lie algebra structure

$$[a_1' \otimes_A (a_1 \otimes l_1), a_2' \otimes_A (a_2 \otimes l_2)]' = a_1' a_2' \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2],$$

$$a_1', a_2' \in A',\ a_1, a_2 \in A,\ l_1, l_2 \in L,$$

on $A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L)$. Here A' is regarded as an A-module with the structure $a'a = a'\varphi(a)$, and the operation $[\,,\,]$ in the right hand side of the formula refers to the Lie algebra structure λ on $A \otimes L$.

The push-out of a formal deformation is defined in a similar way.

1.2. For completeness' sake, we recall the definition of Lie algebra cohomology (see [Fu]). We need only the case of cohomology with coefficients in the adjoint representation, and therefore we restrict our definition to this case.
Let

$$C^q(L;L) = \mathrm{Hom}\,(\Lambda^q L, L)$$

be the space of all skew-symmetric q-linear forms on a Lie algebra L with values in L. Define the differential

$$\delta\colon C^q(L;L) \to C^{q+1}(L;L)$$

by the formula

$$(\delta\gamma)(l_1,\dots,l_{q+1}) = \sum_{1\le s<t\le q+1} (-1)^{s+t-1}\gamma([l_s,l_t],l_1,\dots\hat{l}_s\dots\hat{l}_t\dots,l_{q+1}) + \sum_{1\le u\le q+1}(-1)^u[l_u,\gamma(l_1,\dots\hat{l}_u\dots,l_{q+1})],$$

where $\gamma \in C^q(L;L)$, $l_1,\dots,l_{q+1} \in L$. It may be checked that $\delta^2 = 0$, and the cohomology of the complex $\{C^q(L;L),\delta\}$ is denoted by $H^q(L;L)$.
For $\alpha \in C^p(L;L)$, $\beta \in C^q(L;L)$, the Lie product $[\alpha,\beta]$ is defined be the formula

$$[\alpha,\beta](l_1,\dots,l_{p+q-1}) = \sum_{1\le j_1<\dots<j_q\le p+q-1} (-1)^{\Sigma_s(j_s-s)}\alpha(\beta(l_{j_1},\dots,l_{j_q}),l_1,\dots\hat{l}_{j_1}\dots\hat{l}_{j_q}\dots,l_{p+q-1}) -$$
$$(-1)^{(p-1)(q-1)}\sum_{1\le k_1<\dots<k_p\le p+q-1}(-1)^{\Sigma_t(k_t-t)}\beta(\alpha(l_{k_1},\dots,l_{k_p}),l_1,\dots\hat{l}_{k_1}\dots\hat{l}_{k_p}\dots,l_{p+q-1}).$$

If one sets $\mathcal{C}^q = C^{q+1}(L;L)$, $\mathcal{H}^q = H^{q+1}(L;L)$, then this bracket operation (with the differential δ) makes $\mathcal{C} = \bigoplus \mathcal{C}^q$ a differential graded Lie algebra (DGLA), and makes $\mathcal{H} = \bigoplus \mathcal{H}^q$ a graded Lie algebra.

1.3. Here is the fundamental example of an infinitesimal deformation of a Lie algebra. Consider a Lie algebra L which satisfies the condition

$$\dim H^2(L;L) < \infty.$$

This is true, for example, if $\dim L < \infty$.
(There are some ways to weaken if not to completely avoid this condition. For example, if the Lie algebra L is $\mathbb{Z}$-graded, $L = \bigoplus_{q\in\mathbb{Z}} L_{(q)}$, $[L_{(p)},L_{(q)}] \subset L_{(p+q)}$, then $H^2(L;L)$ also becomes graded, $H^2(L;L) = \bigoplus_{q\in\mathbb{Z}} H^2_{(q)}(L;L)$, and the construction will be valid in a slightly modified form, if one supposes that $\dim H^2_{(q)}(L;L) < \infty$ for all q. See the details in 7.4 below.)
Consider the algebra

$$A = \mathbb{K} \oplus H^2(L;L)'$$

with the second summand being an ideal with zero multiplication ($'$ means the dual). Fix some homomorphism

$$\mu\colon H^2(L;L) \to C^2(L;L) = \mathrm{Hom}\,(\Lambda^2 L, L)$$

which takes a cohomology class into a cocycle representing this class. Define a Lie algebra structure on

$$A\otimes L = (\mathbb{K}\otimes L)\oplus(H^2(L;L)'\otimes L) = L \oplus \mathrm{Hom}\,(H^2(L;L),L)$$

by the formula

$$[(l_1,\varphi_1),(l_2,\varphi_2)] = ([l_1,l_2],\psi),$$

where

$$\psi(\alpha) = \mu(\alpha)(l_1,l_2) + [\varphi_1(\alpha),l_2] + [l_1,\varphi_2(\alpha)],$$
$$l_1,l_2\in L,\ \varphi_1,\varphi_2 \in \mathrm{Hom}\,(H^2(L;L),L)),\ \alpha\in H^2(L;L).$$

(The Jacobi identity for this operation is implied by $\delta\mu(\alpha) = 0$.) This determines a deformation of L with base A which is clearly infinitesimal.

Proposition 1.6. *Up to an isomorphism, this deformation does not depend on the choice of* μ.

Proof. Let

$$\mu' \colon H^2(L;L) \to C^2(L;L)$$

be another choice for μ. Then there exists a homomorphism

$$\gamma \colon H^2(L;L) \to C^1(L;L) = \mathrm{Hom}\,(L,L)$$

such that $\mu'(\alpha) = \mu(\alpha) + \delta\gamma(\alpha)$ for all $\alpha \in H^2(L;L)$. Define a linear automorphism ρ of the space $A \otimes L = L \oplus \mathrm{Hom}\,(H^2(L;L),L)$ by the formula

$$\rho(l,\varphi) = (l,\psi),\ \ \psi(\alpha) = \varphi(\alpha) + \gamma(\alpha)(l),$$

$$l \in L,\ \ \varphi \in \mathrm{Hom}\,(H^2(L;L),L),\ \ \alpha \in H^2(L;L).$$

The map ρ is clearly an automorphism. The inverse of ρ is given by replacing γ with $-\gamma$ in the formula. To prove that ρ is an isomorphism between the two Lie algebra structures, one needs to check that for any $l_1,l_2 \in L$, $\varphi_1,\varphi_2 \in \mathrm{Hom}\,(H^2(L;L),L)$, $\alpha \in H^2(L;L)$ one has

$$\begin{aligned} &\mu(\alpha)(l_1,l_2) + [\varphi_1(\alpha),l_2] + [l_1,\varphi_2(\alpha)] + \gamma(\alpha)([l_1,l_1]) \\ &\qquad = \mu'(\alpha)(l_1,l_2) + [\varphi_1(\alpha)+\gamma(\alpha)(l_1),l_2] + [l_1,\varphi_2(\alpha)+\gamma(\alpha)(l_2)]; \end{aligned}$$

but this follows directly from the equality $\mu'(\alpha) = \mu(\alpha) + \delta\gamma(\alpha)$.

We will denote the infinitesimal deformation of L constructed above by η_L.

1.4. The main property of η_L is its (co-)universality in the class of infinitesimal deformations.

Let λ be an infinitesimal deformation of the Lie algebra L with the finite dimensional base A. Take $\xi \in \mathfrak{m}'$, or, equivalently, $\xi \in A'$ and $\xi(1) = 0$. For $l_1,l_2 \in L$ set

$$\alpha_{\lambda,\xi}(l_1,l_2) = (\xi \otimes \mathrm{id})[1\otimes l_1, 1\otimes l_2]_\lambda \in \mathbb{K} \otimes L = L.$$

Lemma 1.7. *The cochain* $\alpha_{\lambda,\xi} \in C^2(L;L)$ *is a cocycle.*

Proof. Let $l_1,l_2,l_3 \in L$. Since $[1\otimes l_1, 1\otimes l_2]_\lambda - 1\otimes[l_1,l_2] \in \mathfrak{m}\otimes L$, we have

$$[1\otimes l_1, 1\otimes l_2]_\lambda = 1\otimes[l_1,l_2] + \sum_i m_i \otimes k_i,$$

where $m_i \in \mathfrak{m}, k_i \in L$. Hence

$$(\xi\otimes\mathrm{id})[1\otimes l_1, 1\otimes l_2, 1\otimes l_3]_\lambda = (\xi\otimes\mathrm{id})[1\otimes[l_1,l_2], 1\otimes l_3]_\lambda + (\xi\otimes\mathrm{id})\sum_i m_i[1\otimes k_i, 1\otimes l_3].$$

The first summand here is $\alpha_{\lambda,\xi}([l_1,l_2],l_3)$. For the second summand

$$m_i[1\otimes k_i, 1\otimes l_3] = m_i(1\otimes[k_i,l_3] + h),$$

where $h \in \mathfrak{m}\otimes L$. Since $\mathfrak{m}^2 = 0$ we have $m_i h = 0$. Hence $m_i[1\otimes k_i, 1\otimes l_3] = m_i \otimes [k_i,l_3]$, and

$$\begin{aligned} (\xi\otimes\mathrm{id})\sum_i m_i[1\otimes k_i, 1\otimes l_3] &= \sum_i (\xi\otimes\mathrm{id})(m_i\otimes[k_i,l_3]) = \sum_i \xi(m_i)[k_i,l_3] \\ &= \sum_i [\xi(m_i)k_i, l_3] = [(\xi\otimes\mathrm{id})\Big(\sum_i m_i\otimes k_i\Big), l_3] \\ &= [(\xi\otimes\mathrm{id})([1\otimes l_1, 1\otimes l_2]_\lambda - 1\otimes[l_1,l_2]), l_3] \\ &= [(\xi\otimes\mathrm{id})[1\otimes l_1, 1\otimes l_2], l_3] = [\alpha_{\lambda,\xi}(l_1,l_2), l_3]. \end{aligned}$$

In the last step above we used that $\xi(1) = 0$. Thus

$$(\xi\otimes\mathrm{id})[[1\otimes l_1, 1\otimes l_2]_\lambda, 1\otimes l_3]_\lambda = \alpha_{\lambda,\xi}([l_1,l_2],l_3) + [\alpha_{\lambda,\xi}(l_1,l_2), l_3],$$

and the Jacobi identity for $[\ ,\ ]_\lambda$ shows that $\alpha_{\lambda,\xi}$ is a cocycle.

Proposition 1.8. *For any infinitesimal deformation* λ *of the Lie algebra* L *with a finite-dimensional base* A *there exists a unique homomorphism* $\varphi \colon \mathbb{K}\oplus H^2(L;L)' \to A$ *such that* λ *is equivalent to the push-out* $\varphi_*\eta_L$.

PROOF. For $\xi \in \mathfrak{m}'$ let $a_{\lambda,\xi} \in H^2(L;L)$ be the cohomology class of the cocycle $\alpha_{\lambda,\xi}$. The correspondences $\xi \mapsto \alpha_{\lambda,\xi}, \xi \mapsto a_{\lambda,\xi}$ define homomorphisms

$$\alpha_\lambda \colon \mathfrak{m}' \to C^2(L;L), \qquad \delta \circ \alpha_\lambda = 0,$$

$$a_\lambda \colon \mathfrak{m}' \to H^2(L;L).$$

We claim that
(i) the deformation λ is fully determined by α_λ;
(ii) the deformations λ, λ' are equivalent if and only if $a_\lambda = a_{\lambda'}$;
(iii) if $\varphi = \mathrm{id} \oplus a'_\lambda \colon \mathbb{K} \oplus H^2(L;L)' \to \mathbb{K} \oplus \mathfrak{m} = L$, then $\varphi_* \eta_L$ is equivalent to λ.
Since (ii) and (iii) obviously imply Proposition, it remains to prove (i)–(iii). The statement (i) is obvious. To prove (ii) notice that an A-automorphism $\rho\colon A \otimes L \to A \otimes L$, that is

$$L \oplus (\mathfrak{m} \otimes L) \to L \oplus (\mathfrak{m} \otimes L),$$

whose $L \to L$ part is the identity (this is the condition of compatibility with $\varepsilon \otimes \mathrm{id}$), is fully determined by its

$$L \to \mathfrak{m} \otimes L$$

part, which we denote by b_ρ, and the latter may be chosen arbitrarily. This is an element of

$$\mathrm{Hom}\,(L, \mathfrak{m} \otimes L) = \mathfrak{m} \otimes \mathrm{Hom}\,(L, L) = \mathfrak{m} \otimes C^1(L;L) = \mathrm{Hom}\,(\mathfrak{m}', C^1(L;L)).$$

It is easy to check that ρ establishes an isomorphism between the Lie algebra structures λ and λ' if and only if

$$\alpha_{\lambda'} - \alpha_\lambda = \delta \circ b_\rho,$$

which proves (ii). Finally, it follows from the definitions that

$$\alpha_{\varphi_* \eta_L} = \mu \circ a_\lambda,$$

which implies that $a_{\varphi_* \eta_L} = a_\lambda$, and hence $\varphi_* \eta_L$ and λ are isomorphic as was stated in (iii).

REMARK 1.9. Technically, the mapping $a_\lambda \colon \mathfrak{m}' \to H^2(L;L)$ constructed in the proof will be more important for us than the map $\varphi = \mathrm{id} \oplus a'_\lambda$.

Let A be a local algebra with $\dim (A/\mathfrak{m}^2) < \infty$. Obviously, $A/\mathfrak{m}^2$ is local with the maximal ideal $\mathfrak{m}/\mathfrak{m}^2$, and $(\mathfrak{m}/\mathfrak{m}^2)^2 = 0$. Recall that the dual space $(\mathfrak{m}/\mathfrak{m}^2)'$ is called the *tangent space* of A; we denote it by TA.

DEFINITION 1.10. Let λ be a deformation of L with base A. Then the mapping

$$a_{\pi_* \lambda} \colon TA = (\mathfrak{m}/\mathfrak{m}^2)' \to H^2(L;L),$$

where π is the projection $A \to A/\mathfrak{m}^2$, is called the *differential* of λ and is denoted by $d\lambda$.
The differential of a formal deformation is defined in a similar way.

It is clear from the construction that equivalent deformations or formal deformations have equal differentials.

1.5. It is not possible to construct a local or formal deformation of a Lie algebra with a similar universality property in the class of local or formal deformations. But it becomes possible for an appropriate weakening of this property.

DEFINITION 1.11. A formal deformation η of a Lie algebra L with base B is called *miniversal* if
(i) for any formal deformation λ of L with any (local) base A there exists a homomorphism $f \colon B \to A$ such that the deformation λ is equivalent to $f_* \eta$;
(ii) in the notations of (i), if A satisfies the condition $\mathfrak{m}^2 = 0$, then f is unique (see [Fi1]).
If η satisfies only the condition (i), then it is called *versal*.

Our goal is to construct a miniversal formal deformation of a given Lie algebra.

2. Harrison cohomology.

2.1. We will need a special cohomology theory for commutative algebras introduced in 1961 by D. K. Harrison [Harr]. The following general definition is contained in the article [B].

Let A be a commutative $\mathbb{K}$-algebra. Consider the standard Hochschild complex for A: $\{C_q(A), \partial\}$. Here $C_q(A)$ is the A-module $A^{q+1} = A \otimes \cdots \otimes A$ ($q+1$ factors), A operates on the last factor, and the differential $\partial\colon C_q(A) \to C_{q-1}(A)$ is defined by the formula

$$\partial[a_1, \dots, a_q] = a_1[a_2, \dots, a_q] + \sum_{i=1}^{q-1}(-1)^i[a_1, \dots, a_i a_{i+1}, \dots, a_q] + (-1)^q a_q[a_1, \dots, a_{q-1}],$$

where $b_0[b_1, \dots, b_n]$ means $b_0 \otimes b_1 \otimes \cdots \otimes b_n \in C_n(A)$. A permutation from $S(q)$ is called a $(p, q-p)$-shuffle if the inverse permutation $(j_1, \dots, j_q)$ satisfies the conditions $j_1 < \cdots < j_p$, $j_{p+1} < \cdots < j_q$. Let $\mathrm{Sh}\,(p, q-p) \subset S(q)$ be the set of all $(p, q-p)$-shuffles. For $a_1, \dots, a_q \in A$ and $0 < p < q$ set

$$s_p(a_1, \dots, a_q) = \sum_{(i_1, \dots, i_q) \in \mathrm{Sh}(p, q-p)} \mathrm{sgn}(i_1, \dots, i_q)[a_{i_1}, \dots, a_{i_q}] \in C_q(A).$$

Let $\mathrm{Sh}_q(A)$ be the A-submodule of $C_q(A)$ generated by the chains $s_p(a_1, \dots, a_q)$ for all $a_1, \dots, a_q \in A$, $0 < p < q$. It may be checked (see [B], Proposition 2.2) that $\partial(\mathrm{Sh}_q(A)) \subset \mathrm{Sh}_{q-1}(A)$, which yields a complex

$$\mathrm{Ch}\,(A) = \{\mathrm{Ch}_q(A) = C_q(A)/\mathrm{Sh}_q(A), \partial\}.$$

This is the *Harrison complex.*

Definition 2.1. For an A-module M we set

$$H_q^{\mathrm{Harr}}(A; M) = H_q(\mathrm{Ch}\,(A) \otimes M),$$
$$H^q_{\mathrm{Harr}}(A; M) = H^q(\mathrm{Hom}\,(\mathrm{Ch}\,(A), M);$$

these are *Harrison homology and cohomology of A with coefficients in M.* (For the relations between Harrison and Hochschild homology and cohomology see [B].)

We will need the following standard fact, which follows directly from the definition.

Proposition 2.2. *Let A be a local commutative $\mathbb{K}$- algebra with the maximal ideal $\mathfrak{m}$, and let M be an A-module with $\mathfrak{m}M = 0$. Then we have the canonical isomorphisms*

$$H_q^{\mathrm{Harr}}(A; M) \cong H_q^{\mathrm{Harr}}(A; \mathbb{K}) \otimes M, \;\; H^q_{\mathrm{Harr}}(A; M) \cong H^q_{\mathrm{Harr}}(A; \mathbb{K}) \otimes M.$$

2.2. We will need only 1- and 2-dimensional Harrison cohomology. Here is their direct description (belonging to Harrison [Harr]). Let A and M be as above. Consider the complex

$$0 \to \mathrm{Ch}^1 \xrightarrow{d_1} \mathrm{Ch}^2 \xrightarrow{d_2} C^3,$$

where

$$\mathrm{Ch}^1 = \mathrm{Hom}\,(A, M), \;\; \mathrm{Ch}^2 = \mathrm{Hom}\,(S^2 A, M), \;\; C^3 = \mathrm{Hom}\,(A \otimes A \otimes A, M),$$

$$d_1\psi(a, b) = a\psi(b) - \psi(ab) + b\psi(a), \;\; d_2\varphi(a, b, c) = a\varphi(b, c) - \varphi(ab, c) + \varphi(a, bc) - c\varphi(a, b),$$
$$m \in M, \;\; a, b, c \in A, \;\; \psi \in \mathrm{Ch}^1, \;\; \varphi \in \mathrm{Ch}^2.$$

Proposition 2.3. (i) $H^1_{\mathrm{Harr}}(A; M)$ *is the space of derivations $A \to M$.* (ii) *Elements of $H^2_{\mathrm{Harr}}(A; M)$ correspond bijectively to isomorphism classes of extensions $0 \to M \to B \to A \to 0$ of the algebra A by means of M.*

Proof. Part (i) is obvious. To prove (ii), consider an extension $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$ and fix a section $q\colon A \to B$ of p. Then $b \mapsto (p(b), i^{-1}(b - q \circ p(b)))$ is an isomorphism $B \to A \oplus M$. Let $(a, m)_q \in B$ be the inverse image of $(a, m) \in A \oplus M$ with respect to this isomorphism. For $a_1, a_2 \in A$ set $\varphi_q(a_1, a_2) = i^{-1}((a_1, 0)_q(a_2, 0)_q - (a_1 a_2, 0)_q) \in M$. Then the multiplication in B is $(a_1, m_1)_q(a_2, m_2)_q = (a_1 a_2, a_1 m_2 + a_2 m_1 +$

$\varphi_q(a_1, a_2))_q$, so it is determined by φ_q. Furthermore, the associativity of the algebra B implies that $\varphi_q \in \mathrm{Ch}^2$ is a cocycle. For any other section $q'\colon A \to B$ one has $i^{-1} \circ (q' - q) \in \mathrm{Ch}^1$, and it is easy to check that $\varphi_{q'} = \varphi_q + d_1(i^{-1} \circ (q' - q))$. This implies (ii).

Corollary 2.4. *If A is a local algebra with the maximal ideal $\mathfrak{m}$, then* $H^1_{\mathrm{Harr}}(A; \mathbb{K}) = (\mathfrak{m}/\mathfrak{m}^2)' = TA$.

Proof Let $\varphi\colon A \to \mathbb{K}$ be a derivation. If $a \in \mathfrak{m}^2$, that is $a = a_1 a_2,\ a_1, a_2 \in \mathfrak{m}$, then $\varphi(a) = \varphi(a_1 a_2) = a_1\varphi(a_2) + a_2\varphi(a_1) = 0$ (since $\mathfrak{m}\mathbb{K} = 0$). Furthermore, $\varphi(1) = \varphi(1 \cdot 1) = 1\varphi(1) + 1\varphi(1) = 2\varphi(1)$, hence $\varphi(1) = 0$. On the other hand, any homomorphism $\varphi\colon A \to \mathbb{K}$ such that $\varphi(\mathfrak{m}^2) = 0, \varphi(1) = 0$, is a derivation. Thus the space of derivations $A \to \mathbb{K}$ is $(\mathfrak{m}/\mathfrak{m}^2)'$.

Proposition 2.5. *Let* $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$ *be an extension of an algebra A* . (i) *If A has an identity, then so does B.* (ii) *If A is local with the maximal ideal $\mathfrak{m}$, then B is local with the maximal ideal $p^{-1}(\mathfrak{m})$.*

Proof. (i) We use the notations of the previous proof. Fix a section $q\colon A \to B$ of p. Then we get a cocycle $\varphi = \varphi_q \in \mathrm{Ch}^2$. For any $a \in A$

$$d_2\varphi(1, 1, a) = \varphi(1, a) - \varphi(1, a) + \varphi(1, a) - a\varphi(1, 1) = 0,$$

which shows that $\varphi(1, a) = a\varphi(1, 1)$. Consider an arbitrary $\psi \in \mathrm{Ch}^1$ with $\psi(1) = \varphi(1, 1)$. Let $\varphi' = \varphi - d_1\psi$. Then for any $a \in A$

$$\begin{aligned}\varphi'(1, a) &= \varphi(1, a) - d_1\psi(1, a)\\ &= \varphi(1, a) - \psi(a) + \psi(a) - a\psi(1)\\ &= \varphi(1, a) - a\varphi(1, 1) = 0.\end{aligned}$$

According to the previous proof, $\varphi' = \varphi_{q'}$ for some section $q'\colon A \to B$, and one has

$$(1, 0)_{q'}(a, m)_{q'} = (a, m + \varphi_q(1, a))_{q'} = (a, m)_{q'}.$$

Hence, $(1, 0)_{q'} \in B$ is the unit element.
(ii) Let $\mathfrak{n} \subset B$ be an ideal, and let $\mathfrak{n} \not\subset p^{-1}(\mathfrak{m})$. Then there is some $b \in \mathfrak{n}$ such that $p(b) = 1$. Choose a section $q\colon A \to B$ with $q(1) = b$. Then $b = (1, 0)_q$. For any $(a, m)_q \in B$ one has

$$(a, m)_q = (1, 0)_q(a, m - \varphi_q(1, a)) \in \mathfrak{n},$$

and hence $\mathfrak{n} = B$.

2.3. The relationship between the second Harrison cohomology of a finite-dimensional local commutative algebra A and extensions of A may be also described in terms of one remarkable extension. This is the extension

$$0 \to H^2_{\mathrm{Harr}}(A; \mathbb{K})' \to C \to A \to 0, \tag{1}$$

where the operation of A on $H^2_{\mathrm{Harr}}(A; \mathbb{K})'$ is induced by the operation of A on $\mathbb{K}$, and the cocycle

$$f_A : S^2A \to H^2_{\mathrm{Harr}}(A; \mathbb{K})'$$

is defined as the dual of a homomorphism

$$\mu : H^2_{\mathrm{Harr}}(A; \mathbb{K}) \to \mathrm{Ch}^2(A; \mathbb{K}) = (S^2A)',$$

which takes a cohomology class to a cocycle from this class. This extension does not depend, up to an isomorphism, on the choice of μ (compare Proposition 1.6) and possesses the following partial (co-)universality property.

Proposition 2.6. *Let M be an A-module with $\mathfrak{m}M = 0$. Then the extension* (1) *admits a unique homomorphism into an arbitrary extension* $0 \to M \to B \to A \to 0$ *of A.*

Proof. The extension $0 \to M \to B \to A \to 0$ corresponds to some element of $H^2_{\mathrm{Harr}}(A;M) = H^2_{\mathrm{Harr}}(A;\mathbb{K}) \otimes M$ (see Proposition 2.2). The latter defines a mapping $H^2_{\mathrm{Harr}}(A;\mathbb{K})' \to M$, which implies, in turn, a mapping $C \to B$. The resulting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2_{\mathrm{Harr}}(A;\mathbb{K})' & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

is an extension homomorphism. Its uniqueness is obvious.

2.4. $H^1_{\mathrm{Harr}}(A;M)$ is also interpreted as the set of automorphisms of any given extension $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$ of A. An automorphism is an algebra automorphism $f\colon B \to B$ such that $f \circ i = i$ and $p \circ f = p$. In previous notations (see Proof of Proposition 2.3), $f(a,m)_q = (f_1(a,m), f_2(a,m))_q$. The condition $p \circ f = p$ means that $f_1(a,m) = a$. The condition $f \circ i = f$ means that $f_2(0,m) = m$, which implies that $f_2(a,m) = f_2(a,0) + f_2(0,m) = m + \psi(a)$ (where $\psi(a) = f_2(a,0)$). The multiplicativity of f implies successively

$$f((a_1,0)_q(a_2,0)_q) = f(a_1,0)_q f(a_2,0)_q,$$
$$f(a_1a_2, \varphi_q(a_1,a_2))_q = (a_1,\psi(a_1))_q(a_2,\psi(a_2))_q,$$
$$(a_1a_2, \varphi_q(a_1,a_2) + \psi(a_1a_2))_q = (a_1a_2, \varphi_q(a_1,a_2) + a_1\psi(a_2) + a_2\psi(a_1))_q,$$
$$\psi(a_1a_2) = a_1\psi(a_2) + a_2\psi(a_1),$$

that is $d_1\psi = 0$. Conversely, any $\psi\colon A \to M$ with $d_1\psi = 0$ determines an algebra automorphism $f\colon B \to B, (a,m)_q \mapsto (a, m + \psi(a))_q$ with the required properties. Notice that $f(1,0)_q = (1,\psi(1))_q = (1,0)_q$, because $\psi(1) = 0$ for any derivation ψ. Hence f takes the unit element of B into the unit element of B (cf. Proposition 2.4).

2.5. In Section 4 we will use the following result due to Harrison.

Proposition 2.7. ([Harr], Theorems 11 and 18). *Let $A = \mathbb{K}[x_1,\dots,x_n]$ be a polynomial algebra, and let $\mathfrak{m}$ be the ideal of polynomials without constant terms. If an ideal I of A is contained in $\mathfrak{m}^2$, then*

$$H^2_{\mathrm{Harr}}(A/I;\mathbb{K}) \cong (I/(\mathfrak{m}\cdot I))'.$$

Harrison's work contains also an explicit construction of the above homomorphism, which implies the following description of the canonical extension

$$0 \to H^2_{\mathrm{Harr}}(B;\mathbb{K})' \to C \to B \to 0$$

of $B = A/I$ (see 2.3).

Proposition 2.8. *If A, $\mathfrak{m}$, and I are as in Proposition 2.7, then the preceding extension for $B = A/I$ is*

$$0 \to I/(\mathfrak{m}\cdot I) \xrightarrow{i} A/(\mathfrak{m}\cdot I) \xrightarrow{p} A/I \to 0,$$

where i and p are induced by the inclusions $I \to A$ and $\mathfrak{m}\cdot I \to I$.

3. Obstructions to extending deformations

3.1. Let λ be a deformation of a Lie algebra L with a finite-dimensional local base A, and let $0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$ be a 1-dimensional extension of A, corresponding to a cohomology class $f \in H^2_{\mathrm{Harr}}(A;\mathbb{K})$.

Let $I = i \otimes \mathrm{id}\colon L = \mathbb{K} \otimes L \to B \otimes L$ and $P = p \otimes \mathrm{id}\colon B \otimes L \to A \otimes L$. Let also $E = \hat{\varepsilon} \otimes \mathrm{id}\colon B \otimes L \to \mathbb{K} \otimes L = L$, where $\hat{\varepsilon}$ is the augmentation of B. The Lie algebra structure $[\,,\,]_\lambda$ in $A \otimes L$ can be lifted to a B-bilinear operation $\{\,,\,\}\colon \Lambda^2 B \to B$ such that

(i) $P\{l_1,l_2\} = [P(l_1), P(l_2)]_\lambda$ for any $l_1, l_2 \in B \otimes L$,
(ii) $\{I(l), l_1\} = I[l, E(l_1)]$ for any $l \in L, l_1 \in B \otimes L$.

The operation $\{\,,\,\}$ partially satisfies the Jacobi identity, that is

$$\varphi(l_1,l_2,l_3) := \{l_1,\{l_2,l_3\}\} + \{l_2,\{l_3,l_1\}\} + \{l_3,\{l_1,l_2\}\} \in \operatorname{Ker} P.$$

Remark that φ is multilinear and skew-symmetric, and $\varphi(l_1, l_2, l_3) = 0$ if $l_1 \in \operatorname{Ker} E$. (Indeed, if $l_1 = ml'_1$, where $m \in \hat{\mathfrak{m}} = \operatorname{Ker} \hat{\varepsilon}$, then $\varphi(l_1, l_2, l_3) = \varphi(ml'_1, l_2, l_3) = m\varphi(l'_1, l_2, l_3) = 0$. Hence φ determines a multilinear form

$$\bar{\varphi}\colon \Lambda^3 L = \Lambda^3((B \otimes L)/\operatorname{Ker} E) \to \operatorname{Ker} P = L,$$

that is an element $\bar{\varphi}$ of $C^3(L; L)$. It is easy to check that $\delta\bar{\varphi} = 0$.
Let $\{\ ,\ \}'$ be another B-bilinear operation $\Lambda^2 B \to B$ satisfying the conditions (i), (ii) above. Then $\{l_1, l_2\}' - \{l_1, l_2\} \in \operatorname{Ker} P$ for any $l_1, l_2 \in B \otimes L$, and if $l_1 \in \operatorname{Ker} E$ then $\{l_1, l_2\}' - \{l_1, l_2\} = 0$ (as above, if $l_1 = ml'_1, m \in \hat{\mathfrak{m}}$, then $\{l_1, l_2\}' - \{l_1, l_2\} = \{ml'_1, l_2\}' - \{ml'_1, l_2\} = m(\{l'_1, l_2\}' - \{l_1, l_2\}) = 0$. Hence the difference $\{\ ,\ \}' - \{\ ,\ \}$ determines a form $\psi\colon \Lambda^2 L = \Lambda^2((B \otimes L)/\operatorname{Ker} E) \to \operatorname{Ker} P = L$, that is determines a cochain $\psi \in C^2(L; L)$. Moreover, an arbitrary cochain $\psi \in C^2(L; L)$ may be obtained as $\{\ ,\ \}' - \{\ ,\ \}$ with an appropriate $\{\ ,\ \}'$.
Using the cocycle f_A, it is easy to check that if $\bar{\varphi}, \bar{\varphi}' \in C^3(L; L)$ are the cochains corresponding to $\{\ ,\ \}, \{\ ,\ \}'$ in the sense of the construction above, then

$$\bar{\varphi}' - \bar{\varphi} = \delta\psi.$$

Let $\mathcal{O}_\lambda(f) \in H^3(L; L)$ be the cohomology class of the cochain $\bar{\varphi}$. It is obvious that

$$\mathcal{O}_\lambda\colon H^2_{\mathrm{Harr}}(A; \mathbb{K}) \to H^3(L; L),\ f \mapsto \mathcal{O}_\lambda(f)$$

is a linear map.
We can summarize the argumentation above in the following

Proposition 3.1. *The deformation λ with base A can be extended to a deformation of L with base B if and only if $\mathcal{O}_\lambda(f) = 0$.*

The cohomology class $\mathcal{O}_\lambda(f)$ is called the *obstruction* to the extension of the deformation λ from A to B.

3.2. Suppose now that $\mathcal{O}_\lambda(f) = 0$, that is the deformation λ is extendible to a deformation with base B. We are going to study the set of all possible extensions.
Let μ, μ' be deformations of L with base B such that $p_*\mu = p_*\mu' = \lambda$. Then, according to 3.1, the difference $[\ ,\ ]_{\mu'} - [\ ,\ ]_\mu$ determines and is determined by a certain cochain $\psi \in C^2(L; L)$. Since $[\ ,\ ]_{\mu'}$ and $[\ ,\ ]_\mu$ both satisfy the Jacobi identity, $\delta\psi = 0$. Moreover, it is easy to check that if we replace any of the structures $[\ ,\ ]_\mu, [\ ,\ ]_{\mu'}$ with an equivalent one (see 1.1), then the cocycle ψ will be replaced by a cohomologous cocycle. Thus the difference between two isomorphism classes of deformations μ of L with base B such that $p_*\mu = \lambda$ is an arbitrary element of $H^2(L; L)$. In other words, $H^2(L; L)$ operates transitively on the set of these equivalence classes.
On the other hand, the group of automorphisms of the extension

$$0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$$

also operates on the set of equivalence classes of deformations μ. According to 2.5, this group is $H^1_{\mathrm{Harr}}(A; \mathbb{K})$, and according to Corollary 2.4, $H^1_{\mathrm{Harr}}(A; \mathbb{K}) = (\mathfrak{m}/\mathfrak{m}^2)' = TA$.

Proposition 3.2. *These two operations are related to each other by the differential $d\lambda\colon TA \to H^2(L; L)$* (see Definition 1.10). *In other words, if $r\colon B \to B$ determines an automorphism of the extension $0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$ which corresponds to an element $h \in H^1_{\mathrm{Harr}}(A; \mathbb{K}) = TA$, then for any deformation μ of L with base B such that $p_*\mu = \lambda$, the difference between $[\ ,\ ]_{r_*\mu}$ and $[\ ,\ ]_\mu$ is a cocycle of the cohomology class $d\lambda(h)$.*

Proof is obvious.

Corollary 3.3. *Suppose that the differential $d\lambda\colon TA \to H^2(L; L)$ is onto. Then the group of automorphisms of the extensions $0 \to \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \to 0$ operates transitively on the set of equivalence classes of deformations μ of L with base B such that $p_*\mu = \lambda$. In other words, μ is unique up to an isomorphism and an automorphism of the extension $0 \to \mathbb{K} \to B \to A \to 0$.*

3.3. The results of 3.1 and 3.2 may be generalized from the case of extension $0 \to \mathbb{K} \to B \to A \to 0$ to a more general case of extensions $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$, where M is a finite-dimensional A-module satisfying the condition $\mathfrak{m}M = 0$. The construction of 3.1 applied to a deformation λ of L with base A yields an element of $H^3(L; M \otimes L) = M \otimes H^3(L; L)$.

The same element may be obtained from the previous construction in a more direct way. Let $h \in M'$. We set $B_h = (B \oplus \mathbb{K})/\mathrm{Im}\,(i \oplus h)$ (that is $B_h = B/i(\mathrm{Ker}\, h)$ if $h \neq 0$, and $B_0 = A \oplus \mathbb{K}$). There is an obvious extension $0 \to \mathbb{K} \to B_h \to A \to 0$; let $f_h \in H^2_{\mathrm{Harr}}(A; \mathbb{K})$ be the corresponding cohomology class. The formula $h \mapsto \mathcal{O}_\lambda(f_h)$ defines an element of $\mathrm{Hom}\,(M', H^3(L; L)) = M \otimes H^3(L; L)$ which coincides with the obstruction constructed above.

Proposition 3.4. *A deformation μ of L with base B such that $p_*\mu = \lambda$ exists if and only if the element of $M \otimes H^3(L; L)$ constructed above is equal to* 0. *If $d\lambda\colon TA \to H^2(L; L)$ is onto then the deformation μ, if it exists, is unique up to an isomorphism and an automorphism of the extension $0 \to M \to B \to A \to 0$.*

Proof is as above (see 3.1).

4. Construction of a miniversal deformation

4.1. Suppose that $\dim H^2(L; L) < \infty$.

Let $C_0 = \mathbb{K}$, $C_1 = \mathbb{K} \oplus H^2(L; L)'$, and let

$$0 \to H^2(L; L)' \xrightarrow{i_1} C_1 \xrightarrow{p_1'} \mathbb{K} \to 0$$

be the canonical splitting extension. The deformation η_L of L with base C_1 constructed in 1.3 will be denoted here by η_1. Suppose that for some $k \geq 1$ we have already constructed a finite-dimensional commutative algebra C_k and a deformation η_k of L with base C_k. Consider the extension

$$0 \to H^2_{\mathrm{Harr}}(C_k; \mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}'_{k+1}} C_k \to 0 \tag{2}$$

constructed in 2.3 using the cocycle f_{C_k} (the notation was different there). According to 3.3, we obtain the obstruction

$$\mathcal{O}_{\eta_k}(f_{C_k}) \in H^2_{\mathrm{Harr}}(C_k, \mathbb{K})' \otimes H^3(L; L)$$

to the extension of η_k. This gives us a map

$$\omega_k\colon H^2_{\mathrm{Harr}}(C_k, \mathbb{K}) \to H^3(L; L).$$

Set

$$C_{k+1} = \bar{C}_{k+1}/\bar{i}_{k+1} \circ \omega_k'(H^3(L; L)').$$

Obviously, the extension (2) factorizes to an extension

$$0 \to (\mathrm{Ker}\, \omega_k)' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p'_{k+1}} C_k \to 0. \tag{3}$$

Notice that all the algebras C_k are local. Since C_k is finite-dimensional, the cohomology $H^2_{\mathrm{Harr}}(C_k; \mathbb{K})$ is also finite-dimensional, and hence C_{k+1} is finite-dimensional.

Proposition 4.1. *The deformation η_k admits an extension to a deformation with base C_{k+1}, and this extension is unique up to an isomorphism and an automorphism of an extension* (3).

Proof. According to Proposition 3.4, the obstruction to the extension of the deformation η_k of L from C_k to C_{k+1} is a homomorphism $\mathrm{Ker}\, \omega_k \to H^3(L; L)$, and it is easy to show that it is precisely the restriction of ω_k. Hence it is equal to 0. The uniqueness of the extension is stated explicitly in Proposition 3.4.

We choose an extended deformation and denote it by η_{k+1}.

The induction yields a sequence of finite-dimensional algebras

$$\mathbb{K} \xleftarrow{p_1'} C_1 \xleftarrow{p_2'} \dots \xleftarrow{p_k'} C_k \xleftarrow{p'_{k+1}} C_{k+1} \xleftarrow{p'_{k+2}} \dots,$$

and a sequence of deformations η_k of L such that $(p'_{k+1})_*\eta_{k+1} = \eta_k$.

Taking the projective limit, we obtain a formal deformation η of L with base $C = \varprojlim_{k\to\infty} C_k$. In Theorem 4.5 below we will show that η is a miniversal deformation of L.

4.2. Denote the space $H^2(L;L)$ briefly by $\mathbf{H}$. Below we assume that $\dim \mathbf{H} < \infty$. Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{K}[[\mathbf{H}']]$.

PROPOSITION 4.2. *$C_k = \mathbb{K}[[\mathbf{H}']]/I_k$ where*

$$\mathfrak{m}^2 = I_1 \supset I_2 \supset \dots,\ I_k \supset \mathfrak{m}^{k+1}.$$

PROOF. By construction,

$$C_1 = \mathbb{K} \oplus \mathbf{H}' = \mathbb{K}[[\mathbf{H}']]/\mathfrak{m}^2.$$

Suppose that we already know that

$$C_k = \mathbb{K}[[\mathbf{H}']]/I_k,\ \mathfrak{m}^2 \supset I_k \supset \mathfrak{m}^{k+1}.$$

Then, according to Proposition 2.8,

$$\bar{C}_{k+1} = \mathbb{K}[[\mathbf{H}']]/(\mathfrak{m} \cdot I_k),$$

and by construction C_{k+1} is the quotient of $\bar{C}_{k+1}$ over an ideal contained in $I_k/(\mathfrak{m} \cdot I_k) \subset \mathfrak{m}^2/(\mathfrak{m} \cdot I_k)$. Hence

$$C_{k+1} = \mathbb{K}[[\mathbf{H}']]/I_{k+1},\ \text{where } \mathfrak{m}^2 \supset I_{k+1} \supset \mathfrak{m} \cdot I_k \supset \mathfrak{m}^{k+2}.$$

This completes the proof.

COROLLARY 4.3. *For $k \geq 2$ the projection $p_k' \colon C_k \to C_{k-1}$ implies an isomorphism $TC_k \to TC_{k-1}$. In particular, the space TC_k does not depend on k; $TC_k = TC_1 = H^2(L;L)$. More precisely, for any $k \geq 1$ the differential $d\eta_k \colon TC_k \to H^2(L;L)$ is an isomorphism.*

PROPOSITION 4.4. *$C = \mathbb{K}[[\mathbf{H}']]/I$, where I is an ideal contained in $\mathfrak{m}^2$. Note that since $\mathbb{K}[[\mathbf{H}']]$ is Noetherian, then I is finitely generated.*

PROOF. By construction, $C = \overleftarrow{\lim\limits_{k\to\infty}} C_k$ (see 4.1). Proposition 4.2 gives an epimorphism

$$\overleftarrow{\lim_{k\to\infty}} \mathbb{K}[[\mathbf{H}']]/\mathfrak{m}^{k+1} \to \overleftarrow{\lim_{k\to\infty}} C_k,$$

that is

$$\mathbb{K}[[\mathbf{H}']] \to C,$$

and

$$C = \mathbb{K}[[\mathbf{H}']]/I,\ \text{where } I = \cap I_k = \overleftarrow{\lim} I_k.$$

4.3 THEOREM 4.5. *If $\dim H^2(L;L) < \infty$, then the formal deformation η is a miniversal formal deformation of L.*

PROOF. Since $TC_k = H^2(L;L)$ and $d\eta_k = \mathrm{id}$, then $TC = H^2(L;L)$ and $d\eta = \mathrm{id}$. Let A be a complete local algebra with the maximal ideal $\mathfrak{m}$, and let λ be a deformation of L with base A. We put $A_0 = A/\mathfrak{m} = \mathbb{K}$ and $A_1 = A/\mathfrak{m}^2 = \mathbb{K} \oplus (TA)'$. Then we fix a sequence of 1-dimensional extensions

$$0 \to \mathbb{K} \xrightarrow{j_{k+1}} A_{k+1} \xrightarrow{q_{k+1}} A_k \to 0,\ k \geq 1$$

such that $A = \overleftarrow{\lim\limits_{k\to\infty}} A_k$. Let $Q_k \colon A \to A_k$ be the projection; we suppose that Q_1 is the natural projection $A \to A/\mathfrak{m}^2$. Let $\lambda_k = (Q_k)_*\lambda$; it is a deformation of L with base A_k. Obviously, $\lambda_k = (q_{k+1})_*\lambda_{k+1}$. We will construct inductively homomorphisms $\varphi_j \colon C_j \to A_j$, $j = 1, 2, \dots$ compatible with the projections $C_{j+1} \to C_j$, $A_{j+1} \to A_j$ and such that $(\varphi_j)_*\eta_j = \lambda_j$.

Define $\varphi_1 \colon C_1 \to A_1$ as $\mathrm{id} \oplus (d\lambda)' \colon \mathbb{K} \oplus H^2(L;L)' \to \mathbb{K} \oplus (TA)'$; by definition of the differential, $(\varphi_1)_*\eta_1 = \lambda_1$. Suppose that $\varphi_k \colon C_k \to A_k$ with $(\varphi_k)_*\eta_k = \lambda_k$ has been already constructed. The homomorphism $\varphi_k^* \colon H^2_{\mathrm{Harr}}(A_k;\mathbb{K}) \to H^2_{\mathrm{Harr}}(C_k;\mathbb{K})$ induced by φ_k takes the class of extension $0 \to \mathbb{K} \to A_{k+1} \to A_k \to 0$ into the class of some extension $0 \to \mathbb{K} \to B \to C_k \to 0$, and we have a homomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \psi} & & \big\downarrow{\scriptstyle \varphi_k} & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & A_{k+1} & \longrightarrow & A_k & \longrightarrow & 0 \end{array}$$

Obviously, there exists a deformation ξ of L with base B which extends η_k (because the deformations λ_k and η_k have the same obstruction to extension) and such that $\psi_*\xi = \lambda_{k+1}$ (extensions of λ_k and η_k are both parameterized by $H^2(L;L)$).
According to Proposition 2.6, there exists a homomorphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2_{\mathrm{Harr}}(C_k;\mathbb{K})' & \xrightarrow{\bar{i}_{k+1}} & \bar{C}_{k+1} & \xrightarrow{\bar{p}'_{k+1}} & C_k & \longrightarrow & 0 \\ & & \downarrow r & & \downarrow \bar{\chi} & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0 \end{array}$$

and since the deformation η_k is extended to B, it follows that the composition

$$r \circ \omega'_k \colon H^3(L;L)' \to \mathbb{K}$$

is zero. Hence the last diagram may be factorized to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\operatorname{Ker}\omega_k)' & \longrightarrow & C_{k+1} & \longrightarrow & C_k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \chi & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & \mathbb{K} & \longrightarrow & B & \longrightarrow & C_k & \longrightarrow & 0 \end{array}$$

Since $d\eta_k \colon TC_k \to H^2(L;L)$ is an epimorphism (see 1.4.1), the two deformations $\chi_*\eta_{k+1}$ and ξ are related by some automorphism $f \colon B \to B$ of the extension $0 \to \mathbb{K} \to B \to C_k \to 0$. It remains to set $\varphi_{k+1} = \psi \circ f \circ \chi \colon C_{k+1} \to A_{k+1}$; indeed, $(\varphi_{k+1})_*\eta_{k+1} = \psi_* \circ f_* \circ \chi_*\eta_{k+1} = \psi_*\xi = \lambda_{k+1}$.
The limit map $\varphi \colon C \to A$ obviously satisfies the condition $\varphi_*\eta = \lambda$. The uniqueness property (ii) in Definition 1.11 follows from the uniqueness in Proposition 1.8.

4.4 Theorem 4.6. *If $\dim H^2(L;L) < \infty$, then the base of the miniversal formal deformation of L is formally embedded in $H^2(L;L)$, that is, it may be described in $H^2(L;L)$ by a finite system of formal equations.*

Proof. Follows directly from Proposition 4.4.

To make the computation of C more specific, we need an appropriate theory of Massey products.

5. Massey products

5.1. The obstructions

$$\omega_k \colon H^2_{\mathrm{Harr}}(C_k;\mathbb{K}) \to H^3(L;L)$$

which arise in the construction of the miniversal formal deformation of the Lie algebra L (see 4.1) may be described in terms of Massey products in $H^*(L;L)$. The appropriate theory of Massey products was developed by the second author and Lang [FuL]. We briefly recall this theory.

Definition 5.1. A differential graded Lie algebra (DGLA) is a vector space $\mathcal{C}$ over $\mathbb{K}$ with $\mathbb{Z}$ or $\mathbb{Z}_2$ grading $\mathcal{C} = \bigoplus_i \mathcal{C}^i$ and with commutator operation $\mu \colon L \otimes L \to L$, $\mu(\alpha \otimes \beta) = [\alpha, \beta]$ of degree 0 and a differential $\beta \colon \mathcal{C} \to \mathcal{C}$ of degree +1 satisfying the conditions

$$[\alpha,\beta] = -(-1)^{\alpha\beta}[\beta,\alpha],$$
$$\delta[\alpha,\beta] = [\delta\alpha,\beta] + (-1)^{\alpha}[\alpha,\delta\beta],$$
$$[[\alpha,\beta],\gamma] + (-1)^{\alpha(\beta+\gamma)}[[\beta,\gamma],\alpha] + (-1)^{\gamma(\alpha+\beta)}[[\gamma,\alpha],\beta] = 0,$$

where the degree of a homogeneous element is denoted by the same letter as this element.

Our main example of DGLA was introduced in 1.2: $\mathcal{C}^i = C^{i+1}(L;L)$.
The cohomology of $\mathcal{C}$ with respect to δ is denoted as $\mathcal{H} = \bigoplus_i \mathcal{H}^i$. It is a graded Lie algebra.

5.2. The construction of Massey products in $\mathcal{H}$ given below requires the following data. First, a graded cocommutative coassociative coalgebra, that is a $\mathbb{Z}$ or $\mathbb{Z}_2$ graded vector space F over $\mathbb{K}$ with a degree 0 mapping $\Delta \colon F \to F \otimes F$ (comultiplication) satisfying the conditions $S \circ \Delta = \Delta$, where $S \colon F \otimes F \to F \otimes F$ is defined as $S(\varphi \otimes \psi) =$

$(-1)^{\varphi\psi}(\psi \otimes \varphi)$, and $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$. Second, a filtration $F_0 \subset F_1 \subset F$ such that $F_0 \subset \operatorname{Ker} \Delta$ and $\operatorname{Im} \Delta \subset F_1 \otimes F_1$.

Proposition 5.2 (see [FuL], Proposition 3.1). *Suppose a linear mapping $\alpha\colon F_1 \to \mathcal{C}$ of degree 1 satisfies the condition*

$$\delta\alpha = \mu \circ (\alpha \otimes \alpha) \circ \Delta. \tag{4}$$

Then

$$\mu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset \operatorname{Ker} \delta.$$

(The right-hand side of the last formula is well defined because $\Delta(F)$ is contained in $F_1 \otimes F_1$, the domain of $\alpha \otimes \alpha$).

Definition 5.3. Let $a\colon F_0 \to \mathcal{H}$ and $b\colon F/F_1 \to \mathcal{H}$ be linear maps of degrees 1 and 2. We say that b is contained in the Massey F-product of a, and write $b \in \langle a \rangle_F$, or $b \in \langle a \rangle$, if there exists a degree 1 linear mapping $\alpha\colon F_1 \to \mathcal{C}$ satisfying condition (4), and such that the diagrams

$$\begin{array}{ccc} F_0 & \xrightarrow{\alpha|_{F_0}} & \operatorname{Ker}\delta \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} \\ F_0 & \xrightarrow{a} & \mathcal{H}, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\mu\circ(\alpha\otimes\alpha)\circ\Delta} & \operatorname{Ker}\delta \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ F/F_1 & \xrightarrow{b} & \mathcal{H} \end{array}$$

are commutative, where the vertical maps labeled by π denote the projections of each space onto the quotient space.

Note that the upper horizontal maps of the diagrams are well defined, since $\alpha(F_0) \subset \alpha(\operatorname{Ker} \Delta) \subset \operatorname{Ker} \delta$ by virtue of (4), and $\mu \circ (\alpha \otimes \alpha) \circ \Delta(F) \subset \operatorname{Ker} \delta$ by Proposition 5.2. Note also that the definition makes sense even in the case, when $F_1 = F$. In this case we do not need to specify any b, and we will simply say that *a satisfies the condition of triviality of Massey F-products.*

Example 5.4. Let F be the dual of the maximal ideal of $\mathbb{K}[t]/(t^{n+1})$, F_0 and F_1 be the duals of maximal ideals of $\mathbb{K}[t]/(t^2)$ and $\mathbb{K}[t]/(t^n)$. Then F_0 and F/F_1 are 1-dimensional and are generated respectively by t and t^n. In this case $a\colon F_0 \to \mathcal{H}$ and $b\colon F/F_1 \to \mathcal{H}$ are characterized by $a(t) \in \mathcal{H}$ and $b(t^n) \in \mathcal{H}$, and it is easy to check that $b \in \langle a \rangle_F$ if and only if $b(t^n)$ belongs to the n-th Massey power of $a(t)$ in the classical sense. In particular, for $n = 2$, $b \in \langle a \rangle_F$ if and only if $b(t^2) = [a(t), a(t)]$.

5.3. The relationship between Massey products and Lie algebra deformations was established in the article [FuL] by the following result.

Let A be a finite-dimensional local algebra with the maximal ideal $\mathfrak{m}$. Put $F = F_1 = \mathfrak{m}'$ and $F_0 = TA = (\mathfrak{m}/\mathfrak{m}^2)'$.

Proposition 5.5 ([FuL], Theorem 4.2). *A linear map $a\colon F_0 \to H^2(L; L)$ is a differential of some deformation with base A if and only if $-\frac{1}{2}a$ satisfies the condition of triviality of Massey F-products.*

A similar result holds for formal deformations.

6. Calculating obstructions

6.1. Adopt the notations of 4.1. Consider the sequence

$$\mathbb{K} \xrightarrow{p_1} C_1 \xrightarrow{p_2} \dots \xrightarrow{p_k} C_k \xrightarrow{\bar{p}_{k+1}} \bar{C}_{k+1}.$$

Recall that all $C_i, \bar{C}_i$ are finite-dimensional algebras,

$$C_1 = \mathbb{K} \oplus H^2(L; L)',$$

and there is an extension

$$0 \to H^2_{\mathrm{Harr}}(C_k; \mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}'_{k+1}} C_k \to 0$$

and an obstruction homomorphism

$$\omega_k\colon H^2_{\mathrm{Harr}}(C_k; \mathbb{K}) \to H^3(L; L).$$

Recall also that

$$C_{k+1} = \bar{C}_{k+1}/\mathrm{Im}\,(\bar{i}_{k+1} \circ \omega'_k).$$

Let $\mathfrak{m}_i, \bar{\mathfrak{m}}_i$ be the maximal ideals in $C_i, \bar{C}_i$. Then we also have the sequence

$$\mathfrak{m}_1 \xrightarrow{p_2} \mathfrak{m}_2 \xrightarrow{p_3} \dots \xrightarrow{p_k} \mathfrak{m}_k \xrightarrow{\bar{p}_{k+1}} \bar{\mathfrak{m}}_{k+1}.$$

Consider the dual sequence

$$\mathfrak{m}'_1 \xleftarrow{p'_2} \mathfrak{m}'_2 \xleftarrow{p'_3} \dots \xleftarrow{p'_k} \mathfrak{m}'_k \xleftarrow{\bar{p}'_{k+1}} \bar{\mathfrak{m}}'_{k+1}.$$

This is a sequence of successively embedded cocommutative coassociative coalgebras. Put $\bar{\mathfrak{m}}'_{k+1} = F$, $\mathfrak{m}'_1 = F_0$, $\mathfrak{m}'_k = F_1$. Then

$$F_0 = H^2(L; L),\; F/F_1 = H^2_{\mathrm{Harr}}(C_k; \mathbb{K}).$$

We choose the grading in F to be trivial ($\deg \varphi = 0$ for any $\varphi \in F$).

6.2. THEOREM 6.1. $2\omega_k \in \langle \mathrm{id} \rangle_F$ *(this inclusion refers to the Massey product in the sense of Definition* 5.3 *in the cohomology* $\mathcal{H} = \bigoplus_i \mathcal{H}^i$, $\mathcal{H}^i = H^{i+1}(L; L)$, *of the DGLA* $\mathcal{C} = \bigoplus_i \mathcal{C}^i$, $\mathcal{C}^i = C^{i+1}(L; L)$*). Moreover, an arbitrary element of* $\langle \mathrm{id} \rangle_F$ *is equal to* $2\omega_k$ *for an appropriate extension of the deformation* $\eta_1 = \eta_L$ *of* L *with base* C_1 *to a deformation* η_k *of* L *with base* C_k*.*

PROOF. The Lie C_k-algebra structure η_k on $C_k \otimes L$ is determined by the commutators $[l_1, l_2]_{\eta_k} \in C_k \otimes L$ of elements of $L = 1 \otimes L \subset C_k \otimes L$. The difference $[\,,\,]_{\eta_k} - [\,,\,]$ is a linear map $\beta \colon \Lambda^2 L \to \mathfrak{m}_k \otimes L$. This map may be regarded as a map $\mathfrak{m}'_k = F_1 \to \mathrm{Hom}\,(\Lambda^2 L, L) = C^2(L; L)$; we take the last map for α (see Definition 5.3). Obviously, $\alpha | F_0$ represents $a = \mathrm{id} \colon F_0 \to H^2(L; L)$, and the Jacobi identity for $[\,,\,]_{\eta_k}$ means precisely that α satisfies condition (4). Moreover, it is clear, that different α's with these properties correspond precisely to different extensions η_k of η_1.

By definition, a map $b \colon F/F_1 \to H^3(L; L)$ from $\langle a \rangle_F$ is represented by $\mu \circ (\alpha \otimes \alpha) \circ \Delta \colon F \to C^3(L; L)$. On the other hand, the obstruction map $\omega_k \colon H^2_{\mathrm{Harr}}(C_k; \mathbb{K}) = \bar{\mathfrak{m}}'_{k+1}/\mathfrak{m}'_k = F/F_1 \to H^3(L; L)$ is defined by means of lifting the commutator $[\,,\,]_{\eta_k}$ to a skew-symmetric $\bar{C}_{k+1}$-bilinear operation $\{\,,\,\}$ (satisfying some additional conditions – see 3.1). Choose a basis $m_1, \dots, m_s$ in $\mathfrak{m}_k$, and extend it to a basis $\bar{m}_1, \dots, \bar{m}_s$, $\bar{m}_{s+1}, \dots, \bar{m}_{s+t}$ of $\bar{\mathfrak{m}}_{k+1}$. (We will also consider the dual bases $\{m'_i\}$ and $\{\bar{m}'_i\}$ in $\mathfrak{m}'_k$ and $\bar{\mathfrak{m}}'_{k+1}$.) Then

$$[l_1, l_2]_{\eta_k} = [l_1, l_2] + \sum_{i=1}^{s} m_i \otimes [l_1, l_2]_i,$$

and the map α acts by the formula

$$\alpha(m'_i)(l_1, l_2) = [l_1, l_2]_i,\; i = 1, \dots, , s.$$

We define $\{\,,\,\}$ by the formula

$$\{l_1, l_2\} = [l_1, l_2] + \sum_{i=1}^{s} \bar{m}_i \otimes [l_1, l_2]_i.$$

Let the multiplication in $\bar{\mathfrak{m}}_{k+1}$ be

$$\bar{m}_i \bar{m}_j = \sum_{p=1}^{s+t} c^p_{ij} \bar{m}_p;$$

then $\Delta \colon \bar{\mathfrak{m}}'_{k+1} \to \mathfrak{m}_k \otimes \mathfrak{m}_k$ acts by the formula

$$\Delta(\bar{m}'_p) = \sum_{i,j=1}^{s} c^p_{ij} m'_i \otimes m'_j.$$

We have

$$\{\{l_1, l_2\}, l_3\} = [l_1, l_2], l_3] + \dots + \sum_{i,j=1}^{s} \sum_{p=1}^{s+t} c^p_{ij} \bar{m}_p \otimes [[l_1, l_2]_i, l_3]_j,$$

where "..." denotes the part corresponding to $\bar{m}_1, \dots, \bar{m}_s$. Thus the functional $\bar{m}'_p \in \bar{\mathfrak{m}}'_{k+1}$ takes

$$\{\{l_1, l_2\}, l_3\} + \{\{l_2, l_3\}, l_1\} + \{\{l_3, l_1\}, l_2\}$$

into

$$\sum_{i,j=1}^{s} c^p_{ij} \left[\alpha(m'_j)(\alpha(m'_i)(l_1, l_2), l_3) + \alpha(m'_j)(\alpha(m'_i)(l_2, l_3), l_1) + \alpha(m'_j)(\alpha(m'_i)(l_3, l_1), l_2)\right]$$

$$= \frac{1}{2}\mu \circ (\alpha \otimes \alpha) \circ \Delta(\bar{m}'_p),$$

which shows that $\omega_k = \frac{1}{2}b$. Theorem 6.1 follows.

7. Further computations

7.1. The goal of this Section is to provide a scheme of computation of the base of a miniversal deformation of a Lie algebra, convenient for practical use. We begin with the detailed description of the first two steps of this inductive computation.

As in Section 4, we denote $H^2(L; L)$ by $\mathbf{H}$, and also denote by $\mathfrak{m}$ the maximal ideal of the polynomial algebra $\mathbb{K}[\mathbf{H}']$. As before, we assume that $\dim \mathbf{H} < \infty$. Also we adopt the notations $C_k, \bar{C}_k, \mathfrak{m}_k, \bar{\mathfrak{m}}_k$ of 4.1 and 6.1, and to avoid confusion, we denote the map $\alpha\colon \mathfrak{m}'_k \to C^2(L; L)$ of 6.1 by α_k.

According to 4.1,

$$C_1 = \mathbb{K} \oplus \mathbf{H}' = \mathbb{K}[\mathbf{H}']/\mathfrak{m}^2,$$

and hence

$$\mathfrak{m}_1 = \mathfrak{m}/\mathfrak{m}^2 = \mathbf{H}', \ \mathfrak{m}'_1 = \mathbf{H}.$$

According to 4.2,

$$\bar{C}_2 = \mathbb{K}[\mathbf{H}']/\mathfrak{m}^3,$$

and hence

$$\bar{\mathfrak{m}}_2 = \mathfrak{m}/\mathfrak{m}^3, \ \bar{\mathfrak{m}}'_2 = \mathbf{H} \oplus S^2\mathbf{H}.$$

The map

$$\alpha_1 \colon \mathfrak{m}'_1 = \mathbf{H} \to C^2(L; L)$$

takes a cohomology class into a representing cocycle. Hence the map

$$\mu \circ (\alpha_1 \otimes \alpha_1) \circ \Delta \colon \bar{\mathfrak{m}}'_2 \to C^3(L; L), \tag{5}$$

where $\Delta\colon \bar{\mathfrak{m}}'_2 \to \mathfrak{m}'_1 \otimes \mathfrak{m}'_1$ is the comultiplication, acts as zero on $\mathbf{H}$ (because $\Delta|\mathbf{H} = 0$) and takes $\xi\eta \in S^2\mathbf{H}$ (where $\xi, \eta \in \mathbf{H}$) into the product of the chosen cocycles $\alpha_1(\xi), \alpha_1(\eta)$ representing ξ, η. Obviously (and according to Proposition 5.2), the image of the map (5) belongs to $\operatorname{Ker}\delta$, and the composition of this map with the projection $\pi\colon \operatorname{Ker}\delta \to H^3(L; L)$ acts as zero on $\mathbf{H}$ and coincides with the multiplication $[\,,\,]\colon S^2\mathbf{H} \to H^3(L; L)$ on $S^2\mathbf{H}$. Hence

$$\mathfrak{m}'_2 = \mathbf{H} \oplus \operatorname{Ker}([\,,\,]\colon S^2\mathbf{H} \to H^3(L; L)),$$

$$\mathfrak{m}_2 = \frac{\mathfrak{m}}{\mathfrak{m}^3 + J_2}, \text{ where } J_2 = \operatorname{Im}([\,,\,]')$$

$$C_2 = \frac{\mathbb{K}[\mathbf{H}']}{\mathfrak{m}^3 + J_2}.$$

Note that if $\dim H^3(L; L) = q$, then J_2 is an ideal in $\mathbb{K}[\mathbf{H}']$ generated by (at most) q quadratic polynomials.

Furthermore, according to 4.2,

$$\bar{C}_3 = \frac{\mathbb{K}[\mathbf{H}']}{\mathfrak{m}^4 + (\mathfrak{m} \cdot J_2)},$$

and hence

$$\bar{\mathfrak{m}}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (\mathfrak{m} \cdot J_2)}, \ \bar{\mathfrak{m}}'_3 = \mathbf{H} \oplus S^2\mathbf{H} \oplus K,$$

where $K \subset S^3\mathbf{H}$ is the intersection of kernels of the maps

$$f_\varphi\colon S^3\mathbf{H} \to H^3(L;L),\ \varphi \in \mathbf{H}',$$
$$f_\varphi(\xi\eta\zeta) = \varphi(\xi)[\eta,\zeta] + \varphi(\eta)[\zeta,\xi] + \varphi(\zeta)[\xi,\eta].$$

The map

$$\alpha_2\colon \mathfrak{m}_2' = \mathbf{H} \oplus \operatorname{Ker}[\,,\,] \to C^2(L;L)$$

coincides with α_1 on $\mathbf{H}$ and takes $\sum \xi_i\eta_i \in \operatorname{Ker}[\,,\,]$ $(\xi_i, \eta_i \in \mathbf{H})$ into a two-dimensional cochain whose coboundary is $\sum[\alpha_1(\xi_i), \alpha_1(\eta_i)]$. Hence the composition

$$\mu \circ (\alpha_2 \otimes \alpha_2) \circ \Delta\colon \bar{\mathfrak{m}}_3' \to C^3(L;L), \tag{6}$$

where $\Delta\colon \bar{\mathfrak{m}}_3' \to \mathfrak{m}_2' \otimes \mathfrak{m}_2'$ is the comultiplication, coincides with the map (5) on $\mathbf{H} \oplus S^2\mathbf{H}$ and takes $\sum \xi_i\eta_i\zeta_i$ into

$$[\alpha_1(\xi_i), \alpha_2(\eta_i,\zeta_i)] + [\alpha_1(\eta_i), \alpha_2(\zeta_i,\xi_i)] + [\alpha_1(\zeta_i), \alpha_2(\xi_i,\eta_i)].$$

According to Proposition 2.8, the latter is a cocycle, and the composition of the map (6) and the projection $\pi\colon \operatorname{Ker}\delta \to H^3(L;L)$ acts as zero on $\mathbf{H}$, as $[\,,\,]$ on $S^2\mathbf{H}$, and as the "triple Massey product" on K. The kernel of this composition is $\mathfrak{m}_3'$, and $\mathfrak{m}_3$ is the dual of this kernel. Thus, by construction,

$$\mathfrak{m}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + J_3},\ C_3 = \frac{\mathbb{K}[\mathbf{H}']}{\mathfrak{m}^4 + J_3},$$

where $J_3 \cap S^2\mathbf{H}' = J_2 \cap S^2\mathbf{H}'$.

7.2. Describe now the k-th induction step. Suppose that we have already constructed

$$C_k = \frac{\mathbb{K}[\mathbf{H}']}{\mathfrak{m}^{k+1} + J_k},\ \mathfrak{m}_k = \frac{\mathfrak{m}}{\mathfrak{m}^{k+1} + J_k},\ \alpha_k : \mathfrak{m}_k' \to C^2(L;L).$$

Then, according to 4.2,

$$\bar{C}_{k+1} = \frac{\mathbb{K}[\mathbf{H}']}{\mathfrak{m}^{k+2} + (\mathfrak{m}\cdot J_k)},\ \bar{\mathfrak{m}}_{k+1} = \frac{\mathfrak{m}}{\mathfrak{m}^{k+2} + (\mathfrak{m}\cdot J_k)},$$
$$\bar{\mathfrak{m}}_{k+1}' \subset \mathbf{H} \oplus S^2\mathbf{H} \oplus \cdots \oplus S^{k+1}\mathbf{H}.$$

The image of the composition

$$\mu \circ (\alpha_k \otimes \alpha_k) \circ \Delta\colon \bar{\mathfrak{m}}_{k+1}' \to C^3(L;L), \tag{7}$$

where $\Delta\colon \bar{\mathfrak{m}}_{k+1}' \to \mathfrak{m}_k' \otimes \mathfrak{m}_k'$ is the comultiplication, is contained in $\operatorname{Ker}\delta$ (Proposition 5.2), and the composition

$$\pi \circ \mu \circ (\alpha_k \otimes \alpha_k) \circ \Delta\colon \bar{\mathfrak{m}}_{k+1}' \to H^3(L;L)$$

acts as zero on $\mathfrak{m}_k'$. We put

$$\mathfrak{m}_{k+1}' = \operatorname{Ker}(\pi \circ \mu \circ (\alpha_k \otimes \alpha_k) \circ \Delta) \supset \mathfrak{m}_k'.$$

The map $\alpha_k\colon \mathfrak{m}_k' \to C^2(L;L)$ is extended to the map $\alpha_{k+1}\colon \mathfrak{m}_{k+1}' \to C^2(L;L)$ such that $\delta \circ \alpha_{k+1}$ is the restriction of the map (7). The dual to $\mathfrak{m}_{k+1}'$ is

$$\mathfrak{m}_{k+1} = \frac{\mathfrak{m}}{\mathfrak{m}^{k+2} + J_{k+1}},$$

and we put

$$C_{k+1} = \mathbb{K} \oplus \mathfrak{m}_{k+1} = \frac{\mathbb{K}[\mathbf{H}']}{\mathfrak{m}^{k+2} + J_{k+1}}.$$

This completes the construction.

7.3. Two following useful observations are easily derived from the description of the construction given in 7.1 – 7.2.

Proposition 7.1. *For $l \le k$,*

$$J_{k+1} \cap S^l\mathbf{H}' = J_k \cap S^l\mathbf{H}'.$$

Proof. We use induction with respect to k. For $k = 2$ this was proved in 7.1. Suppose that $J_k \cap S^{l-1}\mathbf{H}' = J_{k-1} \cap S^{l-1}\mathbf{H}'$. Then $(\mathfrak{m}\cdot J_k) \cap S^l\mathbf{H}' = (\mathfrak{m}\cdot J_{k+1}) \cap S^l\mathbf{H}'$. Hence $\bar{\mathfrak{m}}_{k+1}'$ and $\bar{\mathfrak{m}}_k'$ have the same $S^l\mathbf{H}'$ component. Since Δ has degree 0 with respect to

$\mathbf{H}'$, and α_k coincides with α_{k-1} on $\mathfrak{m}'_{k-1}$, we may conclude that $\mathfrak{m}'_{k+1}$ and $\mathfrak{m}'_k$ also have the same $S^l\mathbf{H}'$ component. Proposition 7.1 follows.

PROPOSITION 7.2. *If* $\dim H^3(L;L) = q$, *then the ideal* $I = \overleftarrow{\lim}\, I_k = \overleftarrow{\lim}\, J_k$ *from Proposition* 4.4 *has at most q generators. Less formally, the base of the miniversal deformation of L is the zero locus of a formal map* $H^2(L;L) \to H^3(L;L)$.

PROOF. By construction, $\mathfrak{m}_k = \bar{\mathfrak{m}}_k/G_k$, where G_k is generated by the image of a certain map $\beta_k\colon H^3(L;L)' \to \bar{\mathfrak{m}}_k$ (namely, $\beta_k = (\pi\circ\mu\circ(\alpha_{k-1}\otimes\alpha_{k-1})\circ\Delta)'$, see 7.2). Actually, $\bar{\mathfrak{m}}_k$ is a quotient of $\bar{\mathfrak{m}}_{k+1}$, and β_{k+1} is a lift of β_k. Put

$$\bar{\mathfrak{m}}_\infty = \overleftarrow{\lim_{k\to\infty}}\, \bar{\mathfrak{m}}_k,\ \mathfrak{m}_\infty = \overleftarrow{\lim_{k\to\infty}}\, \mathfrak{m}_k.$$

Then $\mathfrak{m}_\infty = \bar{\mathfrak{m}}_\infty/G_\infty$, where G_∞ is generated by the image of

$$\beta_\infty = \overleftarrow{\lim_{k\to\infty}}\, \beta_k\colon H^3(L;L)' \to \bar{\mathfrak{m}}_\infty.$$

Furthermore,

$$\mathfrak{m}_\infty = \mathfrak{m}/I,\ \bar{\mathfrak{m}}_\infty = \mathfrak{m}/(\mathfrak{m}\cdot I),$$

where $I = \overleftarrow{\lim}\, I_k$. Hence

$$G_\infty = I/(\mathfrak{m}\cdot I),$$

and it is clear that generators of G_∞ are lifted to generators of I. Since G_∞ is generated by (at most) q generators, Proposition 7.2 follows.

7.4. We conclude this section with a brief discussion of the graded case. Suppose that the Lie algebra L is G-graded, where G is an Abelian group: $L = \bigoplus_{g\in G} L_q$, $[L_g, L_h] \subset L_{g+h}$. In this case the cochains $\mathcal{C}$ and the cohomology $\mathcal{H}$ get an additional grading: $C^q(L;L) = \bigoplus_{g\in G} C^q_{(g)}(L;L)$ ($\varphi \in C^q_{(g)}(L;L)$, if $\varphi(l_1,\dots,l_q) \in L_{g_1+\dots+g_q-g}$ for $l_1 \in L_{g_1},\dots,l_q \in L_{g_q}$), and $H^q(L;L) = \bigoplus_{g\in G} H^q_{(g)}(L;L)$. The condition $\dim H^2(L;L) < \infty$ may be replaced in this case by a weaker condition: $\dim H^2_{(g)}(L;L) < \infty$ for each g. We preserve the notation $\mathbf{H}$ for $H^2(L;L)$, but $\mathbf{H}'$ will denote $\bigoplus_{g\in G} H^2_{(g)}(L;L)'$. All the spaces $\mathbf{H}, \mathbf{H}', C_k, \mathfrak{m}_k, \bar{\mathfrak{m}}_k, \mathfrak{m}'_k, \bar{\mathfrak{m}}'_k$ have natural G-gradings, and all the maps α_k have degree 0. The whole construction is modified correspondingly. We restrict ourselves to the modified version of Proposition 7.2.

PROPOSITION 7.3. *The ideal I from Proposition* 4.4 *is always generated by homogeneous elements. Moreover, if* $\dim H^3_{(g)}(L;L) = q_g$, *then I has at most* q_g *generators of degree g. Less formally, the base of the miniversal deformation of L is the zero locus of a formal map* $H^2(L;L) \to H^3(L;L)$ *of degree* 0.

8. EXAMPLE: DEFORMATIONS OF THE LIE ALGEBRA $\mathbf{L_1}$

8.1. Let L_1 be the complex Lie algebra of polynomial vector fields $p(x)\dfrac{d}{dx}$ on the line such that $p(0) = p'(0) = 0$. The deformations of this Lie algebra were studied by the first author ([Fi2], [Fi3]), and its formal miniversal deformation was completely described in our joint paper [FiFu]. It turned out that geometrically the base of this deformation is the union of three algebraic curves with a common point: two non-singular, having a common tangent, and one with a cusp, where the tangent at the cusp coincides with the tangent to the smooth components.
Below we show how these results can be obtained by the methods of this article. We will need some (surprisingly little) information about the cohomology and deformations of the Lie algebra L_1. All this information is contained in the articles [FeFu], [Fi2], [Fi3], [FiFu].

8.2. As a complex vector space, the Lie algebra L_1 has the basis $\{e_i | i \geq 1\}$, $e_i = x^{i+1}\dfrac{d}{dx}$, and the commutator operation is $[e_i, e_j] = (j-i)e_{i+j}$. This Lie algebra is $\mathbb{Z}$-graded, $\deg e_i = i$.

PROPOSITION 8.1 ([FeFu], [Fi2]). *The dimensions of* $H^2(L_1; L_1)$ *and* $H^3(L_1; L_1)$ *are equal to* 3 *and* 5. *Moreover,*

$$\dim H^2_{(q)}(L_1; L_1) = \begin{cases} 1 & \text{if } 2 \le q \le 4, \\ 0 & \textit{otherwise}; \end{cases}$$

$$\dim H^3_{(q)}(L_1; L_1) = \begin{cases} 1 & \text{if } 7 \le q \le 11, \\ 0 & \textit{otherwise} \end{cases}$$

PROPOSITION 8.2 ([Fi3]). *Let* $0 \ne \alpha \in H^2_{(2)}(L_1; L_1), 0 \ne \beta \in H^2_{(3)}(L_1; L_1), 0 \ne \gamma \in H^2_{(4)}(L_1; L_1)$. *Then* $0 \ne [\beta, \gamma] \in H^3_{(7)}(L_1; L_1), 0 \ne [\gamma, \gamma] \in H^3_{(8)}(L_1; L_1)$. *Furthermore,* $0 \ne \langle \beta, \beta, \beta \rangle \in H^3_{(9)}(L_1; L_1)$.

The latter means that if $b \in C^2_{(3)}(L_1; L_1)$ is a representative of β, and if $[b, b] = \delta g$, $g \in C^2_{(6)}(L_1; L_1)$, then the cohomology class of the cocycle $[b, g] \in C^3_{(9)}(L_1; L_1)$ (which does not depend on the choice of b and g) is not equal to 0.

8.3. Here are some explicit constructions of deformations of the Lie algebra L_1.

PROPOSITION 8.3 ([Fi2]). *The formulas*

$$[e_i, e_j]^1_t = (j - i)(e_{i+j} + te_{i+j-1});$$

$$[e_i, e_j]^2_t = \begin{cases} (j - i)e_{i+j} & \text{if } i \ne 1, j \ne 1, \\ (j - 1)e_{j+1} + tje_j & \text{if } i = 1, j \ne 1; \end{cases}$$

$$[e_i, e_j]^3_t = \begin{cases} (j - i)e_{i+j} & \text{if } i \ne 2, j \ne 2, \\ (j - 2)e_{j+2} + tje_j & \text{if } i = 2, j \ne 2 \end{cases}$$

determine three one-parameter deformations of the Lie algebra L_1. *All the three deformations are pairwise not equivalent. Moreover, if* L_1^1, L_1^2, L_1^3 *are Lie algebras from the three families corresponding to arbitrary non-zero values of the parameter (up to an isomorphism, they do not depend on the non-zero parameter value), then neither two of* L_1^1, L_1^2, L_1^3 *are isomorphic to each other.*

COROLLARY 8.4. *The base of any versal deformation of the Lie algebra* L_1 *contains at least three different irreducible curves.*

8.4. We will use the notations of Section 7. Let α, β, γ be a basis of $\mathbf{H} = H^2(L_1; L_1)$ (as in Proposition 8.2), and let x, y, z be the dual basis in $\mathbf{H}'$. The algebra $S^*\mathbf{H}' = \mathbb{C}[x, y, z]$ has the monomial basis $\{x^p y^q z^r\}$. Let $\{\alpha^p \beta^q \gamma^r\}$ be the dual basis in the coalgebra $S^*\mathbf{H}$; the comultiplication $\Delta\colon S^*\mathbf{H} \to S^*\mathbf{H} \otimes S^*\mathbf{H}$ acts by the formula

$$\Delta(\alpha^p \beta^q \gamma^r) = \sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} \alpha^i \beta^j \gamma^k \otimes \alpha^{p-i} \beta^{q-j} \gamma^{r-k}.$$

Choose cocycles $a \in C^2_{(2)}(L_1; L_1),\ b \in C^2_{(3)}(L_1; L_1),\ c \in C^2_{(4)}(L_1; L_1)$ representing α, β, γ. Then

$$\alpha_1\colon \mathfrak{m}'_1 = \mathbf{H} \to C^2(L_1; L_1)$$

is defined by the formulas

$$\alpha_1(\alpha) = a,\ \alpha_1(\beta) = b,\ \alpha_1(\gamma) = c.$$

Since $H^3_{(q)}(L_1; L_1) = 0$ for $q < 7$, there exist $d \in C^2_{(4)}(L_1; L_1), e \in C^2_{(5)}(L_1; L_1), f \in C^2_{(6)}(L_1; L_1), g \in C^2_{(6)}(L_1; L_1)$, such that $[a, a] = \delta d, [a, b] = \delta e, [a, c] = \delta f, [b, b] = \delta g$ (the notation g has been already used in 8.2). Since $c \in C^2_{(4)}(L_1; L_1)$ is a cocycle, we can replace d with $d + tc$, where t is an arbitrary complex number. Finally, since $\delta[a, d] = 0$, we also have $[a, d] = \delta h$ for some $h \in C^2_{(6)}(L_1; L_1)$.

The space $\bar{\mathfrak{m}}'_2 = \mathbf{H} \oplus S^2\mathbf{H}$ is spanned by $\alpha, \beta, \gamma, \alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \beta\gamma, \gamma^2$. The map $\mu \circ (\alpha_1 \otimes \alpha_1) \circ \Delta\colon \bar{\mathfrak{m}}'_2 \to C^3(L_1; L_1)$ acts in the following way:

$$\alpha, \beta, \gamma \mapsto 0;\ \alpha^2 \mapsto \delta d,\ \alpha\beta \mapsto 2\delta e,\ \alpha\gamma \mapsto 2\delta f,\ \beta^2 \mapsto \delta g,$$
$$\beta\gamma \mapsto 2[b, c] \notin \operatorname{Im}\delta, \gamma^2 \mapsto [c, c] \notin \operatorname{Im}\delta.$$

Hence $\mathfrak{m}_2'$ is generated by $\alpha, \beta, \gamma, \alpha^2, \alpha\beta, \alpha\gamma, \beta^2$, and

$$\alpha_2 \colon \mathfrak{m}_2' \to C^2(L_1; L_1)$$

is defined as α_1 on $\mathbf{H}$ and

$$\alpha_2(\alpha^2) = d + tc, \;\; \alpha_2(\alpha\beta) = 2e, \;\; \alpha_2(\alpha\gamma) = 2f, \;\; \alpha_2(\beta^2) = g.$$

Furthermore,

$$\mathfrak{m}_2 = \frac{\mathfrak{m}}{\mathfrak{m}^3 + (yz, z^2)},$$

$$\bar{\mathfrak{m}}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (\mathfrak{m} \cdot (yz, z^2))} = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (xyz, xz^2, y^2z, yz^2, z^3)},$$

and

$$\bar{\mathfrak{m}}_3' = \mathbf{H} \oplus S^2\mathbf{H} \oplus K,$$

where K is the subspace of $S^3\mathbf{H}$ spanned by $\alpha^3, \alpha^2\beta, \alpha^2\gamma, \alpha\beta^2, \beta^3$. The map $\mu \circ (\alpha_2 \otimes \alpha_2) \circ \Delta \colon \bar{\mathfrak{m}}_3' \to C^3(L_1; L_1)$ acts as $\mu \circ (\alpha_1 \otimes \alpha_1) \circ \Delta$ on $\mathbf{H} \oplus S^2\mathbf{H}$ (see above), and acts on K in the following way:

$$\begin{aligned}
\alpha^3 &\mapsto 2[a, d + tc] = \delta(h + tf), \\
\alpha^2\beta &\mapsto 4[a, e] + 2[b, d] + 2t[b, c], \\
\alpha^2\gamma &\mapsto 4[a, f] + 2[c, d] + 2t[c, c], \\
\alpha\beta^2 &\mapsto 2[a, g] + 4[b, e], \\
\beta^3 &\mapsto 2[b, g] \notin \operatorname{Im}\delta.
\end{aligned}$$

Since $4[a, e] + 2[b, d] \in \operatorname{Ker}\delta$, $[b, c] \notin \operatorname{Im}\delta$, and $\dim H^3_{(7)}(L_1; L_1) = 1$, we can choose t in such a way that the image of $\alpha^2\beta$ is cohomologous to 0,

$$\alpha^2\beta \mapsto \delta k, \;\; k \in C^2_{(7)}(L_1; L_1).$$

Since $4[a, f]+2[c, d]+2t[c, c]$, $2[a, g]+4[b, e] \in \operatorname{Ker}\delta$, $\dim H^3_{(8)}(L_1; L_1) = 1$, and $[c, c] \notin \operatorname{Im}\delta$, there exist complex numbers A, B such that the images of $\alpha^2\gamma - A\gamma^2$, $\alpha\beta^2 - B\gamma^2$ are cohomologous to 0,

$$\begin{aligned}
\alpha^2\gamma - A\gamma^2 &\mapsto \delta l, \;\; l \in C^2_{(8)}(L_1; L_1), \\
\alpha\beta^2 - B\gamma^2 &\mapsto \delta m, \;\; m \in C^2_{(8)}(L_1; L_1).
\end{aligned}$$

Hence $\mathfrak{m}_3'$ is generated by the generators of $\mathfrak{m}_2'$ (see above) and also $\alpha^3, \alpha^2\beta, \alpha^2\gamma - A\gamma^2$, $\alpha\beta^2 - B\gamma^2$. Thus

$$\mathfrak{m}_3 = \frac{\mathfrak{m}}{\mathfrak{m}^4 + (yz, z^2 + Ax^2z + Bxy^2, y^3)}.$$

To complete this description of the base of the miniversal deformation of L_1, we need to continue the induction to calculate $\mathfrak{m}_4$ and $\mathfrak{m}_5$. This would require more information about the multiplications in the cohomology of L_1. It turns out, however, that we can avoid any additional computations if we use Corollary 8.4.

8.5. According to Propositions 7.3 and 8.1, the base of the miniversal deformation of L_1 is $\mathbb{C}[[x, y, z]]/(F_1, F_2, F_3, F_4, F_5)$, where $F_1, \ldots, F_5$ are polynomials in x, y, z of degrees $7, \ldots, 11$ (with $\deg x = 2, \deg y = 3, \deg z = 4$). The calculations of 8.4 show that

$$\begin{aligned}
F_1 &= yz + \ldots, \\
F_2 &= z^2 + Ax^2z + Bxy^2 + \ldots, \\
F_3 &= y^3 + \ldots,
\end{aligned}$$

where "...", and F_4, F_5 as well, are linear combinations of 4- and 5-fold products of x, y, z having appropriate degrees. These products are the following monomials.

$$\begin{aligned} \text{degree } 7&: \text{ none},\\ \text{degree } 8&: x^4,\\ \text{degree } 9&: x^3y,\\ \text{degree } 10&: x^5, x^3z, x^2y^2,\\ \text{degree } 11&: x^4y, x^2yz. \end{aligned}$$

We exclude the monomial x^2yz, because it can be extinguished by adding a constant times x^2F_1, and get the following intermediate result.

LEMMA 8.5. *The base of the miniversal deformation of L_1 is described in $H^2(L_1; L_1)$ by a system of formal equations*

$$\begin{aligned} &\beta\gamma = 0,\\ &\gamma^2 + A\alpha^2\gamma + B\alpha\beta^2 + C\alpha^4 = 0,\\ &\beta^3 + D\alpha^3\beta = 0,\\ &E\alpha^5 + F\alpha^3\gamma + G\alpha^2\beta^2 = 0,\\ &H\alpha^4\beta = 0. \end{aligned} \tag{8}$$

Consider the zero locus X of the first three equations (8).

LEMMA 8.6. *If $C = BD$, $A^2 \neq 4C$, and $D \neq 0$, then X is the union of three irreducible curves. Otherwise X does not contain three different irreducible curves.*

PROOF. Let $(\alpha, \beta, \gamma) \in X$. The first equation (8) says that either $\beta = 0$, or $\gamma = 0$. If $\beta = 0$, then the third equation holds, and the second equation becomes

$$\gamma^2 + A\alpha^2\gamma + C\alpha^4 = (\gamma + u\alpha^2)(\gamma + v\alpha^2) = 0, \tag{9}$$

where $u \neq v$ if $A^2 \neq 4C$. Hence $X \cap \{\beta = 0\}$ is the union of two parabolas. If $\gamma = 0$, then the second and the third equations become

$$\begin{aligned} \alpha(B\beta^2 + C\alpha^3) &= 0,\\ \beta(\beta^2 + D\alpha^3) &= 0, \end{aligned}$$

which describes just one point $\alpha = 0, \beta = 0$ if $C \neq BD$, the semicubic parabola $\beta^2 + D\alpha^3 = 0$ if $0 \neq C = BD$, and the union of the same semicubic parabola and the line $\beta = 0$ if $0 = C = BD$. In the last case one of the curves (9) is also the line $\beta = 0, \gamma = 0$. Lemma 8.6 follows.

THEOREM 8.7. *The base of the miniversal deformation of the Lie algebra L_1 is described in $H^2(L_1; L_1)$ by the system of formal equations*

$$\begin{aligned} &\beta\gamma = 0,\\ &\gamma^2 + A\alpha^2\gamma + B\alpha(\beta^2 + D\alpha^3) = 0,\\ &\beta(\beta^2 + D\alpha^3) = 0, \end{aligned}$$

where $A^2 \neq 4BD$, and $D \neq 0$.

PROOF. Corollary 8.4 and Lemma 8.6 imply that in equations (8) $C = BD, A^2 \neq 4C$, and $D \neq 0$. Hence the three curves, which are contained in the base of the miniversal deformation according to Corollary 8.4, are

$$\begin{aligned} &\beta = 0,\ \gamma + u\alpha^2 = 0;\\ &\beta = 0,\ \gamma + v\alpha^2 = 0;\\ &\gamma = 0,\ \beta^2 + D\alpha^3 = 0, \end{aligned}$$

where $u \neq v$, $u+v = A$, $uv = BD$. Hence the left hand sides of the last two equations (8) should be equal to 0 on these curves. The monomial $\alpha^4\beta$ is not equal to 0 on the third of the curves; hence $H = 0$. If $\beta = 0$, then the fourth equation becomes

$\alpha^3(E\alpha^2 + F\gamma) = 0$, which cannot hold on *both* parabolas $\gamma + u\alpha^2 = 0$, $\gamma + v\alpha^2 = 0$ unless $E = F = 0$. Finally, if $\gamma = 0$, then the fourth equation (with $E = F = 0$) becomes $G\alpha^2\beta^2 = 0$ which does not hold on the third curve unless $G = 0$.

8.6. Note that the computations made in the article [FiFu] let us find the constants A, B, D from Theorem 8.7. Since these constants depend on a particular choice of cocycles $a \in C^2_{(2)}(L_1; L_1)$, $b \in C^2_{(3)}(L_1; L_1)$, $c \in C^2_{(4)}(L_1; L_1)$ representing generators of $H^2(L_1; L_1)$, we need to specify these cocycles first.
Let W be an L_1-module spanned by e_j with all $j \in \mathbb{Z}$ and with the L_1-action $e_i(e_j) = (j - i)e_{i+j}$. It is an extension of the adjoint representation. Define a cochain

$$\mu_k \in C^1_{(k)}(L_1; W),\ k \geq 2,$$

by the formula

$$\mu_k(e_i) = (-1)^{i+1}\binom{k-1}{i-2}e_{i-k}.$$

Proposition 8.8 [FiFu]. *If $k = 2, 3, 4$, then $\delta\mu_k$ belongs to $C^2_{(k)}(L_1; L_1)$ and is a cocycle not cohomologous to* 0.

Proposition 8.9 [FiFu]. *If one chooses a, b, c to be $\delta\mu_2$, $\delta\mu_3$, $\delta\mu_4$, then*

$$A = -\frac{2 \cdot 11 \cdot 37}{5 \cdot 13^2},\ B = \frac{4 \cdot 7 \cdot 17}{3 \cdot 25 \cdot 13},\ D = \frac{32 \cdot 27}{13^3}.$$

9. Bibliography

[B] Barr, M., "Harrison homology, Hochschild homology and triples," *J. Algebra*, **8** (1963), 314–323.

[FeFu] Feigin, B., Fuchs, D., "Homology of the Lie algebras of vector fields on the line," *Funct. Anal. Appl.*, **14:3** (1980), 45–60.

[Fi1] Fialowski, A., "Deformations of Lie Algebras," *Math. USSR- Sbornik,* **55:2** (1986), 467–473.

[Fi2] Fialowski, A., "Deformations of the Lie algebra of vector fields on the line," *Russian Math. Surveys*, **38** (1983), 185–186.

[Fi3] Fialowski, A., "An Example of Formal Deformations of Lie Algebras," *NATO Conference on Deformation Theory of Algebras and Applications*, Il Ciocco, Italy, 1986. Proceedings, Kluwer, Dordrecht, 1988, 375–401.

[FiFu] Fialowski, A., Fuchs, D., "Singular deformations of Lie algebras on an example," in *Topics in Singularity Theory: V. I. Arnold's* 60^{th} *Anniversary Collection,* A. Khovanskii, A. Varchenko, V. Vassiliev (Eds.), Transl. A.M.S. Ser. 2, **180**, pp. 77–92, Amer. Math. Soc., Providence, RI, 1997.

[Fu] Fuchs, D., *Cohomology of infinite-dimensional Lie Algebras*, Consultants Bureau, NY, London (1986).

[FuL] Fuchs, D., Weldon, L., "Massey brackets and deformations," *Journal of Pure and Applied Algebra*, **156** (2001), pp. 215–229.

[G] Gerstenhaber, M., "On the deformation of rings and algebras, I, III," *Ann. Math.* **79** (1964), 59–103; **88** (1968), 1–34.

[GoM] Goldman, W. M., Millson, J. J., "The Deformation Theory of Representations of Fundamental Groups of Compact Kähler Manifolds," *IHES Pub. Math.*, **67** (1988), 43–96.

[Harr] Harrison, D.K., "Commutative algebras and cohomology," *Trans. Amer. Math. Soc.*, **104** (1962), 191–204.

[Hart] Hartshorne, R., *Algebraic Geometry*, Springer (1977).

[I] Illusie, L., “Complexe cotangent et déformations I,” *Lect Notes in Math.* **239**, Springer (1971).

[K] Kontsevich, M., *Topics in Algebra: Deformation Theory*, Lecture Notes, Univ. Calif. Berkeley (1994).

[La] Laudal, O. A., “Formal Moduli of Algebraic Structures,” *Lect. Notes 754*, Springer (1979).

[NR] Nijenhuis, A., Richardson, R. W., “Cohomology and Deformations in Graded Lie Algebras,” *Bull. Amer. Math. Soc.*, **72** (1966), 1–29.

[P] Palamodov, V.P., “Deformations of complex spaces,” *Russian Math. Surveys*, **31** (1976).

[Sch] Schlessinger, M., “Functors of Artin rings,” *Trans. Amer. Math. Soc.* **130** (1968), 208–222.

Alice Fialowski, *email*: `fialowsk@cs.elte.hu`
Dmitry Fuchs, *email*: `fuchs@math.ucdavis.edu`

thesis.ps

Part II: Nilpotent Lie Algebras and Cohomology

APPEARED IN: *Vestnik Moskovskogo Universiteta Matematika*, **38** (1983) No. 2, 62-64 English translation: *Moscow University Mathematics Bulletin*, **38** (1983) No. 2, 76-79

Classification of Graded Lie Algebras with Two Generators

Alice Fialowski
Moscow State University

Abstract This article considers infinite-dimensional Lie algebras over a field of characteristic 0 with basis $e_1, e_2, \dots$ which satisfy the condition

$$[e_i,\, e_j] = c_{ij} e_{i+j}.$$

A complete description is given of such algebras with two generators. In particular, it follows from the proposed classification that if the number of independent relations between the generators of a Lie algebra of this type is finite, then it is equal to 2.

In this paper we classify the graded Lie algebras $\mathfrak{g} = \oplus_{i=1}^{\infty}\mathfrak{g}_i$ over a field $\mathbb{K}$ of characteristic 0, for which $\dim \mathfrak{g}_i = 1$, with minimum possible number of generators. Obviously this number is equal to 2.

There are three well-known Lie algebras of the above type: $L_1, \mathfrak{n}_1, \mathfrak{n}_2$. The Lie algebra L_1 consists of vector fields on the real line with polynomial coefficients which vanish together with their first derivative at the coordinate origin [1]. The algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are maximal nilpotent subalgebras in Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$, respectively [2].

The problem of classification of algebras such as $L_1, \mathfrak{n}_1, \mathfrak{n}_2$ is naturally related to the problem formulated by V. Kac in [3], which involves the classification of all simple graded Lie algebras $\mathcal{L} = \oplus_{i\in\mathbb{Z}}\mathcal{L}_i$, where $\dim \mathcal{L}_i = 1$.

In the algebra $\mathfrak{g} = \oplus_{i=1}^{\infty}\mathfrak{g}_i$ we choose a basis of homogeneous elements $e_i \in \mathfrak{g}_i$. The generators of $\mathfrak{g}$ are e_1 and e_2. Note that $[e_1,\, e_2] \neq 0$ and $[e_1,\, [e_1,\, e_2]] \neq 0$. We specify the explicit form of the commutator in the algebras $L_1, \mathfrak{n}_1, \mathfrak{n}_2$ as follows.

: L_1: $[e_i,\, e_j] = (j-i)e_{i+j}$,

: $\mathfrak{n}_1$: $[e_i,\, e_j] = a_{ij}e_{i+j}$, where $a_{ij} = \begin{cases} 1 & \text{if } j-i \equiv 1 \mod 3 \\ 0 & \text{if } j-i \equiv 0 \mod 3 \\ -1 & \text{if } j-i \equiv -1 \mod 3 \end{cases}$,

: $\mathfrak{n}_2$: $[e_i,\, e_j] = b_{ij}e_{i+j}$, where the numbers b_{ij} depend only on the residue obtained when dividing i and j by 8 according to the rule $b_{ij} + b_{i'j'} = 0$ if the numbers $i+i'$ and $j+j'$ are divisible by 8. The accompanying table gives the numbers b_{ij} (the remaining b_{ij} are determined from the relations $b_{ij} = -b_{ji}$ and $b_{ij} + b_{8-i,8-j} = 0$.

$i \bmod 8$ \ $j \bmod 8$	1	2	3	4	5	6	7
0	1	-2	-1	0	1	2	-1
1	0	1	-1	3	-2	0	1
2	-1	0	0	0	1	-1	0
3	1	0	0	-3	1	1	-2

In addition to these three algebras, we will need two particular algebras and also a special family of algebras. These are:

: $\mathfrak{m}_0$: The algebra in which $[e_1,\, e_i] = e_{i+1}$ for $i > 1$ and $[e_i,\, e_j] = 0$ for $i, j > 1$,

: $\mathfrak{m}_2$: The algebra in which the commutator is set up as follows: $[e_i,\, e_j] = 0$ for $i, j > 2$, while $[e_1,\, e_j] = e_{j+1}$ for $j \geq 2$ and $[e_2,\, e_j] = e_{j+2}$ for $j > 2$.

: $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$: A family of Lie algebras with countably many parameters $\lambda_{4k} \in \mathbb{K}P^1$. The commutator is defined as follows: $[e_1, e_4] = 0$, $[e_3, e_4] = 0$, $[e_i, e_j] = 0$ if i is even but not 2 and j is any positive integer. Furthermore,

$$[e_1, e_{4k-1}] = \alpha_{4k} e_{4k}, \quad \text{and} \quad [e_2, e_{4k-2}] = \beta_{4k} e_{4k}, \quad k = 2, 3, 4, \ldots,$$

where the α_{4k} and β_{4k} are the homogeneous coordinates of the point $\lambda_{4k} \in \mathbb{K}P^1$. The remaining commutators can be uniquely reconstructed from the above formulas. Their structural constants are homogeneous polynomials of α_{4k} and β_{4k}. See the Appendix for some explicit formulas for the commutators.

EXAMPLE: For the algebra $\mathfrak{g}(1, 1, 1, \ldots)$ the commutators are

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4$$
$$[e_1, e_{2k+1}] = [e_2, e_{2k}] = e_{2k+2} \quad \text{if} \quad k \geq 2$$
$$[e_2, e_{2k-1}] = e_{2k+1} \quad \text{if} \quad k \geq 2$$

the other commutators are 0.

Theorem . *Let $\mathfrak{g} = \oplus_{i=1}^{\infty} \mathfrak{g}_i$ be an $\mathbb{N}$-graded Lie algebra, where* $\dim \mathfrak{g}_i = 1$, *with basis $e_1, e_2, e_3, \ldots$, generated by e_1 and e_2. Then $\mathfrak{g}$ is one of the following.*

: *a) Assume $[e_1, e_4] \neq 0$ and $[e_2, e_3] \neq 0$. If $[e_3, e_4] \neq 0$, then $\mathfrak{g} \cong L_1$ while if $[e_3, e_4] = 0$, then $\mathfrak{g} \cong \mathfrak{m}_2$.*

: *b) Assume $[e_2, e_3] = 0$. If $[e_3, e_4] \neq 0$, then $\mathfrak{g} \cong \mathfrak{n}_2$ while if $[e_3, e_4] = 0$, then $\mathfrak{g} \cong \mathfrak{m}_0$.*

: *c) Assume $[e_1, e_4] = 0$. If $[e_3, e_4] \neq 0$, then $\mathfrak{g} \cong \mathfrak{n}_1$ while if $[e_3, e_4] = 0$, then $\mathfrak{g} \cong \mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$ for some choice of the $\lambda_8, \lambda_{12}, \lambda_{16}, \ldots$.*

PROOF: (sketch) The Jacobi identity yields a system of linear equations for the usual structural constants c_{ij} ($[e_i, e_j] = c_{ij} e_{i+j}$). The numbers c_{12} and c_{13} can be regarded as arbitrary but non-zero. To be specific, assume that $c_{12} = 1$. Then the coefficients c_{14} and c_{23} can no longer be chosen arbitrarily since they must satisfy a linear equation. We fix some solution of this equation after which c_{15} and c_{24} are determined uniquely. At the next step we obtain two equations for c_{16}, c_{25} and c_{34}, etc.
As an example we consider the case in which $c_{14} \neq 0$, $c_{23} \neq 0$, and $c_{34} \neq 0$. We can assume that $c_{13} = 2$, $c_{23} = c_{34} = 1$. Solving this system step-by-step we find that if $c_{14}c_{25} \neq 9$, then the system of equations for c_{ij} with $i + j = 16$ does not have a non-zero solution. If, however, $c_{14}c_{25} = 9$, then all the equations can be solved non-trivially and uniquely so we obtain an algebra that is isomorphic to L_1.
We can similarly consider the cases in which some of the numbers c_{14}, c_{23}, and c_{34} are zero.

Of interest are the relations that link the generators e_1 and e_2. It is easy to show that these generators should satisfy at least two independent relations of weights 5 and 7 of the form

$$\begin{gathered} \lambda[e_1, [e_1, [e_1, e_2]]] + \mu[e_2, [e_2, e_1]] = 0, \\ \alpha[e_1, [e_1, [e_1, [e_1, [e_1, e_2]]]]] + \beta[e_2, [e_2, [e_2, e_1]]] + \gamma[e_1, [e_1, [e_2, [e_2, e_1]]]] = 0. \end{gathered} \tag{$*$}$$

In the algebras L_1, $\mathfrak{n}_1$, $\mathfrak{n}_2$, and $\mathfrak{m}_2$, the relations $(*)$ make up a complete system of defining relations while for the algebras $\mathfrak{m}_0$ and $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$ this system is infinite. Specifically, in the algebra $\mathfrak{n}_1$ we should add relations of weights 13, 17, 21, ... to $(*)$; the relations between generators in the algebras $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$ are not yet computed.

Corollary 0.1. *If the number of relations between the generators of the Lie algebra $\mathfrak{g}$ is finite, then $\mathfrak{g}$ is isomorphic to one of the four algebras L_1, $\mathfrak{n}_1$, $\mathfrak{n}_2$, and $\mathfrak{m}_2$, and the number of relations is 2.*

Let us consider a Lie algebra with generators e_1 and e_2 and relations $(*)$. It turns out that for most points $(\lambda, \mu, \alpha, \beta, \gamma) \in \mathbb{K}^5$ this algebra is finite-dimensional. One can compute the following. If $6\lambda = \mu$, $120\alpha = 3\beta + 20\gamma$, then the Lie algebra is isomorphic to L_1; if $\mu = 0$, $\lambda = 0$, then it is isomorphic to $\mathfrak{n}_1$; if $\lambda = 0$, $\gamma = 0$, it is isomorphic to $\mathfrak{n}_2$; and if $\lambda = \mu$, $\alpha = \beta + \gamma$, it is isomorphic to $\mathfrak{n}_2$. If, however, $\lambda = 0$, $\alpha = 0$ or $\mu = 0$, $\beta = 0$, then the dimension of the space of weight i in the Lie algebra increases exponentially with i, and adding additional relations converts this algebra into one of the algebras of the family $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$.

In conclusion, we should note that the cohomology of the Lie algebras L_1, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ with trivial coefficients are known [4], [5]. In all cases, for $i > 0$ the i-th cohomology space is two-dimensional. It would be interesting to calculate the cohomology of the other algebras considered in this article.

The author is grateful to A.A. Kirillov for his attention.

References

[1] I. M. Gel´fand, B. L. Feigin, and D. B. Fuchs, "Cohomologies of infinite-dimensional Lie algebras and Laplace operators," *Funkts. Analiz i Ego Pril.*, **12**, No. 4 (1978), pp. 1–5. English translation: *Funct. Anal. Appl.*, **12**, pp. 243–247.

[2] J. Lepowsky and S. Milne, "Lie algebraic approaches to classical partition identities," *Adv. Math.*, **29** (1978), pp. 15–59.

[3] V. G. Kac, "Some problems on infinite dimensional Lie algebras," *Lie Algebras and Related Topics*, Lecture Notes in Mathematics **933**, Springer-Verlag 1982.

[4] L. V. Goncharova, "Cohomologies of Lie algebras of formal vector fields on a straight line," *Funkts. Analiz i Ego Pril.*, **7**, No. 2 (1973), pp. 6–14. English translation: *Funct. Anal. Appl.*, **7**, pp. 91–97.

[5] H. Garland and J. Lepowsky, "Lie algebra homology and the Macdonald-Kac formulas," *Invent. Math.*, **34** (1976), pp. 37–76.

Appendix

(added June 1997)

Here are some details for the structural constants of the algebras $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \ldots)$. We compute $[e_i, e_j]$ for all levels $i + j \leq 17$. For these levels, we list only the (generically) non-zero brackets. Using a computer one can extend these much further.

$i + j = 3:$ $[e_1, e_2] = e_3$

$i + j = 4:$ $[e_1, e_3] = e_4$

$i + j = 5:$ $[e_2, e_3] = e_5$

$i + j = 6:$ $[e_1, e_5] = [e_2, e_4] = e_6$

$i + j = 7:$ $[e_2, e_5] = e_7$

$i + j = 8:$ $[e_1, e_7] = \alpha_8 e_8, \quad [e_2, e_6] = \beta_8 e_8, \quad [e_3, e_5] = (\alpha_8 - \beta_8)e_8$

$i + j = 9:$ $[e_2, e_7] = e_9$

$i + j = 10:$ $[e_2, e_8] = e_{10}, \quad [e_1, e_9] = (2\alpha_8 - \beta_8)e_{10}, \quad [e_3, e_7] = (\alpha_8 - \beta_8)e_{10}$

$i + j = 11:$ $[e_2, e_9] = e_{11}$

$i + j = 12:$ $[e_1, e_{11}] = \alpha_{12} e_{12}, \quad [e_2, e_{10}] = \beta_{12} e_{12}$
$[e_3, e_9] = (\alpha_{12} + (\beta_8 - 2\alpha_8)\beta_{12})e_{12},$
$[e_5, e_7] = ((3\alpha_8 - 2\beta_8)\beta_{12}) - \alpha_{12})e_{12}$

$i + j = 13:$ $[e_2, e_{11}] = e_{13}$

$i + j = 14:$ $[e_2, e_{12}] = e_{14},$
$[e_1, e_{13}] = (3\alpha_{12} + (3\beta_8 - 5\alpha_8)\beta_{12})e_{14},$
$[e_3, e_{11}] = (2\alpha_{12} + (3\beta_8 - 5\alpha_8)\beta_{12})e_{14},$
$[e_5, e_9] = ((3\alpha_8 - 2\beta_8)\beta_{12} - \alpha_{12})e_{14}$

$i + j = 15:$ $[e_2, e_{13}] = e_{15}$

$i + j = 16:$ $[e_1, e_{15}] = \alpha_{16} e_{16}, \quad [e_2, e_{14}] = \beta_{16} e_{16}$
$[e_3, e_{13}] = (\alpha_{16} - 3\beta_{16}\alpha_{12} - \beta_{16}\beta_{12}(3\beta_8 - 5\alpha_8))e_{16}$
$[e_5, e_{11}] = (-\alpha_{16} + 5\beta_{16}\alpha_{12} + 2\beta_{16}\beta_{12}(3\beta_8 - 5\alpha_8))e_{16}$
$[e_7, e_9] = (\alpha_{16} - 6\beta_{16}\alpha_{12} + \beta_{16}\beta_{12}(13\alpha_8 - 8\beta_8))e_{16}$

$i + j = 17:$ $[e_2, e_{15}] = e_{17}$

APPEARED IN: *Studia Sci. Math. Hungar.* **19** (1984), 465–483

Deformations of Nilpotent Kac–Moody Algebras

Alice Fialowski
Alfréd Rényi Institute of Mathematics
Budapest

The main goal of this article is the calculation of the one and two-dimensional cohomology of maximal nilpotent subalgebras of affine Kac–Moody type Lie algebras. This calculation allows us to classify the exterior derivations and deformations of the indicated algebras.
The article consists of two sections: The first section contains basic definitions and statements of the results, while the second one contains the proofs.[‡]
The author would like to thank Professor D. Fuchs for stimulating discussions and friendly help.

1. 1. Definitions and the statements of the results

1. Let $A = \|a_{ij}\|$ be an integer $n \times n$ matrix with $a_{11} = \dots = a_{nn} = 2$ and $a_{ij} \leqq 0$ for $i \neq j$. Suppose that A is symmetrisable, i.e. there exist positive numbers $\varrho_1, \dots, \varrho_n$ such that the matrix $\|\varrho_i a_{ij}\| = \varrho A$ is symmetric. From now on $\varrho_1, \dots, \varrho_n$ denote the minimal positive integers with the property above. Define the *Kac–Moody Lie algebra* $\mathfrak{g}^A$ *with the Cartan matrix* A as a complex Lie algebra with the generators $e_1, \dots, e_n$, $f_1, \dots, f_n$, $h_1, \dots, h_n$ and the relations

$$[e_i, f_j] = \delta_{ij} h_j, \qquad [h_i, h_j] = 0,$$
$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j,$$
$$\big[\underbrace{e_i, [e_i, \dots, [e_i}_{-a_{ij}+1}, e_j] \dots]\big] = 0, \qquad \big[\underbrace{f_i, [f_i, \dots, [f_i}_{-a_{ij}+1}, f_j] \dots]\big] = 0 \quad (i \neq j).$$

Define in $\mathfrak{g}^A$ a (multi-) gradation by

$$\deg h = (\underbrace{0, \dots, 0}_{n}), \quad \deg e_i = (\underbrace{0, \dots, 0, \overset{i}{1}, 0, \dots, 0}_{n}),$$
$$\deg f_i = (\underbrace{0, \dots, 0, \overset{i}{-1}, 0, \dots, 0}).$$

Here n is called the rank of $\mathfrak{g}^A$.
Suppose that A is nondecomposable, i.e. it can not become of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ under any simultaneous permutation of rows and columns.
The Weyl group $W = W^A$ of $\mathfrak{g}^A$ is defined as the subgroup of $GL(n, \mathbf{Z})$, generated by the matrices $\sigma_i = E - A_i$ where E is the unity and in A_i the ith row coincides with the ith row of A, while the other rows are zeros. (The elements of W may be considered as transformations of the "weight lattice" $\mathbf{Z}^n$, which grades $\mathfrak{g}^A$.)
Remind some facts about the Kac–Moody Lie algebras (see [1], [2], [3]).
(i) $\mathfrak{g}^A = \mathfrak{n}_+(A) + \mathfrak{h} + \mathfrak{n}_-(A)$, where $\mathfrak{n}_+(A)$ and $\mathfrak{n}_-(A)$ are subalgebras of $\mathfrak{g}^A$, generated by $e_1, \dots, e_n$ and $f_1 \dots, f_n$ respectively, while $\mathfrak{h}$ is n-dimensional (commutative) subalgebra, spanned by $h_1, \dots, h_n$.
(ii) The defining relation system for the generators $e_1, \dots, e_n$ of $\mathfrak{n}_+(A)$ consists of

$$\big[\underbrace{e_i, [e_i, \dots, [e_i}_{-a_{ij}+1}, e_j] \dots]\big] = 0.$$

*1980 Mathematics Subject Classification. Primary 17B56; Secondary 17B65.
‡Key words: Deformation, cohomology, Kac–Moody algebras, spectral sequence.
‡For another proof of a part of these results see in [7].

The similar relations are true for $\mathfrak{n}_-(A)$.
It is natural to divide the Kac–Moody Lie algebras into three classes: algebras with positive definite matrix ϱA, algebras with nonnegative definite matrices of rank $n-1$ and the remaining algebras.
(iii) The class of algebras $\mathfrak{g}^A$ with positive definite matrices ϱA coincides with the class of simple finite-dimensional complex Lie algebras.
In this paper we restrict ourselves to the so called *affine algebras* of the second type. The nondecomposable matrices corresponding to these algebras are listed in Tables 1 and 2.
The vertices in Tables 1–2 correspond to the rows of A. The ith vertex is joined with the jth one by $a_{ij}a_{ji}$ edges; if $|a_{ij}| > |a_{ji}|$, these edges have an arrow, pointing towards the ith vertex. Numerical marks are the coefficients of linear dependence between the corresponding columns of the Cartan matrix A. Fix for these numbers the notation $\omega_1 \dots \omega_n$.
(iv) Let A be a positive definite Cartan matrix, corresponding to certain Dynkin diagram and $\widetilde{A}$ be the Cartan matrix of the extended Dynkin diagram from Table 1. Then $\mathfrak{g}^{\widetilde{A}}$ is the central extension of the *current algebra* $\mathfrak{g}^A \otimes \mathbf{C}[t, t^{-1}]$.
By this the canonical generators $e_1, \dots, e_n$ of $\mathfrak{g}^{\widetilde{A}}$ correspond to the products $e_1 \otimes 1, \dots, e_{n-1} \otimes 1$, $f \otimes t$, where $e_1, \dots, e_{n-1}$ are canonical generators of $\mathfrak{g}^A$ and f is the root vector of $\mathfrak{g}^A$, corresponding to the negative root of maximal length. Moreover, for $(m_1, \dots, m_n) \neq (0, \dots, 0)$

$$\mathfrak{g}^{\widetilde{A}}_{(m_1,\dots,m_n)} = \mathfrak{g}^A_{(m_1 - m_2\alpha_1, \dots, m_{n-1} - m_n\alpha_{n-1})} \otimes t^{m_n}$$

where $(\alpha_1, \dots, \alpha_{n-1})$ is the weight of f.

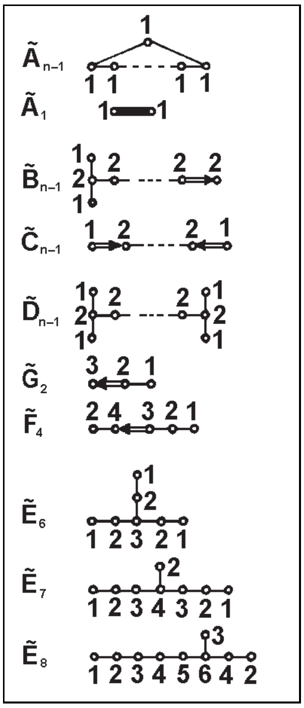


Table 1

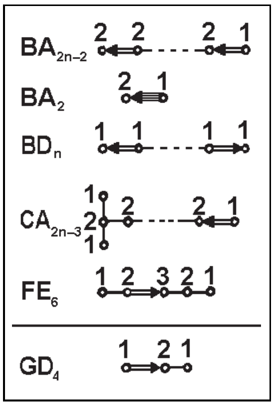


Table 2

We notice also that $\mathfrak{n}_+(\widetilde{A}) = (\mathfrak{n}_+(A) \otimes 1) \oplus \Big(\bigoplus_{m>0} (\mathfrak{g}^A \otimes t^m) \Big)$ and similar is true for $\mathfrak{n}_-(A)$.
Algebras, corresponding to matrices from Table 2 are defined by means of finite order exterior automorphisms of finite-dimensional simple algebras. Namely, if $\varphi : \mathfrak{g} \to \mathfrak{g}$ is such an automorphism and l is its order, then we define $\mathfrak{g}_\varphi$ as the subalgebra

$\bigoplus\limits_{\lambda=-\infty}^{\infty} \mathfrak{g}(\lambda)\otimes t^\lambda$ of $\mathfrak{g}\otimes \mathbf{C}[t,t^{-1}]$, where $\mathfrak{g}(\lambda)$ is the root subspace of the automorphism φ, corresponding to the eigenvalue $e^{2\pi i\lambda/l}$.
(v) The algebras from Table 2 are central extensions of the algebras $\mathfrak{g}_\varphi$. Namely, the first 5 cases correspond to two-order automorphisms, while the last one to three-order automorphism.
The homology of $\mathfrak{n}_+(A)$ with trivial coefficients is known [4], [5]. Let

$$Q_A(x_1,\dots,x_n) = -\frac{1}{2}\sum \varrho_i a_{ij} x_i x_j + \sum \varrho_i x_i.$$

(vi) If $Q_A(m_1,\dots,m_n)\neq 0$, then

$$H_k^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A)) = 0$$

for arbitrary k. If $Q_A(m_1,\dots,m_n)=0$, then there is a unique $k(m_1,\dots,m_n)$, for which

$$H_k^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A)) = \begin{cases}\mathbf{C} & \text{for } k=k(m_1,\dots,m_n),\\ 0 & \text{for the others.}\end{cases}$$

For the practical computation of the number $k(m_1,\dots,m_n)$ it is convenient to use the transformations $s_i:\mathbf{Z}^n\to\mathbf{Z}^n$, defined by

$$s_i(m) = \sigma_i(m) + (0,\dots,0,\overset{i}{\varrho_i},\dots,0).$$

(The transformations s_i also define an action of W in $\mathbf{Z}^n$.) It is easy to show that $Q_A\circ s_i = Q_A$ and that an arbitrary sequence $(m_1,\dots,m_n)$ with $Q_A(m_1,\dots,m_n)=0$ may be obtained from $(0,\dots,0)$ by means of finite number of transformations s_i. The minimal number of these transformations is $k(m_1,\dots,m_n)$.
In particular,

$$H_0(\mathfrak{n}_+(A)) = H_0^{(0,\dots,0)}(\mathfrak{n}_+(A)) = \mathbf{C},$$
$$H_1(\mathfrak{n}_+(A)) = H_1^{(1,0,\dots,0)}(\mathfrak{n}_+(A))\oplus\dots\oplus H^{(0,\dots,0,1)}(\mathfrak{n}_+(A)) = \mathbf{C}^n.$$

2. Let A be a Cartan matrix from Tables 1, 2. The main result of this paper is the computation of one- and two-dimensional cohomologies of $\mathfrak{n}_+(A)$ with coefficients in the adjoint representation. Remind that the computation of one-dimensional cohomology is equivalent to the classification of exterior derivations, and it is that language, in which we formulate here the result. The calculation of two-dimensional cohomology allows us to classify the deformations of the considered algebras.

Theorem 16. *The next derivations form a basis in the space of exterior derivations of* $\mathfrak{n}_+(A)$*:*

$$\overline{h}_i : g\to [h_i,g],\quad i=1,\dots,n-1;$$
$$\tau_i : t^{il+1}\frac{d}{dt},\quad i=0,1,2,\dots\ .$$

Here l and t have the same sense as in (iv) *and* (v) *of subsection 1.*

We describe now some concrete deformations of $\mathfrak{n}_+(A)$.
1°. Let $\alpha\in H^1(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$, $\beta\in H^1(\mathfrak{n}_+(A))$. The element α corresponds to the right extension

$$0\to\mathfrak{n}_+(A)\to\tilde{\mathfrak{n}}_+(A)\to\mathbf{C}\to 0$$

(the elements of $H^1(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$ may be interpreted not only as exterior derivations, but also as right extensions – see [5]), β to a functional $\varphi:\mathfrak{n}_+(A)\to\mathbf{C}$. For $t\in\mathbf{C}$ denote η_t the embedding $\mathfrak{n}_+(A)\to\tilde{\mathfrak{n}}_+(A)\cong\mathfrak{n}_+(A)\oplus\mathbf{C}$ defined by $\eta_t(g)=(g,t\varphi(g))$. It may be easily checked that $\eta_t(\mathfrak{n}_+(A))$ is a subalgebra of $\tilde{\mathfrak{n}}_+(A)$, that this subalgebra is connected with $\mathfrak{n}_+(A)$ by a natural linear isomorphism, and that for $t=0$ this isomorphism is compatible with the bracket operation. Thus we have a deformation of $\mathfrak{n}_+(A)$. The corresponding infinitesimal deformation is evidently the product

$$\alpha\beta\in H^2(\mathfrak{n}_+(A);\mathfrak{n}_+(A)).$$

(By all means, this construction may be applied to an arbitrary Lie algebra.)

2°. Let $1 \leqq i \leqq n$. The algebra $\mathfrak{n}_+(A)$ deforms inside $\mathfrak{g}^A$. The deformed algebra is spanned by the spaces $\mathfrak{g}^A_{(m_1,\dots,m_n)}$ with $(m_1,\dots,m_n) \neq (0,\dots,0,\overset{(i)}{1},\dots,0)$ and by the vector $e_i + tf_i$, where t is a parameter. (Informally speaking, e_i deforms into $e_i + tf_i$, while the other additive generators of $\mathfrak{n}_+(A)$ do not change.)
The number of such deformations is equal to the rank of $\mathfrak{g}^A$.
3°. Let $1 \leqq i, j \leqq n$; consider the entry $a_{ij} = -1$ and if $a_{ij} = a_{ji}$, then $i < j$. The algebra $\mathfrak{n}_+(A)$ deforms again inside $\mathfrak{g}^A$. The deformed algebra is generated by the spaces $\mathfrak{g}^A_{(m_1,\dots,m_n)}$ with

$$(m_1,\dots,m_n) \neq (0,\dots,0,\overset{(i)}{1},0,\dots,0),\ (0,\dots,0,\overset{(i)}{1},0,\dots,0,\overset{(j)}{1},0,\dots,0)$$

and the vectors $e_i + tf_j$ and $[e_i, e_j] - th_j$. (Informally speaking, e_i and $[e_i, e_j]$ deform into $e_i + tf_j$ and $[e_i, e_j] - th_j$, while the other additive generators of $\mathfrak{n}_+(A)$ are not deformed.)
The number of this type deformations is equal to the number of nonzero pairs (a_{ij}, a_{ji}) with $i \neq j$; this number we denote below by p.
Remark that the equality $a_{ij} = -1$ is necessary for the verification of the fact that the deformed algebras are closed under the bracket and that with the only exception of the case $\widetilde{A}_1$, at least one of two nontrivial nondiagonal entries of the Cartan matrix a_{ij}, a_{ji} is equal to -1. This specific property of $\widetilde{A}_1$ compels us to consider the case $\mathfrak{n}_+(\widetilde{A}_1)$ separately.

Theorem 17. *Suppose that $A \neq \widetilde{A}_1$. Then*
(i) *All the homogeneous infinitesimal deformations of $\mathfrak{n}_+(A)$ may be extended to its real deformations.*
(ii) *The space of infinitesimal deformations $H^2(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$ is spanned by deformations, corresponding to the above types* 1°, 2°, 3°. *In other words, the mapping*

$$\psi : \left[H^1(\mathfrak{n}_+(A);\mathfrak{n}_-(A)) \otimes H^1(\mathfrak{n}_+(A))\right] \oplus \mathbf{C}^n \oplus \mathbf{C}^p \to H^2(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$$

defined by the infinitesimal deformations listed above is epimorphism.
(iii) *The kernel of the mapping ψ is contained in*

$$H^1(\mathfrak{n}_+(A);\mathfrak{n}_+(A)) \otimes H^1(\mathfrak{n}_+(A))$$

and its dimension is n. It is spanned by the elements $\kappa_1,\dots,\kappa_n$ defined as follows. Let $1 \leqq i \leqq n$. Choose the numbers $\beta_1,\dots,\beta_{n-1}$ so that $\sum_1^{n-1} \beta_j a_{kj} = 1$ for $k \neq i$ (such numbers can be found, because the rank of the Cartan matrix with one column removed equals to $n-1$). Then

$$\kappa_i = \overline{h}_i \otimes \overline{e}_i, \qquad i = 1,\dots,n-1,$$

$$\kappa_n = \Big(\sum_1^{n-1} \beta_j \overline{h}_j\Big) \otimes \overline{e}_i,$$

where $\overline{e}_i$ is the class of the cocycle from $\mathbf{C}^1(\mathfrak{n}_+(A))$, assigning 1 to e_i and 0 to other e_k's, while the $\overline{h} - s$ were introduced in Theorem 16.

Now turn to the case $A = \widetilde{A}_1$. In this case the Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, and this excludes the possibility of applying the construction 3°. Mention also that it is not true for this case that all infinitesimal deformations may be extended to real deformations.

Theorem 18. (i) *Infinitesimal deformations, corresponding to deformations of type* 1°, 2° *span in $H^2\big(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1)\big)$ a codimension 2 subspace. The complementary subspace is spanned by elements from $H^2_{(-1,-2)}$ and $H^2_{(-2,-1)}$ respectively. These elements can not be extended to the deformation of $\mathfrak{n}_+(\widetilde{A}_1)$.* (Cocycles representing these two classes are given in subsection 2.2).

(ii) *The kernel of the mapping*

$$\left[H^1(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1))\otimes H^1(\mathfrak{n}_+(\widetilde{A}_1))\right]\oplus\mathbf{C}^2\to H^2(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1))$$

may be described just as the kernel of ψ *in part* (iii) *of Theorem 17.*

2. 2. Proofs

1. Let $\mathfrak{g}=\bigoplus_{i>0}\mathfrak{g}_i$ be a nilpotent graded Lie algebra and $B=\oplus B_j$ be a graded $\mathfrak{g}$-module. The space $C_k^{(m)}(\mathfrak{g};B)$ is spanned by "monomials", i.e. by the chains

$$g_1\wedge\cdots\wedge g_k\otimes b,\quad\text{where } g_s\in\mathfrak{g}_{i_s},\quad b\in B_j,\ i_1+\dots i_k+j=m.$$

Denote by $F_pC_k^{(m)}(\mathfrak{g};B)$ the subspace of $C_k^{(m)}(\mathfrak{g};B)$, generated by monomials with $i_1+\cdots+i_k\leqq p$. Evidently, $\{F_p\}$ is a decreasing filtration in $C_*^{(m)}(\mathfrak{g};B)$. The spectral sequence corresponding to this filtration we will call *Feigin–Fuchs spectral sequence* and denote it by $\mathcal{E}(\mathfrak{g},B,m)$. Here $E^0_{p,q}=C^{(p)}_{p+q}(\mathfrak{g};B_{m-p})$, where B_{m-p} is considered as trivial $\mathfrak{g}$-module and $d^0_{p,q}$ is the differential

$$d_{p+q}:C^{(p)}_{p+q}(\mathfrak{g};B_{m-p})\to C^{(p)}_{p+q-1}(\mathfrak{g};B_{m-p});$$

hence

$$E^1_{p,q}=H^{(p)}_{p+q}(\mathfrak{g};B_{m-p})=H^{(p)}_{p+q}(\mathfrak{g})\otimes B_{m-p}.$$

For the algebra L_1 of polynomial vector fields on the line with trivial 1-jets in the point 0 this spectral sequence was considered in [6]. In the cases interesting for us the algebra $\mathfrak{g}$ has multigradation $\mathfrak{g}=\bigoplus_{(i_1,\dots,i_k)>(0,\dots,0)}\mathfrak{g}_{(i_1,\dots,i_k)}$. In this case the spectral sequence $\mathcal{E}(\mathfrak{g},B,m)$ decomposes into the sum of spectral sequences $\mathcal{E}(\mathfrak{g},B,m_1,\dots,m_k)$, $m_1+\cdots+m_k=m$. The initial term of the last spectral sequence is given by the formula

$$E^1_{p,q}=\bigoplus_{p_1+\cdots+p_k=p}H^{(p_1,\dots,p_k)}_{p+q}(\mathfrak{g})\otimes B_{m_1-p_1,\dots,m_k-p_k}.$$

We apply the above spectral sequence to the computation of the one- and two-dimensional homology of the algebra $\mathfrak{n}_+(A)$ with coefficients in the coadjoint representation $\mathfrak{n}_+(A)'$. (This is equivalent to the computation of the cohomology of $\mathfrak{n}_+(A)$ with coefficients in the adjoint representation.) For each of the matrices from Tables 1, 2 the terms and differentials of the spectral sequence $\mathcal{E}(\mathfrak{n}_+(A),\mathfrak{n}_+(A)',m)$ may be explicitly determined, and this leads to the calculation of the indicated homology. All computations are similar, and we shall give details only for the cases $\widetilde{A}_{n-1}$ and BA_2.

2. Let us begin with $\widetilde{A}_1$. There is a convenient explicit description of the quotient algebra of $\mathfrak{g}^{\widetilde{A}_1}$ by its (one-dimensional) center. Namely, it contains an additive basis ε_i ($i\in\mathbf{Z}$) such that

$$[\varepsilon_i,\varepsilon_j]=\alpha_{ij}\varepsilon_{i+j},\quad\text{where}\quad\alpha_{ij}\begin{cases}=-1,0,1,\\ \equiv(j-i)\bmod 3.\end{cases}$$

(In this notation $\varepsilon_1,\varepsilon_2,\varepsilon_{-1},\varepsilon_{-2}$ correspond to e_1,e_2,f_1,f_2, defined in Section 1.) (Bi-)gradation in this basis is given by

$$\deg\varepsilon_{3m}=(m,m),\quad\deg\varepsilon_{3m-1}=(m,m-1),\quad\deg\varepsilon_{3m+1}=(m,m+1).$$

The subspace $\mathfrak{n}_+(\widetilde{A}_1)$ of $\mathfrak{g}^{\widetilde{A}_1}$ is spanned by ε_i, where $i>0$.
According to (vi) in Section 1, for $k>0$

$$H_k(\mathfrak{n}_+(\widetilde{A}_1))=H_k^{((k(k-1))/2,(k(k+1))/2)}\oplus H_k^{((k(k+1))/2,(k(k-1))/2)}=\mathbf{C}\oplus\mathbf{C}$$

(see Fig. 1), moreover, nontrivial elements of the spaces

$$H_k^{((k(k-1))/2,(k(k+1))/2)},H_k^{((k(k+1))/2,(k(k-1))/2)}$$

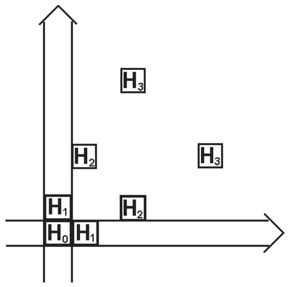


Fig. 1

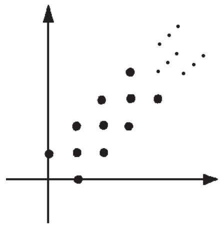


Fig. 2

are represented by cycles $\varepsilon_1 \wedge \varepsilon_4 \wedge \cdots \wedge \varepsilon_{3k-2}$, $\varepsilon_2 \wedge \varepsilon_5 \wedge \cdots \wedge \varepsilon_{3k-1}$ (see [5]). Since

$$\dim(\mathfrak{n}_+(\widetilde{A}_1))_{(m_1,m_2)} = \begin{cases} 1 & \text{if } |m_2 - m_1| \leqq 1, \quad m_2 + m_1 > 0, \\ 0 & \text{in all other cases} \end{cases}$$

(see Fig. 2), in the spectral sequence

$$\mathcal{E}(m_1, m_2) = \mathcal{E}(\mathfrak{n}_+(\widetilde{A}_1), \mathfrak{n}_+(\widetilde{A}_1)', m_1, m_2)$$

$$\dim E^1_k = \begin{cases} 2 & \text{if } k = 1,\ m_1 = m_2 \leqq 0, \\ 1 & \text{if } k - 1 \leqq |m_2 - m_1| \leqq k + 1, \quad m_1 + m_2 < k^2, \\ 0 & \text{in all other cases.} \end{cases}$$

(See Fig. 3; the circles and points show the degrees of the homology with trivial coefficients and the degrees of the nontrivial spaces E^1_k, respectively.)

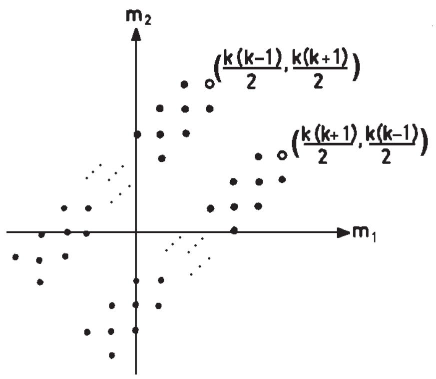


Fig. 3

So, the term E^1 of the spectral sequence $\mathcal{E}(m_1, m_2)$ is constructed in the following way. Let $l = |m_2 - m_1|$ and $m = \min(m_1, m_2)$. If $l > 0$, then the dimensions of the spaces E^1_k are given by the table

$k =$	$\dots$	$l-2$	$l-1$	l	$l+1$	$l+2$	$\dots$	
	$\dots$	0	1	1	1	0	$\dots$	for $m \leqq \frac{l^2-3l}{2}$,
	$\dots$	0	0	1	1	0	$\dots$	for $\frac{l^2-3l}{2} < m < \frac{l^2-l}{2}$,
	$\dots$	0	0	0	1	0	$\dots$	for $\frac{l^2-l}{2} \leqq m \leqq \frac{l^2+l}{2}$,
	$\dots$	0	0	0	0	0	$\dots$	for $\frac{l^2+l}{2} < m$,

and if $l = 0$, then by the table

$$\begin{array}{llllll} k = & \underline{0} & \underline{1} & \underline{2} & \underline{\dots} & \\ & 1 & 2 & 0 & \dots & \text{for } m < 0 \\ & 0 & 2 & 0 & \dots & \text{for } m = 0 \\ & 0 & 0 & 0 & \dots & \text{for } m > 0. \end{array}$$

Lemma. *The non-trivial differentials d_k^1 are the following ones:*

$$d_l^1 : E_l^1 \to E_{l-1}^1, \quad \text{if } l \neq 0, \quad m \leqq \frac{l^2 - 3l}{2},$$
$$d_1^1 : E_1^1 \to E_0^1, \qquad \text{if } l = 0, \quad m < 0;$$

the differentials d_k^r with $r > 1$ are all trivial.

From this lemma it follows

Proposition 2.1.

$$H_0(\mathfrak{n}_+(\widetilde{A}_1); \mathfrak{n}_+(\widetilde{A}_1)') = 0;$$

$$\dim H_1^{(m_1,m_2)}(\mathfrak{n}_+(\widetilde{A}_1); \mathfrak{n}_+(\widetilde{A}_1)') = \begin{cases} 2 & \text{if } m_1 = m_2 = 0, \\ 1 & \text{if } m_1 = m_2 < 0, \\ 0 & \text{in the other cases;} \end{cases}$$

if $k > 1$, then

$$\dim H_1^{(m_1,m_2)})(\mathfrak{n}_+(\widetilde{A}_1); \mathfrak{n}_+(\widetilde{A}_i)') =$$
$$= \begin{cases} 1 & \text{if } |m_1 - m_2| = k - 1, \ m_1 + m_2 < k^2 - 1 \\ & \text{and if } |m_1 - m_2| = k, \ (k-1)^2 < m_1 + m_2 \leqq k^2 - 2, \\ 0 & \text{in the other cases.} \end{cases}$$

(See Fig. 4, on which the weights of one- and two-dimensional homologies are shown.)

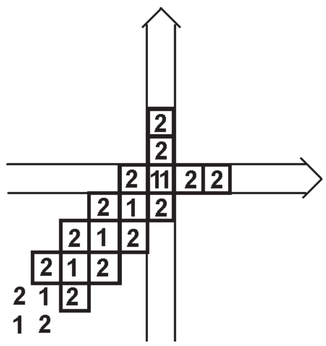


Fig. 4

Lemma may be proved by performing a straight but not particularly short calculation. Since we are interested only in homology of dimension one and two, we give the proof here only for the cases $k \leqq 2$. We have to show that the differentials

(i) d_1^1 for $m_2 = m_1 < 0$,
(ii) d_1^1 for $|m_2 - m_1| = 1$, $\min(m_1, m_2) \leqq -1$,
(iii) d_2^1 for $m_2 - m_1| = 2$, $\min(m_1, m_2) \leqq -1$,
(iv) d_3^1 for $|m_2 - m_1| = 3$, $\min(m_1, m_2) \leqq 0$

are non-trivial and

(v) d_3^1 for $m_1 = 2$, $m_2 = 0$ and $m_1 = 0$, $m_2 = 2$

is trivial. Since the roles of m_1 and m_2 are symmetric, we may consider only the case $m_1 \leqq m_2$. The differential d_k^1 in the spectral sequence $\mathcal{E}(m_1, m_2)$ is non-trivial if there exists a chain

$$c \in C_k^{(m_1,m_2)}(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1)') \text{ such that}$$
$$c = \varepsilon_1 \wedge \cdots \wedge \varepsilon_{3k-2} \otimes \varepsilon_1' + \dots$$
$$\partial c = \mu\varepsilon_1 \wedge \cdots \wedge \varepsilon_{3k-5} \otimes \varepsilon_j' + \dots$$

where $\mu \neq 0$ and dots in the general case stand for terms of smaller filtration. We find such chains for the cases (i)–(iv), putting $m = -m_2$.

(i) $c = \varepsilon_1 \otimes \varepsilon_{3m+1}'$; $\partial c = \varepsilon_{3m}'$,

(ii) $c = \varepsilon_1 \otimes \varepsilon_{3m}'$; $\partial c = -\varepsilon_{3m-1}'$,

(iii) $c = \varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_{3m}' - \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_{3m-1}'$; $\partial c = 2\varepsilon_1 \otimes \varepsilon_{3m-4}'$,

(iv) $c = \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon_{3m}' - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 \otimes \varepsilon_{3m-1}' - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6 \wedge \otimes \varepsilon_{3m-1}' + {}$
$+ \frac{3}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 \otimes \varepsilon_{3m-4}'$; $\partial c = -\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_{3m-7}'$.

The differential d_k^1 is trivial, if there is a chain c of the above form, for which $\partial c = 0$. For the case (v) such a chain is the following:

$$\text{(v)} \quad \begin{aligned} c = {} & \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon_{10}' + \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 \otimes \varepsilon_9' + \frac{1}{2}\varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6 \otimes \varepsilon_9' - \\ & -\frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_6 \otimes \varepsilon_8' - \varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 \otimes \varepsilon_6' - \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 \otimes \varepsilon_5'. \end{aligned}$$

Now we describe cycles, representing bases in $H_k(\mathfrak{n}_+(A_1);\mathfrak{n}_+(A_1)')$ for $k = 1, 2$.
In $C_1^{(0,0)}$: $\varepsilon_1 \otimes \varepsilon_1'$, $\varepsilon_2 \otimes \varepsilon_2'$.
In $C_1^{(m,m)}$, $m < 0$: $\varepsilon_1 \otimes \varepsilon_{-3m+1}' + \varepsilon_2 \otimes \varepsilon_{-3m+2}'$.
In $C_2^{(0,2)}$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_3' - \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_2'$.
In $C_2^{(1,2)}$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_1'$.
In $C_2^{(m,m+1)}$, $m \leqq 0$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_{-3m+4}' + \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_{-3m+3}' + \varepsilon_1 \wedge \varepsilon_2 \otimes \varepsilon_{-3m+2}'$.
Cycles in $C_2^{(2,0)}, C_2^{(m+1,m)}$ are given similarly, by substituting $\varepsilon_1 \leftrightarrow \varepsilon_2$, $\varepsilon_4 \leftrightarrow \varepsilon_5, \dots$.
Since $\dim H^k_{(m_1,m_2)} = \dim H_k^{(-m_1,-m_2)}$, the cohomology needed for us is completely computed. It is easy to see that the above result agrees with the corresponding parts of Theorems 1, 3.
Cocycles, representing basis elements of the cohomology spaces are indicated in the next table.

weight	cocycle	
$(0,-2)$	$(\varepsilon_1, \varepsilon_{3j} \mapsto \varepsilon_{3j-1}, (\varepsilon_1, \varepsilon_{3j+1}) \mapsto -\varepsilon_{3j}$ the rest $\mapsto 0$.	for $j > 0$,
$(-2,0)$	$(\varepsilon_2, \varepsilon_{3j}) \mapsto \varepsilon_{3j-2}$, $(\varepsilon_2, \varepsilon_{3j+2}) \mapsto -\varepsilon_{3j}$ the rest $\mapsto 0$.	for $j > 0$,
$(m, m-1)$ $m \geqq 0$	$(\varepsilon_1, \varepsilon_j) \mapsto j\varepsilon_{j+3m}$ the rest $\mapsto 0$.	for $j \neq 1$,
$(m-1, m)$ $m \geqq 0$	$(\varepsilon_2, \varepsilon_j) \mapsto j\varepsilon_{j+3m}$ the rest $\mapsto 0$.	for $j \neq 1$,
$(-1,-2)$	$(\varepsilon_1, \varepsilon_4) \mapsto 9\varepsilon_1$, $(\varepsilon_1, \varepsilon_j) \mapsto j\varepsilon_{j-3}$ $(\varepsilon_3, \varepsilon_{3j}) \mapsto 2\varepsilon_{3j-1}$, $(\varepsilon_3, \varepsilon_{3j-2}) \mapsto -2\varepsilon_{3j-3}$ $(\varepsilon_4, \varepsilon_{3j-1}) \mapsto 5\varepsilon_{3j-1}$, $(\varepsilon_4, \varepsilon_{3j+4}) \mapsto -5\varepsilon_{3j+4}$ the rest $\mapsto 0$.	for $j \geqq 5$, for $j \geqq 2$, for $j \geqq 1$,
$(-2,-1)$	$(\varepsilon_2, \varepsilon_5) \mapsto 9\varepsilon_2$, $(\varepsilon_2, \varepsilon_j) \mapsto j\varepsilon_{j-3}$ $(\varepsilon_3, \varepsilon_{3j}) \mapsto \varepsilon_{3j-2}$, $(\varepsilon_3, \varepsilon_{3j+2}) \mapsto -\varepsilon_{3j}$ $(\varepsilon_5, \varepsilon_{3j-2}) \mapsto 4\varepsilon_{3j-2}$, $(\varepsilon_5, \varepsilon_{3j+5}) \mapsto -4\varepsilon_{3j+5}$ the rest $\mapsto 0$.	for $j = 4, 6, 7, \dots$, for $j \geqq 1$, for $j \geqq 1$,

We can easily verify that the indicated cochains are really cocycles and they do not vanish on the above cycles.
It remained to show that infinitesimal deformations, determined by two-dimensional cocycles of weight $(0,-2)$, $(-2,0)$ and $(m+1,m)$, $(m,m+1)$ with $m \geqq -1$ can be

extended to real deformations, while infinitesimal deformations of weight $(-1,-2)$, $(-2,-1)$ can not. The extensions in question are explicitly given in Section 1. On the other hand, the cocycles of weight $(-1,-2)$, $(-2,-1)$ have nontrivial squares; for instance the first of them takes the value 135 at the cycle

$$\varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon_4' + \frac{1}{2}(\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 + \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6) \otimes \varepsilon_3' - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_6 \otimes \varepsilon_2'.$$

3. Let us now consider the case BA_2. The corresponding Cartan matrix is $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. The quotient algebra of $\mathfrak{g}^{BA_2}$ by its center has explicit description. Namely, it contains an additive basis ε_i $(i \in \mathbf{Z})$ with $[\varepsilon_i, \varepsilon_j] = \alpha_{ij}\varepsilon_{i+j}$, where α_{ij} depends only on i, j mod 8, $\alpha_{i,j} + \alpha_{i',j'} = 0$ if $i + i'$ and $j + j'$ are multiples of 8, and for $0 \leqq i,\ j \leqq 7$ it is given in the following table:

i mod 8 \ j mod 8	1	2	3	4	5	6	7
0	1	−2	−1	0	1	2	−1
1		1	−1	3	−2	0	1
2			0	0	1	−1	
3				−3	−1		

Gradation is given by formulas

$$\deg \varepsilon_{8m} = (2m, 4m), \quad \deg \varepsilon_{8m+1} = (2m, 4m+1), \quad \deg \varepsilon_{8m+2} = (2m+1, 4m),$$
$$\deg \varepsilon_{8m+3} = (2m+1, 4m+1), \quad \deg \varepsilon_{8m+4} = (2m+1, 4m+2),$$
$$\deg \varepsilon_{8m+5} = (2m+1, 4m+3)$$
$$\deg \varepsilon_{8m+6} = (2m+1, 4m+4), \quad \deg \varepsilon_{8m+7} = (2m+2, 4m+3).$$

The subalgebra $\mathfrak{n}_+(BA_2)$ is spanned by ε_i; with $i > 0$.

By (vi) from Section 1, for $k > 0$

$$H_{2k-1}(\mathfrak{n}_+(BA_2)) = H_{2k-1}^{((3k^2-k)/2, 3k^2-4k+1)} \oplus H_{2k-1}^{((3k^2-5k+2)/2, 3k^2-2k)} = \mathbf{C} \oplus \mathbf{C},$$

$$H_{2k}(\mathfrak{n}_+(BA_2)) = H_{2k}^{((3k^2+k)/2, 3k^2-2k)} \oplus H_{2k}^{((3k^2-k)/2, 3k^2+2k)} = \mathbf{C} \oplus \mathbf{C}$$

(see Fig. 5) and nontrivial elements of the homology in question are represented with the cycles

$$(\varepsilon_2 \wedge \varepsilon_{10} \wedge \cdots \wedge \varepsilon_{8k-6}) \wedge (\varepsilon_3 \wedge \varepsilon_7 \wedge \cdots \wedge \varepsilon_{4k-5}),$$
$$(\varepsilon_6 \wedge \varepsilon_{14} \wedge \cdots \wedge \varepsilon_{8k-10}) \wedge (\varepsilon_1 \wedge \varepsilon_5 \wedge \cdots \wedge \varepsilon_{4k-3}),$$
$$(\varepsilon_2 \wedge \varepsilon_{10} \wedge \cdots \wedge \varepsilon_{8k-6}) \wedge (\varepsilon_3 \wedge \varepsilon_7 \wedge \cdots \wedge \varepsilon_{4k-1}),$$
$$(\varepsilon_6 \wedge \varepsilon_{14} \wedge \cdots \wedge \varepsilon_{8k-2}) \wedge (\varepsilon_1 \wedge \varepsilon_5 \wedge \cdots \wedge \varepsilon_{4k-3}).$$

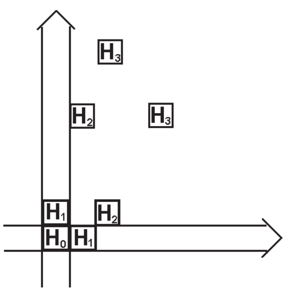


Fig. 5

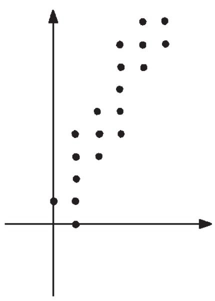


Fig. 6

The dimensions of the spaces $\mathfrak{n}_+(BA_2)_{(m_1,m_2)}$ equal to 0 and 1; the points (m_1, m_2) corresponding to spaces of dimension 1 are shown on Fig. 6.

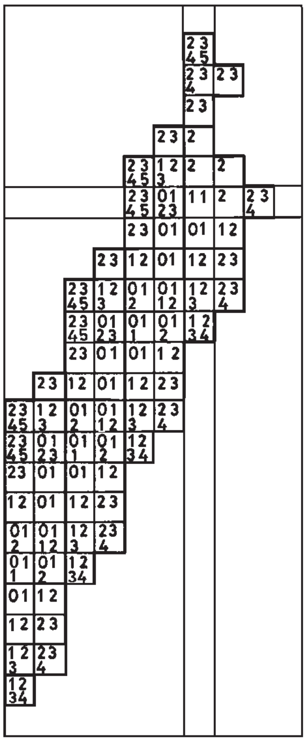


Fig. 7

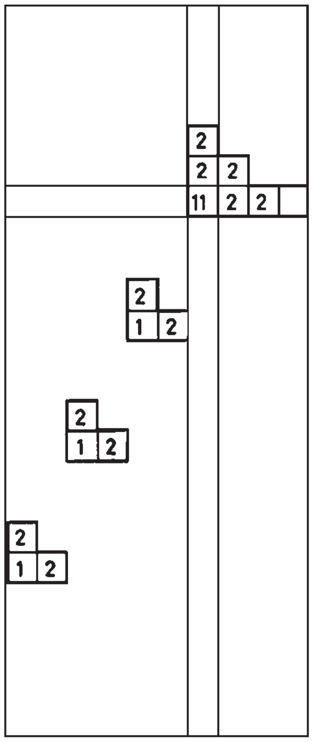


Fig. 8

In this way we can determine the dimensions of the spaces, forming the initial terms of the spectral sequences $\mathcal{E}_{(m_1,m_2)} = \mathcal{E}(\mathfrak{n}_+(BA_2), \mathfrak{n}_+(BA_2)', m_1, m_2)$. We restrict ourselves to (m_1, m_2) such that the space $\bigoplus_{k=0}^{2} E_k^1$ is nontrivial. These (m_1, m_2) are represented by small cells on Fig. 7. On this figure the cell (m_1, m_2) contains as many k's as the dimension of E_k^1 (for instance, in the spectral sequence $\mathcal{E}(-1,-3)$ the dimensions of E_k^1 are $1, 2, 1, 0, 0, \dots$). We remark that the left half plane on Fig. 7 is periodic with period 2 on the abscissa axis and with period 4 on the ordinate axis. The action of the differentials in these spectral sequences may be calculated in the same way as in subsection 2.2. The result of the computations is shown on Fig. 8: the number of the 1's and 2's in the cell (m_1, m_2) equals to the dimension of $H_1^{(m_1,m_2)}$ and $H_2^{(m_1,m_2)}$, respectively.

Now we describe the cycles, which represent the basis in $H_k(BA_2, BA_2')$, $k = 1, 2$.

In $C_1^{(0,0)}$: $\varepsilon_1 \otimes \varepsilon_1'$, $\varepsilon_2 \otimes \varepsilon_2'$.

In $C_1^{(2m,4m)}$, $m < 0$: $2\varepsilon_1 \otimes \varepsilon'_{-8m+1} + \varepsilon_2 \otimes \varepsilon'_{-8m+2}$.

In $C_2^{(0,2)}$: $\varepsilon_1 \wedge \varepsilon_6 \otimes \varepsilon_5' + \frac{2}{3}\varepsilon_1 \wedge \varepsilon_5 \otimes \varepsilon_4' + \frac{2}{9}\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_3' - \frac{2}{9}\varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_2'$.

In $C_2^{(1,1)}$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon_2'$.

In $C_2^{(2,0)}$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon_1'$.

In $C_2^{(2m,4m+1)}$, $m \leqq 0$: $\varepsilon_1 \wedge \varepsilon_6 \otimes \varepsilon'_{-8m+6} - \varepsilon_1 \wedge \varepsilon_5 \otimes \varepsilon'_{-8m+5} + \varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon'_{-8m+4} -$

$$-\varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon'_{-8m+3} + \varepsilon_1 \wedge \varepsilon_2 \otimes \varepsilon'_{-8m+2}.$$

In $C_2^{(2m+1,4m)}$, $m \leqq 0$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon'_{-8m+3} - \varepsilon_2 \wedge \varepsilon_1 \otimes \varepsilon'_{-8m+1}$.

So, the cohomology needed for us is computed. It is easy to see that the above result agrees with Theorems 1, 2.

Cocycles, representing basis elements of the cohomology spaces are indicated in the next table.

weight	cocycle
$(0,-2)$	$(\varepsilon_1, \varepsilon_{8j} \mapsto -\varepsilon_{8j-1}$, $(\varepsilon_1, \varepsilon_{8j+1}) \mapsto -\varepsilon_{8j}$ $(j \geqq 0)$, $\left.\begin{array}{l}(\varepsilon_1, \varepsilon_{8j+3}) \mapsto -2\varepsilon_{8j+2},\ (\varepsilon_1, \varepsilon_{8j+4}) \mapsto 3\varepsilon_{8j+3}\\ (\varepsilon_1, \varepsilon_{8j+5}) \mapsto -\varepsilon_{8j+4},\ \ (\varepsilon_1, \varepsilon_{8j+6}) \mapsto \varepsilon_{8j+5}\end{array}\right\\} j \geqq 0$ the rest $\mapsto 0$.
$(-1,-1)$	$\left.\begin{array}{c}(\varepsilon_1, \varepsilon_{8j-1}) \mapsto \varepsilon_{8j-3},\ (\varepsilon_1, \varepsilon_{8j}) \mapsto -2\varepsilon_{8j-2}\\ (\varepsilon_1, \varepsilon_{8j+2}) \mapsto \varepsilon_{8j},\ (\varepsilon_1, \varepsilon_{8j+3}) \mapsto -\varepsilon_{8j+1}\\ (\varepsilon_3, \varepsilon_{j+1}) \mapsto \varepsilon_{8j+1}\end{array}\right\\} j \geqq 1$ $\left.\begin{array}{l}(\varepsilon_3, \varepsilon_{8j+2}) \mapsto -2\varepsilon_{8j+2},\ (\varepsilon_3, \varepsilon_{8j+5}) \mapsto \varepsilon_{8j+5}\\ (\varepsilon_3, \varepsilon_{8j+6}) \mapsto 2\varepsilon_{8j+6},\ (\varepsilon_3, \varepsilon_{8j+7}) \mapsto -\varepsilon_{8j+7}\end{array}\right\\} j \geqq 0$ $(\varepsilon_1, \varepsilon_3) \mapsto -2\varepsilon_1$, the rest $\mapsto 0$.
$(-2,0)$	$(\varepsilon_2, \varepsilon_{8j}) \mapsto 2\varepsilon_{8j-2}$, $(\varepsilon_2, \varepsilon_{8j+2}) \mapsto -\varepsilon_{8j}$ $(j \geqq 1)$ $(\varepsilon_2, \varepsilon_{8j+3}) \mapsto \varepsilon_{8j+1}$, $(\varepsilon_2, \varepsilon_{8j+7}) \mapsto -\varepsilon_{8j+5}$ $(j \geqq 0)$ the rest $\mapsto 0$.
$(2m, 4m-1)$ $m \geqq 0$	$(\varepsilon_1, \varepsilon_j) \mapsto j\varepsilon_{j+8m}$ for $j \neq 1$, the rest $\mapsto 0$.
$(2m-1, 4m)$ $m \geqq 0$	$(\varepsilon_2, \varepsilon_j) \mapsto j\varepsilon_{j+8m}$ for $j \neq 2$, the rest $\mapsto 0$.

4. Now consider the case $\widetilde{A}_{n-1}$ with $n \geqq 3$. The case $n = 3$ is somewhat different from the general case (the main difference, from our point of view, is in the structure of the three-dimensional homology with trivial coefficients). Nevertheless, the final formula is the same, and the differences in the proofs are not essential. Therefore from now on we shall ignore the specific case $n = 3$, indirectly assuming that $n \geqq 4$.
The Cartan matrix of $\mathfrak{g}^{\widetilde{A}_{n-1}}$ is:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

By (vi) in Section 1

$$\dim H_*^{(m_1,\dots,m_n)}(\mathfrak{n}_+(\widetilde{A}_{n-1})) = \begin{cases} 1, & \text{if } P(m_1,\dots,m_n) = 0, \\ 0 & \text{in the other cases,} \end{cases}$$

where $P(m_1,\dots,m_n) = m_1^2 + \dots + m_n^2 - (m_1m_2 + \dots + m_{n-1}m_n + m_nm_1) - (m_1 + \dots + m_n)$. In more details, if $k = 1, 2, 3$ then the space $H_k^{(m_1\dots,m_n)}(\mathfrak{n}_+(\widetilde{A}_{n-1}))$ has dimension 1 for the following sequences $(m_1,\dots,m_n)$:

$$k = 0: \ (0,\dots,0); \quad k = 1: \ (1, 0, \dots, 0);$$
$$k = 2: \ (2, 1, 0, \dots, 0), \ (1, \underbrace{0,\dots,0}_{>0}, 1, \underbrace{0,\dots,0}_{>0});$$

$$\begin{aligned} k = 3: \ & (2, 2, 0, \dots, 0), (2, 1, 2, 0, \dots, 0), \\ & (3, 2, 1, 0, \dots, 0), (1, 3, 1, 0, \dots, 0), \\ & (2, 1, \underbrace{0,\dots,0}_{>0}, 1, \underbrace{0,\dots,0}_{>0}), (1, \underbrace{0,\dots,0}_{>0}, 1, \underbrace{0,\dots,0}_{>0}, 1, \underbrace{0,\dots,0}_{>0}), \end{aligned}$$

and also for the cases, obtained from these by cyclic permutation and reflection; for the remaining $(m_1,\dots,m_n)$ the named homology is 0.
Next we give cycles which represent generators of the above homology (ε_{ij} here and below stand for the matrix with 1 in the section of ith row and jth column and 0

elsewhere).

$$\begin{array}{ll} 1, & \varepsilon_{12}, \\ \varepsilon_{12}\wedge\varepsilon_{i,i+1}, & \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{23}, \\ \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{23}, & \varepsilon_{12}\wedge\varepsilon_{14}\wedge\varepsilon_{34}, \\ \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{14}, & \varepsilon_{13}\wedge\varepsilon_{23}\wedge\varepsilon_{24}, \\ \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{i,i+1}, & \varepsilon_{12}\wedge\varepsilon_{i,i+1}\wedge\varepsilon_{j,j+1}, \end{array}$$

where $\varepsilon_{n,n+1} = \varepsilon_{n,1}t$ by definition. Similarly, if as the result of cyclic permutation, we find the first index to be larger than the second one, we have to multiply ε by t. Now we can determine the dimensions of the space which form the initial terms of the spectral sequences

$$\mathcal{E}(m_1,\dots,m_n) = \mathcal{E}(\mathfrak{n}_+(\widetilde{A}_{n-1}), \mathfrak{n}_+(\widetilde{A}_{n-1})', m_1,\dots,m_n).$$

$(m_1,\dots,m_n)$	dim E^1_0	dim E^1_1	dim E^1_2	dim E^1_3	
	n-1	n	0	0	*
	0	n	0	0	(m=0)*
	1	n-1	n-1	0	*
	0	0	n-1	0	(m=0)*
	1	2	2	1	
	0	0	2	1	(m=0)*
	1	2	1	0	
	0	2	n-1	n-2	
	0	1	2	1	
	0	1	2	1	
	0	0	2	1	(m=0)
	0	1	n-1	n-1	
	0	1	2	1	
	0	1	2	1	
	0	0	2	≥2	

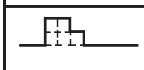
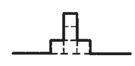
$(m_1,\dots,m_n)$	dim E^1_0	dim E^1_1	dim E^1_2	dim E^1_3
	0	0	1	≥1
	0	0	1	≥1
	0	0	2	2
	0	0	1	≥1
	0	0	1	1
	0	0	2	≥2
	0	0	1	≥1
	0	0	1	≥1
	0	0	1	≥1
	0	0	1	≥1
	0	0	3	≥3

Table 3

We restrict ourselves to such $m_1,\dots,m_n$ that $\bigoplus_{k=0}^{2} E^1_k$ are nontrivial. The dimensions of E^1_k for these sequences are presented in Table 3.

In this table the sequence $(m_1,\dots,m_n)$ is presented as a graph: the thick broken line is the graph of the step function with equally long steps and $m_1, \dots, m_n$ sequence of values. The left end of the line corresponds to the level $-m(m_1 = -m)$. Whenever $m = 0$ it is written at the end of the row. All calculations and dimensions are the same for those $(m_1,\dots,m_n)$ which can be obtained by reflection and cyclic permutation from those ones in the table.

It is easy to compute the differentials of the spectral sequences and it turns out that homologies with dimension 1, 2 occur only in the cases which are marked in the table by stars. We calculate the differentials in these cases.

1° $$(m_1,\dots,m_n) = (-m,\dots,-m).$$

In this case E^1_0 is trivial for $m = 0$; and for $m > 0$ it is spanned by the classes of the chains

$$\alpha_i = \big((\varepsilon_{i,i} - \varepsilon_{i+1,i+1})t^m\big)',$$

and E^1_1 is always spanned by classes of the chains

$$\beta_i = \varepsilon_{i,i+1} \otimes (\varepsilon_{i,i+1}t^m)', \quad i = 1,\dots,n-1, \quad \beta_n = \varepsilon_{n,1}t \otimes (\varepsilon_{n,1}t^{m+1})'.$$

Evidently, $d\beta_i = \alpha_i$ for $i = 1, \dots, n-1$ and $d\beta_n = -\alpha_1 - \cdots - \alpha_{n-1}$. So,

$$\dim H_1^{(-m,\dots,-m)} = \begin{cases} 1 & \text{for } m > 0 \\ n & \text{for } m = 0, \end{cases} \qquad H_2^{(-m,\dots,-m)} = 0.$$

One-dimensional cohomologies for $m > 0$ are spanned by the class of the chain $\beta_1 + \cdots + \beta_n$, and for $m = 0$ by classes of the chains $\beta_1, \dots, \beta_n$.

2°. $$(m_1, \dots, m_n) = (\underbrace{-m, \dots, -m}_{i-1}, -m+1, -m, \dots, -m), \quad 1 \leqq i \leqq n.$$

In this case E_0^1 is trivial for $m = 0$, and for $m > 0$ it is spanned by the class of the chain

$$\alpha = (\varepsilon_{i+1,i} t^m)';$$

E_1^1 is trivial for $m = 0$, and for $m > 0$ it is spanned by the classes of the chains

$$\beta_j = \varepsilon_{i,i+1} \otimes \big((\varepsilon_{j,j} - \varepsilon_{j+1,j+1}) t^m\big)', \quad j = 1, \dots, n-1;$$

E_2^1 is always spanned by the classes of the chains

$$\begin{aligned} \gamma_j &= \varepsilon_{i,i+1} \wedge \varepsilon_{j,j+1} \otimes (\varepsilon_{j,j+1} t^m)', \quad j = 1, \dots, i-2, i+2, \dots, n-1 \\ \gamma_n &= \varepsilon_{i,i+1} \wedge \varepsilon_{n,1} t \otimes (\varepsilon_{n,1} t^{m+1})', \\ \gamma_{i-1} &= \varepsilon_{i-1,i+1} \wedge \varepsilon_{i,i+1} \otimes (\varepsilon_{i-1,i+1} t^m)', \\ \gamma_{i+1} &= \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \otimes (\varepsilon_{i,i+2} t^m)' \end{aligned}$$

(γ_i is absent). The differential $d = d^1$ acts by

$$d\beta_j = \begin{cases} -2\alpha & \text{for } j = i \\ \alpha & \text{for } j = i \pm 1, \\ 0 & \text{in the other cases;} \end{cases}$$

$$d\gamma_j = \begin{cases} \beta_j & \text{for } j \neq i,\ i \pm 1, \\ -\beta_1 - \cdots - \beta_{n-1} & \text{for } j = n, \\ -2\beta_{i-1} - \beta_i & \text{for } j = i-1, \\ \beta_i + 2\beta_{i+1} & \text{for } j = i+1. \end{cases}$$

So,

$$H_1^{(-m,\dots,-m+1,\dots,-m)} = 0,$$

$$\dim H_2^{(-m,\dots,-m+1,\dots,-m)} = \begin{cases} 1 & \text{for } m > 0 \\ n-1 & \text{for } m = 0. \end{cases}$$

The two-dimensional homologies for $m > 0$ are spanned by the class of the cycle $\gamma_1 + \cdots + \gamma_{i-2} + \frac{1}{2}\gamma_{i-1} + \frac{1}{2}\gamma_{i+1} + \gamma_{i+2} + \cdots + \gamma_n$, while for $m = 0$ by the classes of the cycles $\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n$.

3°. $$(m_1, \dots, m_n) = (\underbrace{0, \dots, 0}_{i-1}, 1, 1, 0, \dots, 0), \quad 1 \leqq i \leqq n-1.$$

In this case $E_0^1 = E_1^1 = 0$, E_2^1 is spanned by the classes of the chains

$$\gamma_1 = \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \otimes (\varepsilon_{i,i+1})', \quad \gamma_2 = \varepsilon_{i,i+2} \wedge \varepsilon_{i+1,i+2} \otimes (\varepsilon_{i+1,i+2})',$$

E_3^1 is spanned by the class of

$$\delta = \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \wedge \varepsilon_{i+1,i+2} \otimes (\varepsilon_{i,i+2})';$$

the differential acts by $d\delta = \gamma_1 - \gamma_2$. That means,

$$H_1^{(0,\dots,0,1,1,0,\dots,0)} = 0$$
$$\dim H_2^{(0,\dots,0,1,1,0,\dots,0)} = 1.$$

The two-dimensional homologies are spanned by the class of γ_1 (or γ_2).
The case $(m_1, \dots, m_n) = (1, 0, \dots, 0, 1)$ is similar to the above one.

4°. $$(m_1,\dots,m_n)=(\underbrace{0,\dots,0}_{i-1},2,0,\dots,0),\quad 1\leqq i\leqq n.$$

In this case $E_0^1=E_1^1=0$, E_2^1 is spanned by the classes of

$$\gamma_i=\varepsilon_{i,i+1}\wedge\varepsilon_{i,i+2}\otimes(\varepsilon_{i+1,i+2})',\quad \gamma_2=\varepsilon_{i-1,i+1}\wedge\varepsilon_{i,i+1}\otimes(\varepsilon_{i-1,i})',$$

E_3^1 is spanned by the class of the chain

$$\delta=\varepsilon_{i-1,i+1}\wedge\varepsilon_{i,i+1}\wedge\varepsilon_{i,i+2}\otimes(\varepsilon_{i-1,i+2})';$$

the differential acts by $d\delta=\gamma_1-\gamma_2$. That means,

$$H_1^{(0,\dots,0,2,0,\dots,0)}=0,$$
$$\dim H_2^{(0,\dots,0,2,0,\dots,0)}=1.$$

The two-dimensional homologies are spanned by the class of γ_1 (or γ_2).
As usually, we have isomorphism between the cohomology and homology. As it is clear from the list of deformations given before Theorem 1 in Section 1, all classes of two-dimensional cohomologies are represented by deformations of $\mathfrak{n}_+(\widetilde{A}_{n-1})$.

5. The general case of an affine algebra $\mathfrak{g}^A$ for $A\neq\widetilde{A}_1$ is quite similar to the above case. We restrict ourselves to formulate the final result.

$$\dim H_1^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A);\mathfrak{n}_+(A)')=$$
$$=\begin{cases} n & \text{for } (m_1,\dots,m_n)=(0,\dots,0),\\ 1 & \text{for } (m_1,\dots,m_n)=(-ml\omega_1,\dots,-ml\omega_n),\quad m>0,\\ 0 & \text{in all other cases,}\end{cases}$$

where $\omega_1,\dots,\omega_n$ are the coefficients of linear dependence between the columns of the Cartan matrix, while l equals to 1 for the current algebras (Table 1) and for the matrices from Table 2 is indicated in (v), Section 1.

$$\dim H_2^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A);\mathfrak{n}_+(A)')=$$
$$=\begin{cases} n-1 & \text{for } (m_1,\dots,m_n)=(\underbrace{0,\dots,0}_{i-1},1,0,\dots,0),\quad 1\leqq i\leqq n\\ 1 & \text{for } (m_1,\dots,m_n)=\\ & \quad=(-ml\omega_1,\dots,-ml\omega_{i-1},-ml\omega_i+1,-ml\omega_{i+1},\dots,-ml\omega_n)\\ & \qquad 1\leqq i\leqq n,\ m>0\\ 1 & \text{for } (m_1,\dots,m_n)=(\underbrace{0,\dots,0}_{i-1},2,0,\dots,0),\quad 1\leqq i\leqq n\\ 1 & \text{for } (m_1,\dots,m_n)=(\underbrace{\underbrace{0,\dots,0}_{i-1},1,0,\dots,0}_{j-1},1,0,\dots,0)\\ & \qquad 1\leqq i<j\leqq n,\ a_{ij}\neq 0\\ 0 & \text{in all other cases.}\end{cases}$$

As in the previous case, $H_k^{(m_1,\dots,m_n)}=H^k_{(-m_1,\dots,-m_n)}$, and all the two-dimensional cohomologies are represented as deformations.

References

[1] Kac, V. G., Simple irreducible graded Lie algebras of finite growth, *Izv. Akad. Nauk. SSSR Ser. Mat.* **32** (1968), 1323–1367 (in Russian). MR 41 # 4590.

[2] Moody, R. V., A new class of Lie algebras, *J. Algebra* **10** (1968), 211–230. MR 37 # 5261.

[3] Macdonald, I. G., Affine Lie algebras and modular forms, *Séminaire Bourbaki*, 33e année, 1980–81, no. 577. MR 84i: 17014.

[4] Garland, H., Lepowsky, J., Lie algebra homology and the Macdonald–Kac formulas, *Invent. Math.* **34** (1976), 37–76. MR 54 # 2744.

[5] Fuchs, D. B., *Cohomology of infinite-dimensional Lie algebras*, Moscow, Nauka, 1983 (in Russian).

[6] Feigin, B. L., Fuchs, D. B.: Homology of the Lie algebra of vector fields on the line, *Funkcional Anal. i Priložen.* **14** (1980), 45–60, 96 (in Russian). MR 82b: 17017.

[7] Feigin, B. L. and Fialowski, A., About the cohomology of nilpotent loop algebras, *Dokl. Akad. Nauk SSSR* **271** (1983), 813–816 (in Russian). MR 84k: 17013.

APPEARED IN: *Advances in Math.*, **97** (1993), 267–277

On the Cohomology of Infinite Dimensional Nilpotent Lie Algebras

Alice Fialowski
Department of Mathematics
University of California, Davis

Abstract In the paper one- and two-dimensional cohomology is compared for finite and infinite nilpotent Lie algebras, with coefficients in the adjoint representation. It turns out that, because the adjoint representation is not a highest weight representation in infinite dimension, the considered cohomology shows basic differences.

On my visit to M.I.T., B. Kostant asked the following question: What is the main difference between the cohomology of finite and infinite dimensional nilpotent Lie algebras with coefficients in the adjoint representation, at what points does the generalization of the finite dimensional situation fail?
Understanding this difference is especially important as the nilpotent Lie algebra cohomology is very hard to compute and in both finite and infinite dimensional cases only the one- and two-dimensional cohomology is known so far.

1. Comparison of the One-Dimensional Cohomology Spaces

Let us recall the result on the Lie algebra cohomology $H^1(\mathfrak{n}, \mathfrak{n})$ where $\mathfrak{n}$ is the maximal nilpotent ideal of a Borel subalgebra of a finite dimensional simple Lie algebra $\mathfrak{g}$. The result can be obtained from Theorem 5.14 of [K]. Leger and Luks deduced the structure of $H^1(\mathfrak{n};\mathfrak{n})$ from Kostant's general result.
Suppose that the dimension of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is l.

Theorem 1.1 ([L-L, Theorem 3.1]). *Except for the Lie algebra $s\ell_2$,*

$$H^1(\mathfrak{n};\mathfrak{n}) \cong \mathfrak{h} \oplus \mathfrak{h}.$$

For $s\ell_2$, $\dim H^1(\mathfrak{n};\mathfrak{n}) = 1$.

Let us note that in finite dimension, $\mathfrak{n}$ is a highest weight representation. Consider the root space decomposition of $\mathfrak{n}$:

$$\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha.$$

The Weyl group in this case is generated by reflections on the simple roots $s_{\alpha_1}, \ldots, s_{\alpha_l}$.
The one-dimensional cocycles arise from two different sources:
(i) $D_h = \operatorname{ad} h_{\alpha_i}$. The number of these cocycles is l.
(ii) Let λ be a highest weight of $\mathfrak{g}$.

$$D'_\alpha(e_\beta) = \begin{cases} e_{s_\alpha(\lambda)} & \text{if } \beta = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

The number of such cocycles is l.
The bracket operation in the Lie algebra $H^1(\mathfrak{n};\mathfrak{n})$ is

$$[D_h, D'_\alpha] = \big(s_\alpha(\lambda) - \alpha\big) D'_\alpha.$$

On the other hand, the result in the analogous nilpotent affine Kac–Moody cases is completely different. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the Cartan decomposition of an affine algebra $\hat{\mathfrak{g}}$. The second type (ii) of cocycles does not arise in infinite dimension,

*The work was partly done during a fellowship of the Alexander von Humboldt-Stiftung at the Max-Planck-Institut für Mathematik, Bonn, Germany.

because $\hat{\mathfrak{n}}_+$ is not a highest weight representation. Instead, another algebra – now infinite dimensional – appears.

Theorem 1.2. *For an affine Lie algebra* $\hat{\mathfrak{g}}$,

$$H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \cong \hat{\mathfrak{h}} \oplus L_0,$$

where L_0 *is a subalgebra of the Virasoro algebra, isomorphic to the Lie algebra of polynomial vector fields on the line, vanishing at the origin.*

Remark 1.3. Theorem 1.2 without proof was stated in a previous work of the author with B. Feigin [F-F]. In [F] a proof was given by direct computation, counting explicitly the cocycles in $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ in each affine case. Here we give another proof which shows more of the critical points of the difference between finite and infinite dimensional cases.

Proof. Consider the following exact sequence of $\hat{\mathfrak{n}}_+$-modules:

$$0 \to \hat{\mathfrak{n}}_+ \to \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+ \to 0.$$

Here $\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+$ is isomorphic to $\hat{\mathfrak{h}} + \hat{\mathfrak{n}}_-$ as vector space and $\hat{\mathfrak{n}}_-$ is isomorphic to $\hat{\mathfrak{n}}_+^*$ by means of the Killing form. Consider the induced cohomology sequence:

$$\begin{aligned} H^0(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) &\to H^0(\hat{\mathfrak{n}}_+;\hat{\mathfrak{h}}+\hat{\mathfrak{n}}_+^*) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \\ &\to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{h}}+\hat{\mathfrak{n}}_+^*) \to \cdots . \end{aligned}$$

As $H^0(\hat{\mathfrak{n}}_+;\hat{\mathfrak{h}}+\hat{\mathfrak{n}}_+^*)$ is isomorphic to $\hat{\mathfrak{h}}$ and in $\hat{\mathfrak{g}}$ there are no invariant elements, we have

$$0 \longrightarrow \hat{\mathfrak{h}} \xrightarrow{\partial} H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \xrightarrow{\mu} H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \xrightarrow{\nu} H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{h}}+\hat{\mathfrak{n}}_+^*) \longrightarrow \cdots .$$

Each element $h_{\alpha_i} \in \hat{\mathfrak{h}}$ defines a nontrivial cohomology class in $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ (just as in finite dimension) with the cocycle

$$D_h = \operatorname{ad} h_{\alpha_i}, \qquad h_{\alpha_i} \in \hat{\mathfrak{h}}.$$

The number of such cocycles is the number of simple roots ($= \operatorname{rk}\hat{\mathfrak{g}}$). On the other hand, by the infinite dimensional analogue of the Bott–Kostant Theorem [F-F],

$$H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \cong \mathbb{C}[t,t^{-1}] = \mathbb{C}[t] + t^{-1}\mathbb{C}[t^{-1}].$$

Each cohomology class can be represented by the cocycle of the form

$$f(t)\frac{\partial P}{\partial t},$$

where

$$f(t) \in \mathbb{C}[t,t^{-1}] \text{ and } P(t) = x_0 + tx_1 + t^2x_2 + \cdots \in \hat{\mathfrak{n}}_+.$$

Proposition 1.4. *The kernel of the map* $\nu : H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \to H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+)$ *is* $t\mathbb{C}[t](\partial/\partial t)$.

Proof. Any cocycle corresponding to a polynomial vector field $P(t)(\partial/\partial t)$ is of the form $\omega_p : f(t) \mapsto P(\partial f/\partial t)$. So, if $P \in \mathbb{C}[t](\partial/\partial t)$, then for any $f(t) \in \hat{\mathfrak{n}} = \mathfrak{n} + t\mathfrak{g} + t^2\mathfrak{g} + \cdots$, $P(t)(\partial f/\partial t)$ is in $\hat{\mathfrak{n}}_+$ and therefore $\nu(\omega_p) = 0$.
On the other hand, assume that $P \in \mathbb{C}[t^{-1}]$ and $\nu(\omega_p) = 0$. This means that for some P_0 from $\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+$, ω_p is the differential of P_0. By definition of the differential, for any $f \in \hat{\mathfrak{n}}_+$,

$$P\frac{\partial f}{\partial t} - [P_0, f] \in \hat{\mathfrak{n}}_+ = \mathfrak{n} + t\mathfrak{g} + t^2\mathfrak{g} + \cdots . \tag{1.4}$$

Let $P = \alpha_n t^{-n} + \cdots + \alpha_0$; $P_0 = A_n t^{-n} + \cdots + A_0$ where $\alpha_i \in \mathbb{C}$, $A_i \in \mathfrak{g}$. Apply (1.4) to $f = X_0 \in \mathfrak{n}$ (constant polynomial). Thus, for any $X_0 \in \mathfrak{n}$, $\sum_{i=0}^n [A_i, X_0]t^{-i} \in \hat{\mathfrak{n}}_+$; from this it follows that $[A_i, X_0] = 0$ for $i > 0$ and $[A_0, X_0] \in \mathfrak{n}$. So, A_i for $i > 0$ are the multiples of the highest weight vector $v_\lambda \in \mathfrak{g}$; $A_0 \in \mathfrak{h} + \mathfrak{n} \subset \mathfrak{g}$; $P_0(t) = (t^{-n}\beta_n + \cdots + t^{-1}\beta_1)v_\lambda + A_0$. Now, apply (1.4) to $f(t) = tX_1$ where $X_1 \in \mathfrak{g}$. We get $(\alpha_n t^{-n} + \cdots + \alpha_0)X_1 - [A_0, X_1]t - (\beta_n t^{-n} + \cdots + \beta_1 t^{-1})[v_\lambda, X_1]t \in \hat{\mathfrak{n}}_+$. Comparing coefficients near t^{-i}, we get $\alpha_n = 0$; $\alpha_i X_1 = \beta_{i+1}[v_\lambda, X_1]$ for $i > 0$; but this may be so for any $X_1 \in \mathfrak{g}$ only when $\alpha_i = \beta_{i+1} = 0$. Also we get $\alpha_0 X_1 \equiv \beta_1[v_\lambda, X_1] \pmod{)n}$

for any $X_1 \in \mathfrak{g}$, which implies $\alpha_0 = \beta_1 = 0$. So, we proved that $\ker \nu$ is exactly $t\mathbb{C}[t](\partial/\partial t)$.
Now we have the following cohomology sequence:

$$0 \longrightarrow \hat{\mathfrak{h}} \xrightarrow{\partial} H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \longrightarrow \mathbb{C}[t] \longrightarrow 0.$$

The second type of nontrivial cocycles from $H^1(\hat{\mathfrak{n}}_+,\hat{\mathfrak{n}}_+)$ have the form

$$P \mapsto f(t)P'(t), \qquad \text{where } \ f(t) \in \mathbb{C}[t].$$

The nonequivalent cocycles of this type form a Lie algebra, isomorphic to L_0.

Note. The difference between the finite and infinite dimensional case is that in finite dimension, by the Bott–Kostant Theorem [K] the dimension of $H^1(\mathfrak{n};\mathfrak{g})$ is equal to the elements of length l in the Weyl group, while here we have

$$H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \cong \mathbb{C}[t,t^{-1}].$$

2. Method of Computation for the Two-Dimensional Cohomology

Using Kostant's results [K], Leger and Luks [L-L] computed $H^2(\mathfrak{n};\mathfrak{n})$ for finite dimensional simple Lie algebras $\mathfrak{g}$. Their main idea is the following. Consider the next exact sequences of $\mathfrak{n}$-modules:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathfrak{n} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathfrak{g} & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g}/\mathfrak{n} & \longrightarrow & \mathfrak{n}^* & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

These induce the following exact cohomology sequences:

$$\begin{array}{ccccccccccc}
 & & & & & & H^1(\mathfrak{n};\mathfrak{n}) & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & H^1(\mathfrak{n};\mathfrak{g}) & & & & \\
 & & l & & l^2 & & \downarrow & & l^2+l-1 & & \\
H^0(\mathfrak{n};(\mathfrak{h}+\mathfrak{n})^*) & \to & H^0(\mathfrak{n};\mathfrak{n}^*) & \overset{i}{\hookrightarrow} & H^1(\mathfrak{n};\mathfrak{h}) & \underset{l^2-l}{\overset{\partial}{\longrightarrow}} & H^1(\mathfrak{n};\mathfrak{h}+\mathfrak{n}^*) & \underset{2l-1}{\longrightarrow} & H^1(\mathfrak{n};\mathfrak{n}^*) & \to & H^2(\mathfrak{n};\mathfrak{h}) \quad (*) \\
 & & & & & & \downarrow & & \text{symmetric maps} & & \\
 & & & & & & H^2(\mathfrak{n};\mathfrak{n}) & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & H^2(\mathfrak{n};\mathfrak{g}) & & & & \\
 & & & & & & \downarrow & & & & \\
 & & & & & & H^2(\mathfrak{n};\mathfrak{g}/\mathfrak{n}) & & & &
\end{array}$$

$$B_{\alpha\beta} = 0 \quad \left.\begin{array}{r} \text{if } \alpha \text{ or } \beta \text{ not simple} \\ \text{or} \\ \text{if } \alpha \neq \beta \text{ and } \alpha+\beta \notin \Delta \end{array}\right\}$$

The dimensions of the cohomology spaces are marked above the spaces.

We approach the space $H^2(\mathfrak{n};\mathfrak{n})$ step by step, studying the above diagram.
The elements of $H^0(\mathfrak{n};\mathfrak{n}^*)$ are the invariant elements of $\mathfrak{n}^* : \{e_{-\alpha}\}$, where α is a simple root.

Cocycles in $H^1(\mathfrak{n};\mathfrak{h})$ have the form

$$\varphi(e_\gamma) = 0, \qquad \gamma \neq \alpha$$
$$\varphi(e_\alpha) = h_\beta, \quad \text{if } \alpha, \beta \text{ are simple roots.}$$

There are l^2 such cocycles. Their images look the same in $H^1(\mathfrak{n}, \mathfrak{h} + \mathfrak{n}^*)$. Obviously among them the ones with the form

$$\varphi(e_\alpha) = h_\alpha$$
$$\varphi(e_\gamma) = 0, \qquad \gamma \neq \alpha$$

are zero cocycles.
It is easy to see that i is embedding into $H^1(\mathfrak{n};\mathfrak{h})$.
The space $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is generated by $\langle e_\alpha\rangle$, α simple, and $\mathfrak{h} \cong \langle h_\alpha\rangle$, α simple and the rank of such a homomorphism is l^2.
We compute $\operatorname{im}\partial$: take $L : \big(\langle e_\alpha\rangle, \alpha \text{ simple}\big) \to \big(\langle h_\alpha\rangle, \alpha \text{ simple}\big)$, $L(e_\alpha) = \sum_\beta L_{\alpha\beta} h_\beta$. Then as $\operatorname{im}\partial$ we have to take the factor

$$\underset{\dim = l^2}{\langle L\rangle} \,/\, \underset{\dim = l}{\langle L : L_{\alpha\beta} = 0,\ \alpha \neq \beta\rangle}\,.$$

So from the left we get $l^2 - l$ cocycles.

3. The Spaces $H^1(\mathfrak{n};\mathfrak{n}^*)$ and $H^2(\mathfrak{n};\mathfrak{n})$

The main problem is to compute $H^1(\mathfrak{n};\mathfrak{n}^*)$. The cocycles representing the cohomology classes in $H^1(\mathfrak{n};\mathfrak{n}^*)$ are bilinear forms ϕ on $\mathfrak{n}$ such that

$$\phi([X,Y],Z) = -\phi(Y,[X,Z]) + \phi(X,[Y,Z]).$$

Leger and Luks state [Theorem 4.1] that the cocycles are exactly the ones obtained with the help of a symmetric invariant form B,

$$\phi(X,Y) = B(TX,Z) - B(TY,X),$$

where T is a derivation.
Note that Leger and Luks prove it for any finite dimensional Lie algebra, but the statements are true for any Lie algebra.
Now we have to count the symmetric bilinear forms on $\mathfrak{n}$. Their explicit form is

$$B(e_\alpha, e_\beta) = 1, \quad \alpha, \beta \text{ simple},$$
$$B(e_\gamma, e_\delta) = 0, \quad (\gamma,\delta) \neq (\alpha,\beta)\ (\gamma \text{ or } \delta \text{ is not simple}).$$

The number of such forms in finite dimension is

$$l(\alpha = \alpha) + \frac{l(l-1)}{2}(\alpha \neq \beta \text{ simple}) + \frac{l(l+1)}{2} = \frac{l^2+l}{2}.$$

By Theorem 4.1 of Leger and Luks, in $H^1(\mathfrak{n};\mathfrak{n}^*)$ we have $(l^2+l)/2$ classes. Consider the differential

$$H^1(\mathfrak{n};\mathfrak{n}^*) \xrightarrow{\partial} H^2(\mathfrak{n};\mathfrak{h}) = H^2(\mathfrak{n}) \otimes \mathfrak{h}.$$

Let us compute $\ker\partial$.
Suppose that we have a functional φ on $\mathfrak{n}$ defined by the cochain ϕ,

$$\varphi(X)(Y) = B(TX,Y) - B(TY,X) \quad \text{if } Y \in \mathfrak{n}$$

and assume that $\varphi(X)(H) = 0$ if $H \in \mathfrak{h}$. Let us continue it onto $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}^*$. Then its differential is

$$\begin{aligned}\partial\varphi &= X\varphi(Y)(H) - Y\varphi(X)(H) = \varphi(Y)([X,H]) - \varphi(X)([Y,H]) \\ &= -B(TY,[X,H]) + B(TX,[Y,H]) + B(T[X,H],Y) \\ &\quad - B(T[Y,H],X).\end{aligned}$$

Put $X = e_\alpha$, $Y = e_\beta$, where α, β are simple positive roots.

Then $TY = l(\beta)e_\alpha$ where $l(\beta)$ is the length of the root β and $[e_\alpha, H] = \alpha[H]e_\alpha$. The right hand side looks like

$$-l(\beta)\alpha(H)B(e_\alpha, e_\beta) + l(\alpha)\beta(H)B(e_\alpha, e_\beta) - l(\alpha)\alpha(H)B(e_\alpha, e_\beta) \\ + l(\beta)\beta(H)B(e_\alpha, e_\beta).$$

We get

$$\partial\varphi_{B_{\alpha,\beta}}(H) = (l(\alpha) + l(\beta))(\beta - \alpha)(H)B(e_\alpha, e_\beta).$$

As a consequence, if $\alpha = \beta$ then $\partial\varphi = 0$.
Let us assume $\alpha \neq \beta$. Then

$$\partial\varphi_{B_{\alpha,\beta}}(e_\alpha, e_\beta) = (l(\alpha) + l(\beta))(\check{\beta} - \check{\alpha}), \quad \text{where } \check{\alpha}, \check{\beta} \in \mathfrak{h}.$$

It is easy to see that the cocycle

$$\omega(e_\alpha, e_\beta) = 1, \qquad \omega(e_\gamma, e_\delta) = 0$$

is trivial. The space $H^2(\mathfrak{n})$ is isomorphic to $\oplus\{\mathbb{C}w, l(w) = 2,\, w \in W\}$:

$$\partial\varphi_{B_{\alpha,\beta}} = 0 \Leftrightarrow B(e_\alpha, e_\beta) = 0 \Leftrightarrow \alpha + \beta \in \Delta.$$

So we get $\ker\partial = \{\alpha + \beta \in \Delta \text{ and } \alpha = \beta\}$.
Now we have the exact sequence

$$0 \to \{B_{\alpha\beta}\} \to H^1(\mathfrak{n}; \mathfrak{n}^*) \to H^2(\mathfrak{n}; \mathbb{C}) \to 0.$$

The basis for the image of $H^2(\mathfrak{n}; \mathbb{C})$ in $H^1(\mathfrak{n}; \mathfrak{n}^*)$ is represented by cocycles obtained by symmetric bilinear forms with $\alpha + \beta \in \Delta$, or $\alpha < \beta$ and $\alpha + \beta \notin \Delta$. So we can write

$$\begin{array}{cccc} 0 \to & \underset{\alpha \neq \beta}{(\alpha, \beta)} \to H^1(\mathfrak{n}; \mathfrak{n}^* + \mathfrak{h}) & \to & B_{\alpha\beta} \\ & & & \left\{\begin{array}{l} \quad \alpha + \beta \notin \Delta \\ \text{or} \\ \quad \alpha = \beta \\ \quad \alpha, \beta \text{ simple} \end{array}\right\} \\ & H^1(\mathfrak{n}; \mathfrak{h}) \to H^1(\mathfrak{n}; \mathfrak{n}^* + \mathfrak{h}) & \to & H^1(\mathfrak{n}; \mathfrak{n}^*) \end{array}$$

from which it immediately follows that

$$\dim H^1(\mathfrak{n}; \mathfrak{n}^*) = \frac{l^2 + l}{2} + \frac{l}{2}(l + 2)(l - 1) = l^2 + l - 1$$

[L-L,Theorem 5.4].

Now it is easy to compute the nontrivial cocycles in $H^2(\mathfrak{n}; \mathfrak{n})$. The differential from $H^1(\mathfrak{n}; \mathfrak{g}/\mathfrak{n})$ to $H^2(\mathfrak{n}; \mathfrak{n})$ is a monomorphism. We know that, in finite dimension, the map

$$H^2(\mathfrak{n}; \mathfrak{n}) \to H^2(\mathfrak{n}; \mathfrak{g})$$

is an epimorphism and we also know the space $H^2(\mathfrak{n}; \mathfrak{g})$;

$$\dim H^2(\mathfrak{n}; \mathfrak{g}) = \#\{w : w \in W, l(w) = 2\},$$

and the image is equal to the kernel of the differential. The space $H^2(\mathfrak{n}; \mathfrak{g})$ is represented by the cocycles

$$f_{\alpha\beta}(e_\sigma, e_\tau) = \begin{cases} s_\beta s_\alpha(\lambda) & \text{if } (\sigma, \tau) = (\alpha + r\beta, \beta) \\ 0 & \text{otherwise,} \end{cases}$$

where α, β are simple positive roots such that

$$\alpha + \beta \in \Delta$$

or

$$\alpha < \beta \quad \text{and} \quad \alpha + \beta \notin \Delta.$$

The number of those cocycles is $\frac{1}{2}(l + 2)(l - 1)$.
From this and the previous considerations it follows:

Theorem 3.1 ([L-L, Theorem 6.4]). *If $\mathfrak{g}$ is not of type $A_1, A_2,$ or B_2 then*

$$H^2(\mathfrak{n};\mathfrak{n}) \approx H^2(\mathfrak{n};\mathfrak{g}) \oplus H^1(\mathfrak{n};\mathfrak{g}/\mathfrak{n}).$$

Here

$$\dim H^1(\mathfrak{n};\mathfrak{g}/\mathfrak{n}) = (2l-1) + l^2 - l,$$
$$\dim H^2(\mathfrak{n};\mathfrak{g}) = \frac{1}{2}(l+1)(l-1).$$

4. The Spaces $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+^*)$ and $H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$

In the affine situation there is a different picture for $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$. It is true that all the $B_{\alpha,\beta}$ symmetric bilinear forms define a cocycle, but not only those. There is an infinite series of cocycles, namely each polynomial without constant term defines one (see [F-F]). But they do not lie in the kernel. The differential acts by the following:

$$\partial\varphi_{B_{\alpha,\beta}}(e_\alpha, e_\beta)(H) = (l(\beta) + l(\alpha))(\beta - \alpha)(H)B(e_\alpha, e_\beta)$$

with

$$B(e_\alpha, e_\beta) = \left\langle P(t^{-1})\frac{\partial e_\alpha}{\partial t}, e_\beta \right\rangle + \left\langle P(t^{-1})\frac{\partial e_\beta}{\partial t}, e_\alpha \right\rangle.$$

If $e_{\beta_0} = t \cdot e_{-\lambda}$ where λ is the highest weight of a representation, then

$$B(e_\alpha, te_{-\lambda}) = \langle P(t^{-1})e_{-\lambda}, e_\alpha \rangle = \mathrm{Res}_t P(t^{-1}) \cdot (e_{-\lambda}, e_\alpha).$$

If $-\lambda = \alpha$ then the Killing form is nonzero.
It follows easily that the image of the infinite series from $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}})$ to $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{h}}+\mathfrak{n}_+^*)$, and the preimage of the infinite series of cocycles in $H^1(\hat{\mathfrak{n}}_+,\hat{\mathfrak{n}}_+^*)$ cancel each other in the affine cases, and in the diagram $(*)$ from above and from the right we get no other cocycles in $H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{h}}+\hat{\mathfrak{n}}_+^*)$ but the ones in the finite dimensional cases.
Let us summarize once more what they are. In finite dimension, the differential of the cocycles of $H^1(\mathfrak{n};\mathfrak{g})$ is zero, while from the right in the diagram $(*)$ we get additional nontrivial cocycles in $H^1(\mathfrak{n};\mathfrak{g}/\mathfrak{n})$. Their number is $\frac{1}{2}\#\{\alpha,\beta$ simple and $\alpha+\beta$ is a root$\}+$ the number of diagonal elements in the Cartan matrix of $\mathfrak{g}$.
The space $H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}})$ again differs in infinite dimension. Here we have

$$H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) = H^1(\hat{\mathfrak{n}}_+) \otimes \mathbb{C}[t,t^{-1}] \qquad \text{(see [F-F])}.$$

Proposition 4.1. *$H^1(\hat{\mathfrak{n}}_+)\otimes t\mathbb{C}[t]$ is in the kernel of the differential map. The sequence*

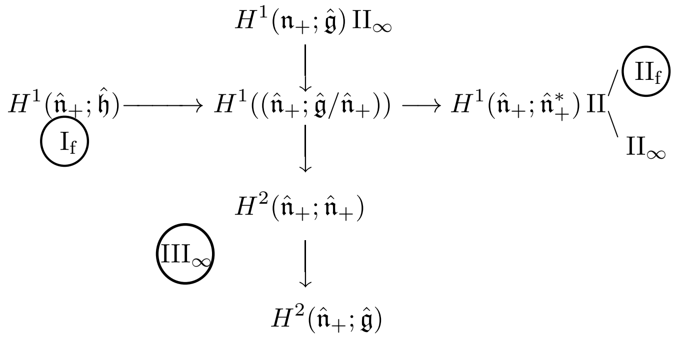
$$H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \to H^1(\hat{\mathfrak{n}}_+) \otimes \mathbb{C}[t] \to 0$$

is exact.

Proof. This follows easily from the corresponding statement for $H^1(\hat{\mathfrak{n}}_+,\hat{\mathfrak{n}}_+^*)$ and from the fact that the maps

$$H^*(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+) \to H^*(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}) \to H^*(\hat{\mathfrak{n}}_+;\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+)$$

are homomorphisms of $H^*(\hat{\mathfrak{n}}_+)$-modules.
Now we are able to compute $H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$. We have the following cohomology sequences:



Theorem 4.2. *With the exception of* $\widehat{s\ell}(2,\mathbb{C})$*, the space* $H^2(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+)$ *is the direct sum of three subspaces, coming from three kinds of cocycles* I–III*. The cocycles of type* $\mathrm{I_f}$ *and* $\mathrm{II_f}$ *are the same as for finite dimensional algebras. The cocycles of type* II_∞ *coming from above and from the right cancel each other. Cocycles of type* III_∞ *only appear in the affine cases. They form a space isomorphic to* $H^1(\hat{\mathfrak{n}}_+)\otimes L_0$.

Remark 4.3. Cocycles of type $\mathrm{I_f}$ and III_∞ form the space

$$H^1(\hat{\mathfrak{n}}_+)\otimes H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+).$$

The number of such deformations is $\dim(H^1(\hat{\mathfrak{n}}_+))\times\dim(H^1(\hat{\mathfrak{n}}_+;\hat{\mathfrak{n}}_+))$. We compare this result with the ones in [F], where they are called the cocycles of type (1°).
In the notation of [F], cocycles of type $\mathrm{II_f}$ give infinitesimal deformations of type 2° and 3°. In [F], cocycles of type (2°) are the following: Let $1\leqslant i\leqslant n$. The algebra $\hat{\mathfrak{n}}_+$ deforms inside $\hat{\mathfrak{g}}=\mathfrak{g}^A$ where A is the Cartan matrix. The deformed algebra is spanned by the spaces $\mathfrak{g}^A_{(m_1,\dots,m_n)}$ with

$$(m_1,\dots,m_n)\neq(0,\dots,0,\overset{(i)}{1},\dots,0)$$

and by the vector e_i+tf_i where t is a parameter. The number of such cocycles is equal to the rank of $\hat{\mathfrak{g}}$.
Similarly cocycles of type (3°) are the following: Let $1\leqslant i\leqslant n$; consider the entry $a_{ij}=-1$ in the Cartan matrix A, and if $a_{ji}=a_{ij}$, then $i<j$. The algebra $\hat{\mathfrak{n}}_+$ deforms again inside $\hat{\mathfrak{g}}$. The deformed algebra is generated by the spaces

$$\mathfrak{g}^A_{(m_1,\dots,m_n)} \text{ with } (m_1,\dots,m_n)$$
$$\neq(0,\dots,0,\overset{(i)}{1},0,\dots,0),(0,\dots,0,\overset{(i)}{1},0,\dots,0,\overset{(j)}{1},0,\dots,0)$$

and the vectors e_i+tf_j and $[e_i,e_j]-th_j$. The number of this type of cocycles is equal to the number of nonzero pairs (a_{ij},a_{ji}) in the Cartan matrix with $i\neq j$.
The exceptional case $\widehat{s\ell}(2,\mathbb{C})$ is also discussed in [F]. In this case, cocycles of type 3° do not exist. Instead, there are two additional cocycles in $H^2(\hat{\mathfrak{n}}_+,\hat{\mathfrak{n}}_+)$. This Lie algebra is really exceptional in the deformation sense also: the two above-mentioned additional cocycles cannot define an extendible deformation of $\hat{\mathfrak{n}}_+$, while all the other types of infinitesimal deformations for any affine Lie algebra are extendible.

References

[F] A. Fialowski, Deformations of nilpotent Kac–Moody algebras, *Studia Sci. Math. Hungar.* **19** (1984), 465–483.

[F-F] B. L. Feigin and A. Fialowski, On the cohomology of the nilpotent algebras of flows, *Soviet Math. Dokl.* **28**, No. 1 (1983), 178–181.

[K] B. Kostant, Lie algebra cohomology and the generalized Borel–Weil theorem, *Ann. Math.* **74**, No. 2 (1961), 329–387.

[L-L] G. Leger and E. Luks, Cohomology of nilradicals of Borel subalgebras, *Trans. Amer. Math. Soc.* **195** (1974), 305–316.

APPEARED IN: *Journal of Algebra*, *Journal of Algebra* **318** (2007), 1002–1026.

Cohomology and Deformations of the Infinite Dimensional Filiform Lie Algebra $\mathfrak{m}_0$

Alice Fialowski
Institute of Mathematics
Eötvös Loránd University, Budapest

Friedrich Wagemann
Lab. de mathématiques Jean Leray
Université de Nantes

Abstract Denote $\mathfrak{m}_0$ the infinite dimensional $\mathbb{N}$-graded Lie algebra defined by basis e_i $i \geq 1$ and nontrivial relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$. We compute in this article the bracket structure on $H^1(\mathfrak{m}_0; \mathfrak{m}_0)$, $H^2(\mathfrak{m}_0; \mathfrak{m}_0)$ and in relation to this, we establish that there are only finitely many true deformations of $\mathfrak{m}_0$ in each nonpositive weight, by constructing them explicitly. It turns out that in weight 0 one gets exactly the other two filiform Lie algebras.

Introduction

Recall the classification of infinite dimensional $\mathbb{N}$-graded Lie algebras $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ with one-dimensional homogeneous components $\mathfrak{g}_i$ and two generators over a field of characteristic zero. A. Fialowski showed in [1] that any Lie algebra of this type must be isomorphic to $\mathfrak{m}_0$, $\mathfrak{m}_2$ or L_1. We call these Lie algebras infinite dimensional filiform Lie algebras in analogy with the finite dimensional case where the name was coined by M. Vergne in [9]. Here $\mathfrak{m}_0$ is given by generators e_i, $i \geq 1$, and nontrivial relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $\mathfrak{m}_2$ with the same generators by nontrivial relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$, and L_1 with the same generators is given by the relations $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \geq 1$. L_1 appears as the positive part of the Witt algebra given by generators e_i for $i \in \mathbb{Z}$ with the same relations $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \in \mathbb{Z}$. The result was also obtained later by Shalev and Zelmanov in [8].

The cohomology with trivial coefficients of the Lie algebra L_1 was studied in [6], the adjoint cohomology in degrees 1, 2 and 3 has been computed in [2] and also all of its non equivalent deformations were given. For the Lie algebra $\mathfrak{m}_0$, the cohomology with trivial coefficients has been studied in [4], but neither the adjoint cohomology, nor related deformations have been computed so far. The reason is probably that - as happens usually for solvable Lie algebras - the cohomology is huge and therefore meaningless. Our point of view is that there still remain interesting features. We try to prove this in the present article by studying the adjoint cohomology of $\mathfrak{m}_0$, while we reserve $\mathfrak{m}_2$ for a forthcoming paper.

Indeed, it is true that the first and second adjoint cohomology of $\mathfrak{m}_0$ are infinite dimensional. The space $H^1(\mathfrak{m}_0; \mathfrak{m}_0)$ becomes already interesting when we split it up into homogeneous components $H^1_l(\mathfrak{m}_0; \mathfrak{m}_0)$ of weight $l \in \mathbb{Z}$, this latter space being finite dimensional for each $l \in \mathbb{Z}$. We compute the bracket structure on $H^1(\mathfrak{m}_0; \mathfrak{m}_0)$ in section 1.

The space $H^2(\mathfrak{m}_0; \mathfrak{m}_0)$ is discussed in section 2. This space is worse as it is infinite dimensional even in each weight separately. The interesting new feature here is that there are only finitely many generators in each negative or zero weight which give rise to *true* deformations. Given a generator of $H^2(\mathfrak{m}_0; \mathfrak{m}_0)$, i.e. an infinitesimal deformation, corresponding to the linear term of a formal deformation, one can try to adjust higher order terms in order to have the Jacobi identity in the deformed Lie

*Keywords: Filiform Lie algebra, cohomology, deformation, Massey product
†Mathematics Subject Classifications (2000): 17B65, 17B56, 58H15

algebra up to order k. If the Jacobi identity is satisfied to all orders, we will call it a true (formal) deformation, see Fuchs' book [5] for details on cohomology and [2] for deformations of Lie algebras.

In section 3 we discuss Massey products, in section 4 describe all true deformations in negative weights. Section 5 deals with deformations in zero and positive weights.

As obstructions to infinitesimal deformations given by classes in $H^2(\mathfrak{m}_0;\mathfrak{m}_0)$ are expressed by Massey powers of these classes in $H^3(\mathfrak{m}_0;\mathfrak{m}_0)$, it is the vanishing of these Massey squares, cubes etc which selects within the $H^2_l(\mathfrak{m}_0;\mathfrak{m}_0)$ of weight l a finite number of cohomology classes. The main result reads

Theorem 0.4. *The true deformations of $\mathfrak{m}_0$ are finitely generated in each weight $l \leq 1$. More precisely, the space of unobstructed cohomology classes is in degree*

- *$l \leq -3$ of dimension two,*
- *$l = 0$ of dimension two,*
- *$l = -2$ of dimension three,*

while there is no true deformation in weight $l = -1$. In weight $l = 0$, these are deformations to $\mathfrak{m}_2$ and L_1. In weight $l = 1$, there are exactly two true deformations, while in weight $l \geq 2$, there are at least two.

We do not have more precise information about how many true deformations there are in positive weight, but there are always at least two. As a deformation in these weights is a true deformation if and only if all of its Massey squares are zero (as cochains !), true deformations are determined by a countable infinite system of homogeneous quadratic equations in countably infinitely many variables. We didn't succeed in determining the space of solutions of this system.

We believe that the discussion of these examples of deformations are interesting as they go beyond the usual approach where the condition that $H^2(\mathfrak{m}_0,\mathfrak{m}_0)$ should be finite dimensional is the starting point for the examination of deformations, namely the existence of a miniversal deformation [3].

Another attractive point of our study is the fact that in some cases the Massey squares and cubes involved are not zero because of general reasons, but because of the combinatorics of the relations. Thus the second adjoint cohomology of $\mathfrak{m}_0$ may serve as an example on which to study explicitly obstruction theory.

Acknowledgments: The work has been partially supported by the grants OTKA T034641 and T043034 and by the Erasmus program between Eötvös Loránd University Budapest and Université Louis Pasteur Strasbourg. Both authors are grateful to IHES where some of this work was accomplished, to Yury Nikolayevsky for useful remarks and to Matthias Borer who helped us to get hold on the weight $l > 0$ case by MUPaD based computations.

1. The space $H^1(\mathfrak{m}_0;\mathfrak{m}_0)$

The Lie algebra $\mathfrak{m}_0$ is an $\mathbb{N}$-graded Lie algebra $\mathfrak{m}_0 = \bigoplus_{i=1}^{\infty}(\mathfrak{m}_0)_i$ with 1-dimensional graded components $(\mathfrak{m}_0)_i$ and generated in degree 1 and 2. Choosing a basis e_i of $(\mathfrak{m}_0)_i$, the only non-trivial brackets (up to skew-symmetry) read $[e_1, e_i] = e_{i+1}$ for all i. We are computing in this section the first cohomology space $H^1(\mathfrak{m}_0,\mathfrak{m}_0)$ of $\mathfrak{m}_0$ with adjoint coefficients. As Lie algebra and module are graded, the cohomology space splits up into homogeneous components, and we will always work with homogeneous cocycles $\omega(e_i) = a_i e_{i+l}$ for a scalar a_i and a given *weight* $l \in \mathbb{Z}$.

Concerning the cocycle identity $d\omega(e_j, e_i) = 0$, let us first suppose that $j = 1$ and $i > 1$ (up to choosing the symmetric case $j > 1$ and $i = 1$). In this case, it reads

$$\omega(e_{i+1}) \;=\; [e_1,\omega(e_i)] - [e_i,\omega(e_1)]$$

or, putting in the expression of ω, for all $l \geq 0$

$$a_{i+1}e_{i+l+1} \;=\; a_i e_{i+l+1} + \delta_{l,0}a_1 e_{i+l+1}.$$

This means that for all $l \geq 0$, we must have

$$a_{i+1} \;=\; a_i + \delta_{l,0}\ a_1,$$

while for $l = -1$, we get the previous equation for $i \geq 3$ and $a_3 = 0$, for $l = -2$, we get the previous equation for $i \geq 4$ and $a_4 = a_3 = 0$, and for $l \leq -3$, we get the previous equation for $i \geq -l + 2$ and $a_{-l+2} = a_{-l+1} = 0$, while there is no equation for $i < -l$.
The second situation where the cocycle identity has non-zero terms is when $l \leq -1$, and i and $j \geq 2$. In this case, there is only one non-zero term in the equation, and we get $a_i = 0$ for $i + l = 1$.
Now let us deduce the possible 1-cocycles in different weights:

case 1: $l \leq -1$
In case $l \leq -3$, the first identity means that all a_i for $i \geq -l + 2$ must be equal and $a_{-l+2} = a_{-l+1} = 0$, therefore all $a_i = 0$ for $i \geq -l + 1$, while there is no constraint on $a_1, a_2, \ldots, a_{-l}$. This is compatible with the second situation.
In case $l = -1$ and $l = -2$, the first constraint implies that all $a_i = 0$ for $i \geq 3$, while there is no constraint on a_1 and a_2. The second identity is then already satisfied for $l \leq -2$, while for $l = -1$, it implies $a_2 = 0$.
But observe that the formula $\omega(e_i) = a_i e_{i+l}$ makes sense for $l \leq -1$ only if $i \geq -l+1$. Therefore all coefficients $a_1, \ldots, a_{-l}$ has to be set zero for $l \leq -1$.
In conclusion, all cohomology is zero in weight $l \leq -1$.

case 2: $l \geq 1$
In this case, the cocycle identity means that all a_i for $i \geq 2$ must be equal, while there is no constraint on a_1.

case 3: $l = 0$
In this case, the first identity means that all a_i for $i \geq 3$ are determined by a_1 and a_2, while there is no constraint on a_1 and a_2.
Let us now examine the coboundaries: an element $x \in \mathfrak{m}_0$ determines a 1-coboundary by $\alpha_x(y) := [x, y]$ for all $y \in \mathfrak{m}_0$. In order to have a homogeneous coboundary, we must take $x = e_i$ for some $i > 0$; α_{e_i} is then homogeneous of weight i. Therefore we have:

- $dC^0_l(\mathfrak{m}_0, \mathfrak{m}_0) = 0$ for $l \leq 0$
- $dC^0_l(\mathfrak{m}_0, \mathfrak{m}_0)$ is generated by $de_l = [e_l, -]$ for $l \geq 1$.

Observe that the coboundaries for $l \geq 2$ are non-zero only on e_1, thus they can modify only the a_1-term of a cocycle. The coboundary for $l = 1$ is zero on e_1 and non-zero and constant on all other e_i. It thus kills the cocycle where all a_i for $i \geq 2$ are equal.
In conclusion, we have

Theorem 1.1.
$$\dim\, H^1_l(\mathfrak{m}_0; \mathfrak{m}_0) \;=\; \begin{cases} 1 & \text{for} \quad l \geq 1 \\ 2 & \text{for} \quad l = 0 \\ 0 & \text{for} \quad l \leq -1 \end{cases}$$

Let us now determine representatives of the non-zero cohomology classes:
In $H^1_0(\mathfrak{m}_0; \mathfrak{m}_0)$, we have the generators ω_1 (corresponding to $a_1 = 1$ and $a_2 = 0$) and ω_2 (corresponding to $a_1 = 0$ and $a_2 = 1$) defined by:

$$\omega_1(e_k) \;=\; \begin{cases} e_1 & \text{for} \quad k = 1 \\ 0 & \text{for} \quad k = 2 \\ (k-2)e_k & \text{for} \quad k \geq 3 \end{cases}$$

$$\omega_2(e_k) \;=\; \begin{cases} 0 & \text{for} \quad k = 1 \\ e_k & \text{for} \quad k \geq 2 \end{cases}$$

In $H^1_l(\mathfrak{m}_0; \mathfrak{m}_0)$ for $l \geq 1$, we have two different kinds of cocycles: there is γ for $l = 1$, and α_l for $l \geq 2$:

$$\gamma(e_k) \;=\; \begin{cases} ce_2 & \text{for} \quad k = 1 \\ 0 & \text{for} \quad k \geq 2 \end{cases}$$

$$\alpha_l(e_k) \;=\; \begin{cases} 0 & \text{for} \quad k = 1 \\ b_l e_{k+l} & \text{for} \quad k \geq 2 \end{cases}$$

It is well known that $H^*(\mathfrak{g};\mathfrak{g})$ carries a graded Lie algebra structure for any Lie algebra $\mathfrak{g}$, and that $H^1(\mathfrak{g};\mathfrak{g})$ forms a graded Lie subalgebra. Let us compute this bracket structure on our generators:
Given $a \in C^p(\mathfrak{g};\mathfrak{g})$ and $b \in C^q(\mathfrak{g};\mathfrak{g})$, define

$$ab(x_1,\ldots,x_{p+q-1}) \;=\; \sum_{\sigma\in \mathrm{Sh}_{p,q}} (-1)^{\mathrm{sgn}\,\sigma} a(b(x_{i_1},\ldots,x_{i_q}),x_{j_1}\ldots,x_{j_{p-1}})$$

for $x_1,\ldots,x_{p+q-1}\in\mathfrak{g}$. The bracket is then defined by

$$[a,b] \;=\; ab-(-1)^{(p-1)(q-1)}ba.$$

It thus reads on $H^1(\mathfrak{g};\mathfrak{g})$ simply

$$[a,b](x) \;=\; a(b(x))-b(a(x)).$$

We compute

$$[\omega_1,\alpha_l](e_k) \;=\; \begin{cases} 0 & \text{for} \quad k=1 \\ b_l(2+l-2)e_{2+l} & \text{for} \quad k=2 \\ (k+l-2)b_le_{k+l}-(k-2)b_le_{k+l} & \text{for} \quad k\geq 3 \end{cases}$$

Therefore $[\omega_1,\alpha_l] \;=\; l\alpha_l$.

$$[\omega_1,\gamma](e_k) \;=\; \omega_1(\delta_{k1}ce_2)-\gamma(\delta_{k1}e_1)$$

Therefore $[\omega_1,\gamma] \;=\; -\gamma$.

$$[\omega_2,\alpha_l](e_k) \;=\; \begin{cases} 0 & \text{for} \quad k=1 \\ \omega_2(b_le_{k+l})-\alpha_l(e_k)=0 & \text{for} \quad k\geq 2 \end{cases}$$

Therefore $[\omega_2,\alpha_l] \;=\; 0$.

$$[\omega_2,\gamma](e_k) \;=\; \begin{cases} \omega_2(ce_2)-0 & \text{for} \quad k=1 \\ 0-\gamma(e_k)=0 & \text{for} \quad k\geq 2 \end{cases}$$

Therefore $[\omega_2,\gamma] \;=\; \gamma$.

$$[\alpha_l,\gamma](e_k) \;=\; \begin{cases} \alpha_l(ce_2)-0 & \text{for} \quad k=1 \\ 0-\gamma(b_le_{k+l}))=0 & \text{for} \quad k\geq 2 \end{cases}$$

This gives $[\alpha_l,\gamma]=\delta_{k1}cb_le_{2+l}$. This is a cocycle in weight $l+1$, $l\geq 2$, but by the list of coboundaries in weight ≥ 2, we see that it is actually a coboundary. Therefore we have $[\alpha_l,\gamma]=0$ in cohomology.

$$[\omega_1,\omega_2](e_k) \;=\; \begin{cases} 0-0=0 & \text{for} \quad k=1 \\ 0-0=0 & \text{for} \quad k=2 \\ (k-2)e_k-(k-2)e_k=0 & \text{for} \quad k\geq 3 \end{cases}$$

Therefore $[\omega_1,\omega_2] \;=\; 0$. It is also rather clear that $[\alpha_l,\alpha_m]=0$.
In summary:

Theorem 1.2. *The bracket structure on $H^1(\mathfrak{m}_0;\mathfrak{m}_0)$ is described as follows: the commuting weight zero generators ω_1 and ω_2 act on the trivial Lie algebra generated by γ in weight 1 and the α_l for weight $l\geq 2$ as grading elements, γ has degree -1 w.r.t. ω_1, degree 1 w.r.t. ω_2, while α_l has degree l w.r.t. ω_1 and degree 0 w.r.t. ω_2.*

2. The space $H^2(\mathfrak{m}_0;\mathfrak{m}_0)$

Let us first compute $H^2(\mathfrak{m}_0;\mathfrak{m}_0)$. We work with homogeneous cocycles $\omega(e_i,e_j)=a_{ij}e_{i+j+l}$ for a fixed weight $l\in\mathbb{Z}$, and for $i,j\geq 1$, $i\neq j$.

2.0 Observe that for weights $l\leq -3$, there are forbidden coefficients $a_{i,j}$, because they show up in front of e_{i+j+l} with $i+j+l\leq 0$. For example in $l=-3$, $a_{1,2}$ must be set to zero, in weight $l=-4$, $a_{1,2}$, $a_{1,3}$ must be set to zero, and so on.

2.1 The cocycle identity reads

$$\begin{aligned} d\omega(e_i,e_j,e_k) &= \omega([e_i,e_j],e_k)+\omega([e_j,e_k],e_i)+\omega([e_k,e_i],e_j) \\ &\quad -[e_i,\omega(e_j,e_k)]-[e_j,\omega(e_k,e_i)]-[e_k,\omega(e_i,e_j)]=0. \end{aligned}$$

Let us first suppose that one index is equal to 1. The identity reads then

$$(a_{i+1,j}+a_{i,j+1})e_{i+j+l+1} = (a_{i,j}-\delta_{j+l,0}\ a_{j,1}-\delta_{i+l,0}\ a_{1,i})e_{i+j+l+1}, \qquad (2.5)$$

where $i,j\geq 2$, $i\neq j$. This identity makes only sense for $i+j+l\geq 2$, because in the above equation, the $a_{i,j}$ term shows up in front of the bracket of e_1 with e_{i+j+l}. It is therefore not valid uniformly for all i,j starting from $l\leq -4$.
For $i+j+l<0$, there is no equation, while there is a special equation for $i+j+l=0,1$, namely

$$a_{i+1,j}+a_{i,j+1}=0.$$

Note that by **2.0**, coefficients $a_{i,j}$ with $i+j+l\leq 0$ are set to zero.

2.2 The cocycle identity for e_i,e_j,e_k for $i,j,k\geq 2$, $i\neq j$, $i\neq k$, and $j\neq k$, gives a non-zero factor only if $j+k+l=1$ or $i+j+l=1$ or $i+k+l=1$ (thus for $l\leq -4$). One can always arrange that only one factor is possibly non-zero for given i, j with $i+j+l=1$ (by choosing $k=\max(i+1,j+1)$, for example). Thus for weight $l\leq -4$, the coefficient $a_{i,j}$ with $i+j=-l+1$, $i\neq j$, $i,j\geq 2$, must be zero (which is compatible with the special equation !).

2.3 Let us now consider coboundaries: expressing that $\omega\in Z^2_l(\mathfrak{m}_0,\mathfrak{m}_0)$ is a coboundary $\omega=d\alpha$ for some 1-cochain $\alpha\in C^1_l(\mathfrak{m}_0,\mathfrak{m}_0)$, $\alpha(e_i)=\alpha_i e_{i+l}$ for all $i\geq 1$, gives by evaluation on e_i and e_j:

$$a_{i,j}e_{i+j+l} = \alpha([e_i,e_j])-[e_i,\alpha(e_j)]+[e_j,\alpha(e_i)].$$

This equation makes sense only for $i+j+l\geq 1$ as all terms are multiples of e_{i+j+l}. Let us first take one index to be 1, then we get

$$a_{1,i} = \alpha_{i+1}-\alpha_i+\alpha_1\ \delta_{l,0}$$

for all $i\geq\max(2,-l+2)$, because α_i appears in front of the bracket of e_1 with e_{l+i}. Thus all $a_{1,i}$, $i\geq\max(2,-l+2)$, can be taken to be zero by adding a coboundary. For $i=-l,-l+1\geq 2$, we have the special equation

$$a_{1,i} = \alpha_{i+1}.$$

It is now clear that, up to a coboundary, we may suppose for any l that the last two terms in equation (2.5) are zero. Observe that the non-coboundary terms $a_{1,i}$ for $l\leq -3$ in a general cocycle, namely the terms with $i=2,\ldots,-l-1$, must be set to zero by **2.0**.

2.4 For weight $l\leq -1$, we have additional coboundaries: indeed, there is a non-zero term in the coboundary equation for e_i, e_j, $i,j\geq 2$, $i\neq j$ yielding

$$a_{-l+1,j} = -\alpha_{-l+1}$$

for all $j\geq 2$, $j\neq -l+1$. Be aware that the coefficient α_{-l+1} of the coboundary $d\alpha$ is linked to $a_{1,-l}$ by the equation $\alpha_{-l+1}=a_{1,-l}$ (cf **2.3**). Thus we cannot choose at the same time to render $a_{1,-l}=0$ and $a_{-l+1,j}=0$ in weight $l\leq -1$ by addition of a coboundary, we can impose only one of these conditions. This means for example that the cocycle given by coefficients $a_{i,j}$ with $a_{2,j}=1$ for all $j\geq 3$ and $a_{i,j}=0$ for all $i,j\neq 2$ ("the 2-family", cf **2.5**) is a coboundary in weight $l=-1$. Here $a_{1,1}=0$ and α_2 are not linked. More generally, the cochain given by coefficients $a_{i,j}$ with $a_{m+1,j}=1$ for all $j\geq m+2$, $a_{i,j}=0$ for all other $i,j>m+1$ (unless those which must be non-zero in order to respect antisymmetry) is cohomologous to the cocycle consisting of the only non-trivial coefficient $a_{1,m}=1$ in weight $l=-m\leq -2$.

Let us now reconsider the equations (2.5) in the stable range, i.e. with i and j such that $i, j \geq 2$, $i \neq j$, and $i + j \geq -l + 2$:

$$a_{i+1,j} + a_{i,j+1} \;=\; a_{ij} \tag{2.6}$$

We will adopt two different points of view on this system of equations:

2.5 First point of view: Call the equations $a_{2,3} = a_{2,4}$, $a_{3,4} = a_{3,5}$, $a_{4,5} = a_{4,6}$, ... *diagonal equations*, and the terms involved *diagonal terms*. The prescription

$$a_{i,i+1} = a_{i,i+2} = \begin{cases} 1 & \text{for} \quad i = k \\ 0 & \text{for} \quad i \neq k \end{cases}$$

specifies uniquely (unicity is shown by induction) a solution to this system, called the kth *family* or k*-series*. For the kth family, all $a_{r,s}$ with $r > k$ are zero, $a_{r,s} = 1$ for $r = k$, $a_{r,s}$ is linear in k for $r = k - 1$ (and s sufficiently big), $a_{r,s}$ is quadratic in k for $r = k - 2$ (and s sufficiently big), and so on.
Let us consider some examples, while we refer to section **4.5** and **5.1** for more information; in the following expressions, all coefficients involving an index 1 are set to zero, and the first non-zero column starting from the RHS (i.e. the non-zero elements of the column $\{a_{m,k}\}_{k \geq m+1}$) is normalized to 1.

The 2-family: $a_{2,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 3$.

The 3-family: $a_{3,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 4$, and $a_{2,3} = a_{2,4} = 0$, $a_{2,k} = -(k-4)$ for all $k \geq 5$.

The 4-family: $a_{4,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 5$, $a_{3,4} = a_{3,5} = 0$, $a_{3,k} = -(k-5)$ for all $k \geq 6$, and $a_{2,3} = a_{2,4} = a_{2,5} = a_{2,6} = 0$, $a_{2,k} = \frac{(k-5)(k-6)}{2}$ for all $k \geq 7$ (even for all $k \geq 5$).

The 5-family: $a_{5,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq 6$, $a_{4,5} = a_{4,6} = 0$, $a_{4,k} = -(k-6)$ for all $k \geq 7$, $a_{3,4} = a_{3,5} = a_{3,6} = a_{3,7} = 0$, $a_{3,k} = \frac{(k-6)(k-7)}{2}$ for all $k \geq 8$ (even for all $k \geq 6$), and $a_{2,3} = a_{2,4} = a_{2,5} = a_{2,6} = a_{2,7} = a_{2,8} = 0$, $a_{2,k} = -\frac{(k-6)(k-7)(k-8)}{3!}$ for all $k \geq 9$ (even for all $k \geq 6$).

2.6 Second point of view: One can specify $a_{2,n}$ for all n, in such a way that the diagonal equations are satisfied. This implies that by choosing pairs $(a_{2,3}, a_{2,4})$, $(a_{2,5}, a_{2,6})$, $(a_{2,7}, a_{2,8})$, and so on, the first member is free, while the second member is determined by the corresponding diagonal equation.
Indeed, in $(a_{2,3}, a_{2,4})$, $a_{2,4}$ is determined by $a_{2,3} = a_{2,4}$, in $(a_{2,5}, a_{2,6})$, $a_{2,6}$ is determined by $a_{2,5} - a_{2,4} = a_{2,6} - a_{2,5}$ (which is just $a_{3,4} = a_{3,5}$), in $(a_{2,7}, a_{2,8})$, $a_{2,8}$ is determined by $((a_{2,7} - a_{2,6}) - (a_{2,5} - a_{2,4})) = ((a_{2,3} - a_{2,7}) - (a_{2,7} - a_{2,6}))$ (which is just $a_{4,5} = a_{4,6}$). All the other coefficients are then uniquely determined.

2.7 In conclusion, it is clear that for each weight $l \in \mathbb{Z}$, there is a countably infinite number of independent 2-cohomology classes. More precisely, in weight $l > -4$, the k-families with $k = 2, 3, \ldots$ represent independent 2-cohomology classes. In weight $l \leq -4$, **2.2** shows that the k family is contradictory for

$$k < \begin{cases} 2 + \frac{-l-3}{2} & \text{for} \quad l \quad \text{odd} \\ 2 + \frac{-l-2}{2} & \text{for} \quad l \quad \text{even} \end{cases}$$

But there is still a countably infinite number of independent 2-cohomology classes in each weight.

Theorem 2.1.

$$\dim\, H^2_l(\mathfrak{m}_0, \mathfrak{m}_0) \;=\; \infty$$

for each weight $l \in \mathbb{Z}$.

3. Massey products and deformations

The 2-cohomology is rather meaningless, as it is infinite dimensional even in each weight separately. We ask now which of these homogeneous 2-cocycles gives rise to a deformation of $\mathfrak{m}_0$. A necessary condition is that the class of the Massey square of the cocycle in question is zero. The first thing we will show is that for a large range of weights, even the condition that the Massey square is zero as a cochain is necessary and sufficient, and we will then determine all 2-cocycles which have zero Massey square and are thus the infinitesimal part of a deformation of $\mathfrak{m}_0$ which is polynomial and of polynomial degree 1. We will show in section 5 that $\mathfrak{m}_0$ deforms (in a homogeneous way) to $\mathfrak{m}_2$ and to L_1, but to no other $\mathbb{N}$-graded Lie algebra (non-isomorphic to $\mathfrak{m}_0$, $\mathfrak{m}_2$, and L_1). This is consistent with the classification of $\mathbb{N}$-graded Lie algebras with 1-dimensional graded components, generated in degrees 1 and 2 [1].
Let ω be a 2-cocycle, given as above by its coefficients $a_{i,j}$. The *Massey square* of ω is by definition

$$M(a)_{ijk} \;=\; (a_{i,j}a_{i+j+l,k} + a_{j,k}a_{j+k+l,i} + a_{k,i}a_{k+i+l,j})e_{i+j+k+2l}. \qquad (3.7)$$

Observe that $M(a)_{ijk} = 0$ if any two indices coincide.

Massey squares and deformations.

Proposition 3.1. *Let $l \geq -1$.*
If there exist i, j, k with $M(a)_{ijk} \neq 0$, then it is a non-trivial 3-cohomology class, and the 2-cocycle ω is obstructed. Thus ω is obstructed if and only if there exist i, j, k with $M(a)_{ijk} \neq 0$.

Proof. Let $\alpha \in C^2(\mathfrak{m}_0, \mathfrak{m}_0)$ be a homogeneous 2-cochain with $\alpha(e_i, e_j) = b_{i,j}e_{i+j+m}$. Then

$$d\alpha(e_i, e_j, e_k) \;=\; \alpha([e_i, e_j], e_k) - [e_i, \alpha(e_j, e_k)] + \text{cyclicpermutations}.$$

Given a 2-cocycle ω with non-zero Massey square $M(a)_{ijk}$, one wants to find α which compensates $M(a)_{ijk}$, i.e. with $d\alpha(e_i, e_j, e_k) = M(a)_{ijk}$. As $M(a)_{ijk}$ is of weight $2l$, one must have $m = 2l$.
For $i, j, k \geq 2$, there is only one non-zero term in the coboundary equation (cf **2.4**), and we have $d\alpha(e_i, e_j, e_k) = b_{i,j}e_{k+1}$ in case $i + j + 2l = 1$ (the cases $j + k + 2l = 1$ or $i + k + 2l = 1$ are similar). Thus we can compensate all Massey squares $M(a)_{ijk}$ with $i + j + 2l = 1$, $j + k + 2l = 1$ or $i + k + 2l = 1$. As $i, j, k \geq 2$, the highest weight case appears for $l = -2$.
On the other hand, for $l \geq -1$ all $a_{1,i}$ can be taken to be zero by adding coboundaries (cf **2.3**). Thus $M(a)_{ijk} = 0$ if one index is equal to 1, and the only squares to compensate are those with $i, j, k \geq 2$. □

An interesting fact to note from the above proof is that the Massey squares that one can compensate by 3-coboundaries are the $M(a)_{ijk}$ with $i + j + 2l = 1$, $j + k + 2l = 1$ or $i + k + 2l = 1$ for $i, j, k \geq 2$ in weight $l \leq -2$. In the following, we will use the notation M_{ijk} for the coefficient of $e_{i+j+k+2l}$ in the corresponding Massey square.

4. Deformations in negative weights

a). **True deformations in weight -1.** We now consider only *square zero cohomology* in weight -1, i.e. those classes in $H^2_{-1}(\mathfrak{m}_0, \mathfrak{m}_0)$ with Massey square equal to zero (not only as a cohomology class, but as a cochain !). By the previous section, this determines all deformations of $\mathfrak{m}_0$ in weight -1.
First of all, the 2-family (cf **2.5**) is a square zero 2-cocycle, and it is not contradictory in weight -1 (cf **2.7**). But the 2-family is actually a coboundary according to **2.4**.
Now assume that $a_{i,j}$ for all $i, j \geq 2$, $i \neq j$, defines a normalized cocycle, i.e. $a_{1,s} = 0$ for all s, which we may assume according to **2.3**. Observe that for $l = -1$, there is no special equation of type $a_{1,-l} = \alpha_{-l+1}$. We assume further that all Massey squares M_{ijk} are zero. Suppose that k is the first integer such that $a_{3,k} \neq 0$, $k \geq 4$.

As all 3 coefficients below $a_{3,k}$ are zero, equation (2.6) shows that the first non-zero 4 coefficient is $a_{4,k-1}$, and that all 2 coefficients are equal up to $a_{2,k}$, while $a_{2,k} \neq a_{2,k+1}$. Denote $a_{2,3} = c$. We will establish a table for the coefficients in order to examine the possible cases:

Lemma 4.1. $a_{3,k} \neq 0$ *implies* $a_{3,k+1} \neq 0$.

Proof. If $a_{3,k+1} = 0$, then $M_{23k} = a_{2,3}a_{4,k} + a_{3,k}a_{k+2,2}$. But by equation (2.6), $a_{3,k} = a_{4,k}$ and $a_{2,k} - a_{3,k} = a_{2,k+1} = a_{2,k+2}$, and therefore $M_{23k} = a_{2,3}a_{3,k} - a_{3,k}(a_{2,3} - a_{3,k}) = a_{3,k}^2 \neq 0$. This contradiction shows that $a_{3,k+1} \neq 0$. □

For $k = 4$, $a_{3,k} = a_{3,k+1}$. Then $M_{234} = a_{3,4}(a_{2,4} - a_{2,6})$. If $a := a_{3,4} \neq 0$, then $a_{2,4} = a_{2,6} =: c$, and we have $a_{2,5} = c - a$ and thus $a - c = a + c$, implying $a = 0$: contradiction.
Let us now suppose $k > 5$. Then $M_{34(k-1)} = a_{4,k-1}a_{k+2,3} = 0$ implies $a_{k+2,3} = 0$, because $a_{4,k-1} = -a_{3,k} \neq 0$ by equation (2.6). Consideration of $M_{23(k+1)} = 0$ gives $a_{2,3} = a_{2,k+2} = a_{2,k+3}$. $M_{23(k+2)} = 0$ gives $a_{4,k+2} = -a_{3,k+3}$. $M_{34k} = 0$ implies that either $a_{3,k+3} = 0$ or $a_{3,k+1} = 2a_{3,k}$. But in this last case, $a_{5,k-1} = -4a_{3,k}$, and $M_{35(k-1)} = 0$ gives $a_{3,k+3} = 0$ anyhow. $M_{34(k+1)} = 0$ gives $a_{3,k+4} = a_{4,k+3}$.
This gives the following table for the coefficients $a_{i,j}$ (where we used i as the column index and j as the row index, contrary to the usual convention) with $a_{3,k} = a$ and $a_{3,k+1} = b$:

	2	3	4	5	6
k-3	c	0	0	0	-a
k-2	c	0	0	a	3a+b
k-1	c	0	-a	-(2a+b)	-3a+b
k	c	a	a-b	a-2b	a-3b
k+1	c+b	b	b	b	b
k+2	c	0	0	0	0

But the relation $a_{2,k} = c = a_{2,k+1} + a_{3,k} = c + b + a$ shows that $a = -b$.
Now, if k is odd, we have $a_{3,k} = a$, $a_{5,k-2} = a$, $a_{7,k-4} = a$, and so on, until we reach the diagonal $a_{j,j} = a = 0$. Thus in this case we have a contradiction.
But if k is even, we will take the line of $a_{i,j}$ given by $i + j = 4 + k$, and go to the diagonal: finally, we will also get $a = 0$, i.e. a contradiction.
In conclusion, a non-zero 3 coefficient for a square zero cocycle leads in weight -1 to a contradiction. But then all 2 coefficients must be equal by equations (2.6), and this gives the 2 family.
In conclusion, we have shown

Proposition 4.2. *There is no non-trivial square-zero cohomology class in weight* $l = -1$. *In particular, there does not exist any non-trivial true cohomology class in weight* $l = -1$.

b). **True deformations in weight** -2. We saw in the last section how to determine all cocycles, here in weight -2, which lead to true deformations. But as Proposition 1 in section 3.1 is not valid in weight -2, we cannot use the vanishing of all Massey squares to get restrictions on our cocycle. Instead, we have to leave out those which are coboundaries and could thus be compensated by higher Massey products.
Let ω be a cocycle given by its coefficients $a_{i,j}$. Note that we can still suppose $a_{1,j} = 0$ for all $j \geq -l+1$ and all $j \leq -l-1$ (cf **2.0**, **2.3**, **2.4**). On the other hand, we may suppose that we are in the stable range and by **2.4** that the first terms in the 3 column (i.e. at least $a_{3,4}$, $a_{3,5}$) are zero up to a coboundary. As we cannot assume simultaneously that $a_{3,4}$, $a_{3,5}$ and $a_{1,-l} = a_{1,2}$ are zero (cf **2.4**), we choose to allow $a_{1,2}$ non-zero.
When writing a Massey square M_{ijk}, we will now always suppose that the indices are ordered $i < j < k$. The Massey squares which may be compensated are those

M_{ijk} with $i+j+2l=1$, according to section **3.1**. This means in weight $l=-2$ that all M_{23k} can be compensated, and that these are the only (ordered) ones. The other M_{ijk} must be zero.
We start now a case study in order to determine which possibilities there are for ω, imposing that all (ordered) Massey squares M_{ijk} with $i+j \neq 5$ are zero.

1st case: Suppose $a_{2,3}=a_{2,4}=0$. Then $a_{2,5}=-a_{3,4}=-a_{3,5}$ and $a_{2,5}=a_{3,5}+a_{2,6}$ implying that $a_{2,6}=-2a_{3,5}$. $M_{245}=a_{4,5}(a_{2,5}-a_{2,7})=0$.

case 1a: $a_{2,5}=0$, and thus $a_{3,4}=a_{3,5}=0$, $a_{2,6}=0$. Then either $a_{4,5}=0$ ($\Rightarrow a_{3,6}=0$, $a_{2,7}=0$), or $a_{2,7}=0$ ($\Rightarrow a_{3,6}=0$, $a_{4,5}=0$). In any case $a_{2,7}=0$, $a_{3,6}=0$ and $a_{4,5}=0$.
Suppose now given r ($r \geq 10$) such that $a_{i,j}=0$ for all $i+j \leq r$. Then $M_{2ij}=a_{i,j}(a_{2,i}-a_{2,i+j-2}+a_{2,j})$ must be zero for $i \geq 4$. Let us suppose $i<j$ (indices ordered !) and $i+j=r+1$. Then, by hypothesis, $a_{2,i}=a_{2,j}=0$ and $M_{2jk}=-a_{i,j}a_{2,r-1}$. Thus either $a_{i,j}=0$, or $a_{2,r-1}=0$. But these two elements are on a new diagonal (in the matrix of coefficients $a_{i,j}$), and all elements with lower indices are zero. By equation (2.6) this implies that two (because approaching the diagonal, one jumps to the next diagonal by the diagonal equations (cf **2.5**) $a_{s,s+1}=a_{s,s+2}$) new diagonals are zero, and by induction, all coefficients are zero in this case.

case 1b: $a_{2,5}=1$, and thus $a_{2,6}=2$, $a_{3,4}=a_{3,5}=-1$. Now $M_{245}=0$ implies $a_{4,5}=0$ or $a_{2,7}=1$. But for $a_{2,7}=1$, we get by repeated use of equation (2.6), $a_{3,6}=1$, $a_{4,5}=-2$, $a_{4,6}=-2$, $a_{3,7}=3$ and finally $a_{2,8}=-2$. Relate this then to $M_{246}=a_{4,6}(-a_{2,8}+a_{2,6}) \neq 0$, to conclude that $a_{2,7} \neq 1$, and therefore $a_{4,5}=0$.
This means that all $a_{i,j}$ with $i+j \leq 10$ are the same as for the 3-family (cf **2.5**). Let us show the induction step in order to conclude that two more diagonals are like in the 3-family. Indeed, suppose now $a_{i,j}$ with $i+j \leq r$ like in the 3-family, and take $j=r-3$. We have $M_{24j}=a_{4,j}(a_{2,j}-a_{2,j+2})=a_{4,r-3}(a_{2,r-3}-a_{2,r-1})=0$. The coefficient $a_{2,r-3}$ must be as in the 3-family by hypothesis. We want to conclude that $a_{4,r-3}=0$ (as in the 3-family), opening up two more diagonals. Therefore we show that $a_{2,r-3} \neq a_{2,r-1}$.
Let us denote $a_{2,r-3}=t$. We have by hypothesis $a_{2,r-2}=t+1$, $a_{3,r-4}=a_{3,r-3}=-1$, and $a_{k,s}=0$ for $k<s$, $k \geq 4$. Suppose $a_{2,r-3}=a_{2,r-1}$, and this will lead to $a_{4,r-2} \neq 0$ while $a_{2,r-2}=t+1$ and $a_{2,r} \neq t+1$. More precisely, in the new diagonal starting from $a_{2,r-1}=t$, we get $a_{3,r-2}=1$, $a_{4,r-3}=-2$, $a_{5,r-4}=2$, and then we always get ± 2, because there are only zeroes one diagonal higher. By construction $r-1$ is odd, say $r-1=2k+1$. Doing in this sense $k-1$ steps on the diagonal towards the diagonal transforms $a_{2,r-1}=a_{2,2k+1}$ into $a_{k-1+2,k+2}=a_{k+1,k+2}$. But by the diagonal equation (cf **2.5**), $a_{k+1,k+2}=a_{k+1,k+3}$, and then we work back $k-3$ steps to get $\pm 2(k-3)$, and finally $a_{2,r}=-(\mp 2(k-3)+1)+t$. This is equal to $t+1$ only if $k=2$ or $k=4$. $k=2$ is already treated, and for $k=4$, one can check directly that $a_{2,8} \neq a_{2,10}$:

2	3	4	5
0			
0	-1		
1	-1	0	
2	-1	0	2
3	-1	-2	2
4	1	-4	
3	5		
-8			

In conclusion, the only non-zero cocycle (making zero the non-compensable Massey squares) compatible with case 1 is the 3-family.

2nd case: Here we can take $a_{2,3} = a_{2,4} = 1$. Recall that we choose to take the first terms in the 3 column (i.e. at least $a_{3,4}$, $a_{3,5}$) to be zero (possibly by adding a coboundary).

case 2a: Suppose as a first subcase $a_{4,5} = a_{4,6} = 0$. Then we have up to $a_{i,j}$ with $i + j = 10$ the 2-family. Set $a_{5,6} = a$. We get then $a_{2,9} = 1 - a$ and $a_{2,10} = 1 - 4a$ by repeated use of equation (2.6). But $M_{247} = a_{4,7}(a_{2,4} - a_{2,9} + a_{2,7}) = 0$ implies $a = 0$ or $a = -1$, while $M_{248} = a_{4,8}(a_{2,4} - a_{2,10} + a_{2,8}) = 0$ implies $a = 0$ or $a = -\frac{1}{4}$. In conclusion, $a = 0$ and the 2-family is reproduced one diagonal higher. Using M_{24j} and $M_{24(j+1)}$, one can show in a similar way that the only solution here is the 2-family.

case 2b: Here $a_{2,3} = a_{2,4} = 1$, $a_{3,4} = a_{3,5} = 0$, but $a := a_{4,5} = a_{4,6} \neq 0$. By $M_{245} = a_{4,5}(a_{2,4} - a_{2,7} + a_{2,5})$, $M_{246} = a_{4,6}(a_{2,4} - a_{2,8} + a_{2,6})$ and $M_{345} = a_{4,5}(a_{3,5} - a_{3,7})$, we get thus $a_{3,5} = a_{3,6} = 0$, $a_{2,4} + a_{2,5} = a_{2,7}$ and $a_{2,4} + a_{2,6} = a_{2,8}$. But then on the one hand $a_{2,3} = a_{2,4} = a_{2,5} = a_{2,6} = a_{2,7}$ by equation (2.6), but also $a_{2,4} + a_{2,5} = a_{2,7} = 2$, which is a contradiction.

As a conclusion of the case study, the only cocycles which can possibly give true deformations are the 2-and the 3-family, but possibly with a non-zero $a_{1,2}$ coefficient.

The 2-family (cf **2.5**) is a square zero 2-cocycle in weight $l = -2$, it is not contradictory in weight -2 (cf **2.7**), and is thus one solution here.
The 3-family ω has a non-zero Massey square, namely $M_{23j} = a_{2,j} - a_{2,j+1} = 1$ for all $j \geq 4$. Let us show that the corresponding Massey cube is then zero, and thus that the 3-family gives indeed rise to a true deformation in weight -2:
We must write M_{23k} as a coboundary. The cochain $\alpha(e_i, e_j) = b_{i,j}e_{i+j-4}$ with $b_{i,j} = M_{23k}$ for $i = 2$ and $j = 3$, $b_{i,j} = M_{32k}$ for $i = 3$ and $j = 2$, and $b_{i,j} = 0$ otherwise satisfies

$$d\alpha(e_i, e_j, e_k) \;=\; M_{23k}e_{k+1}.$$

We must then compute the Massey cube

$$N_{ijk} := \alpha(\omega(e_i, e_j), e_k) + \omega(\alpha(e_i, e_j), e_k) + \text{cycl.} \;=\; a_{i,j}b_{i+j-2,k} + b_{i,j}a_{i+j-2,k} \;+\; \text{cycl.}.$$

But if $b_{i+j-2,k} \neq 0$, then $i + j - 2, k \in \{2, 3\}$. The only possibly non zero term is thus $N_{23k} = M_{23k}a_{1,k} = 0$ ($k \geq 4$ here).
Finally, let us show that we cannot get any information about $a_{1,2}$, neither by the cocycle equations, nor by the vanishing of the Massey squares. This is clear for the cocycle equations. Let us show that we cannot deduce $a_{1,2} = 0$ from Massey squares which have to vanish. Indeed, when writing down the Massey squares which involve $a_{1,2}$, the only possibly non-zero Massey squares M_{ijk} involving $a_{1,2}$ (with ordered indices) have $i = 1$. But then we have to have $j = 2$ in order to involve $a_{1,2}$. One easily checks that $M_{12k} = 0$.
To summarize, we have the following

Proposition 4.3. *The 3-family in weight -2 has a non-zero Massey square, but its Massey cube is zero, and we get consequently a true deformation. In weight -2, the 2- and 3-family with possibly a non-zero term $a_{1,2}$ define the only cohomology classes leading to true deformations.*

c). **True deformations in weight** -3**.** We will determine all cocycles leading to true deformations in weight -3 once again by imposing on a general cocycle ω given by its coefficients $a_{i,j}$ for all $i, j \geq 2$, $i \neq j$, that all Massey squares which cannot possibly be compensated (cf section **3.1**) are zero. The squares which cannot serve to give conditions on the $a_{i,j}$ are those M_{ijk} (with ordered indices $i < j < k$) with $i + j = 7$, $i + k = 7$ or $j + k = 7$.
All 1-coefficients other than $a_{1,2}$ and $a_{1,3}$ may be supposed to be zero by **2.3** (cf **2.4**), $a_{1,2} = 0$ by **2.0**. We choose once again that the first 4-coefficients (i.e. at least $a_{4,5} = a_{4,6}$) are zero, up to a coboundary, according to **2.5**, while not imposing anything on $a_{1,3}$ (cf **2.4**).
Let us draw the table for the coefficients of ω:

2	3	4	5
a			
a	b		
a-b	b	0	
a-2b	b	0	
a-3b	b		
a-4b			

Now we write down the Massey squares that we may use: $M_{236} = a_{2,6}(a_{2,3} - a_{3,6} + a_{3,5}) = a(a-2b)$, $M_{237} = a_{2,7}(a_{2,3} - a_{3,7} + a_{3,6}) = a(a-3b)$, $M_{246} = a_{2,4}a_{3,6} + a_{4,6}a_{7,2} + a_{6,2}a_{5,4} = ab$.
In conclusion, $a = 0$. But then up to $i + j = 10$, the 3-family has built up. Let us show by induction that the 3-family is the only possible solution:
Suppose the $a_{i,j}$ up to $i + j = r$ for $r \geq 10$ are like in the 3-family (cf **2.5**). Consider the Massey square

$$M_{23k} = a_{2,3}a_{2,k} + a_{3,k}a_{k,2} + a_{k,2}a_{k-1,3} = a_{2,k}(a_{2,3} - a_{3,k} + a_{3,k-1}).$$

We may use its vanishing to deduce restrictions on the $a_{i,j}$ as soon as $k \geq 6$. For $k \geq 8$ and with $r = k+2$, $M_{23k} = 0$ implies under the induction hypothesis that $a_{3,k} = a_{3,k-1}$, and we have therefore transmitted the 3-family to two more diagonals, showing the induction step.

In order to conclude, let us show that the 3-family is of Massey square zero:

Recall that the 3-family is defined by $a_{3,k} = a \neq 0$, $a_{j,k} = 0$ for all $j, k \geq 4$, and $a_{2,3} = a_{2,4} = 0$, $a_{2,k} = -(k-4)a$ for all $k \geq 5$.
It is clear that $M(a)_{ijk} = 0$ for all $i, j, k \geq 4$, by definition of the 3-family. Suppose $i = 3$ ($j = 3$ or $k = 3$ would be a symmetric case):

$$M(a)_{3jk} = a_{3,j}a_{j,k} + a_{j,k}a_{j+k-3,3} + a_{k,3}a_{k,j} = a_{j,k}(a_{3,j} + a_{j+k-3,3} + a_{3,k}).$$

This last expression is zero if both j and k are greater or equal to 4 (as then $a_{j,k} = 0$), and also if one of them is equal to 2, because in this case the term in parenthesis is zero. Suppose now that $i = 2$.

$$M(a)_{2jk} = a_{2,j}a_{j-1,k} + a_{j,k}a_{j+k-3,2} + a_{k,2}a_{k-1,j}.$$

In case $j, k \geq 4$, this expression reduces to $a_{2,j}a_{j-1,k} + a_{k,2}a_{k-1,j}$ which is evidently zero if both j and k are greater or equal to 5, and in case $j = 4$, $a_{k-1,j}$ and $a_{2,j}$ are zero. It remains the case where $j = 3$, but then we get $a_{3,k}a_{k,2} + a_{k,2}a_{k-1,3} = 0$.
Finally, let us show that the possibly non-zero coefficient $a_{1,3}$ cannot be shown to be zero using the vanishing of Massey squares. The only M_{ijk} (with ordered indices) involving $a_{1,3}$ have $i = 1$.

$$M_{1jk} = a_{1,j}a_{j+1+l,k} + a_{j,k}a_{j+k+l,1} + a_{k,1}a_{k+1+l,j}.$$

Then for $j = 3$, we get $M_{13k} = 0$, and for $j + k = 6$, we get also $M_{1jk} = 0$, and these are the only combinations (up to reordering) involving $a_{1,3}$.
To summarize, we get the following

Proposition 4.4. *In weight -3, the 3-family, with a possibly non-zero $a_{1,3}$ coefficient, defines the only cohomology class leading to a true deformation.*

d). **True deformations in weight -4.** We will determine all cocycles leading to true deformations in weight -4 once again by imposing on a general cocycle ω given by its coefficients $a_{i,j}$ for all $i, j \geq 2$, $i \neq j$ that all Massey squares which cannot possibly be compensated (cf section **3.1**) are zero. The squares which cannot serve to give conditions on the $a_{i,j}$ are those M_{ijk} (with ordered indices $i < j < k$) with $i+j = 9$, $i+k = 9$ or $j+k = 9$. In weight $l = -4$, we have to be more careful with the conditions as we are not always in the stable range (cf **2.5**). For example, **2.1** implies here that $a_{2,4} = 0$ (and we can not deduce here $a_{2,3} = a_{2,4}$). But for $j > i \geq 3$, and for $i = 2$ and $j \geq 4$ we still have

$$a_{i+1,j} + a_{i,j+1} = a_{i,j}.$$

But then **2.2** implies that $a_{2,3} = 0$. All 1-coefficients other than $a_{1,2}$, $a_{1,3}$ and $a_{1,4}$ may supposed to be zero by **2.3**, while $a_{1,2} = a_{1,3} = 0$ follows from **2.0**. We choose once again according to **2.4** that the first 5-coefficients (i.e. at least $a_{5,6} = a_{5,7}$) are zero, up to a coboundary, while we do not impose anything on $a_{1,4}$, cf **2.4**.
Let us draw the table for the coefficients of ω:

2	3	4	5
0			
0	b		
-b	b	c	
-2b	b-c	c	0
c-3b	b-2c	c	0
3c-4b	b-3c	c	
6c-5b	b-4c		
10c-6b			

Now we write down the Massey squares that we may use: $M_{246} = a_{4,6}a_{6,2} = 2cb$, $M_{238} = a_{3,8}a_{7,2} + a_{8,2}a_{6,3} = (b-3c)(3b-c) + (3c-4b)(b-c) = -b(b+3c)$, $M_{256} = a_{2,5}a_{3,6} + a_{5,6}a_{7,2} + a_{6,2}a_{4,5} = b(-b+3c)$.
Now start a case study: either $b = 0$, and in this case we want to show that the 4-family is built up by induction. Indeed, we have $M_{24k} = a_{2,k}(a_{4,k-2} - a_{4,k})$ which must vanish as soon as $k \geq 8$. In this way we transmit the built up of the 4-family to another two diagonals. Or $c = 0$ and in this case all coefficients are zero. The zero family is also easily shown to be built up from this initial stage.
Let us show that the nullity of $a_{1,4}$ cannot be derived from the nullity of Massey squares. The Massey squares (with ordered indices), where $a_{1,4}$ shows up, have either $i = 1$ or they are M_{234}. The latter is zero anyhow, and the former are shown to be zero as for $l = -1, -2$ and -3.
To summarize, we have the following

Proposition 4.5. *The only cohomology class leading to a true deformation in weight $l = -4$ is represented by the 4-family, with a possibly non-zero coefficient $a_{1,4}$.*

The fact that the 4-family has zero Massey square in weight -4 follows from Proposition 7 in section **4.5**.

e). **True deformations in weight l, $l \leq -5$.** We will show that in degree l, the $-l$ family is of Massey square zero, and that this is the only family for which all Massey squares which cannot be compensated, are zero. Therefore, we will show that in weight l, $l \leq -5$, the $-l =: m$ family, with a possibly non-zero coefficient $a_{1,m}$, is the only cocycle which leads to true deformations.
For this, we need the explicit expression of the non-zero low degree coefficients of the $-l$ family. It is obvious from **2.5** how to deduce the expressions of the coefficients of the general $m := -l$ family from those for the low degree families:

The m-family: $a_{m,k} = 1$, $a_{j,k} = 0$ for all $j, k \geq m+1$, $a_{m-1,m} = a_{m-1,m+1} = 0$, $a_{m-1,k} = -(k-(m+1))$ for all $k \geq m+2$, $a_{m-2,m-1} = a_{m-2,m} = a_{m-2,m+1} = a_{m-2,m+2} = 0$, $a_{m-2,k} = \frac{(k-(m+1))(k-(m+2))}{2}$ for all $k \geq m+3$, and $a_{m-3,m-2} = a_{m-3,m-1} = a_{m-3,m} = a_{m-3,m+1} = a_{m-3,m+2} = a_{m-3,m+3} = 0$, while for all $k \geq m+4$: $a_{m-3,k} = -\frac{(k-(m+1))(k-(m+2))(k-(m+3))}{3!}$ and so on.

Proposition 4.6. *The m-family defines a 2-cocycle in any weight.*

Proof. We have to show that the m-family satisfies the requirements of sections **2.1** (i.e. equation (2.6); observe that with the non-zero coefficients of the m-family, we are always in the stable range) and **2.2**. It is clear that the requirement of **2.2** is met.
For the equation (2.6), take the general expression of the above coefficients

$$a_{m-r,k} \;=\; \pm \frac{(k-(m+1))!}{r!(k-(m+r+1))!}$$

for all $k \geq m+r+1$, and all $r \leq m-2$; $\pm$ denotes an alternating sign with respect to the parity of r. Now

$$a_{m-r,k+1} + a_{m-(r-1),k} = \frac{(k+1-(m+1)-r)(k-(m+1))!}{r!(k-(m+r))!} = a_{m-r,k}.$$

□

Let us show now that the Massey square of the m-family is zero (i.e. not only the non-compensable Massey squares, but all).

Proposition 4.7. *All Massey squares of the m-family are zero in weight $l=-m$.*

Proof. Indeed, we have

$$M_{ijk} \;=\; a_{i,j}a_{i+j+l,k} + a_{j,k}a_{j+k+l,i} + a_{k,i}a_{k+i+l,j}.$$

We will always consider ordered Massey squares, i.e. M_{ijk} with $i<j<k$, and it will be enough to show that these are zero.
First case: $i=m-r$, $j=m-p$, and $k \geq m+1$ with $p,r \geq 0$. These conditions imply $a_{i,j}=0$, and we get

$$\begin{aligned} M_{ijk} &= a_{j,k}a_{j+k+l,i} + a_{k,i}a_{k+i+l,j} \\ &= (-1)^{p+1}\frac{(k-(m+1))!}{p!(k-(m+p+1))!}(-1)^r\frac{(k-p-(m+1))!}{r!(k-p-(m+p+1))!} \\ &+ (-1)^{r+1}\frac{(k-(m+1))!}{r!(k-(m+r+1))!}(-1)^{p+1}\frac{(k-r-(m+1))!}{p!(k-r-(m+p+1))!}. \end{aligned}$$

Suppose now first that $k+i+l=k-r>j$. In this case we get by taking out common factors

$$M_{ijk} = \frac{(-1)^{p+r+1}(k-(m+1))!}{r!p!(k-p-(m+r+1))!}\left(\frac{(k-p-(m+1))!}{(k-(m+p+1))!} - \frac{(k-r-(m+1))!}{(k-(m+r+1))!}\right)$$

which is zero. On the other hand, in case $k+i+l=k-r=j$, we get $k=m+q$, $i=m-r$, $j=m-p$. Then the only possibly non-zero term is $M_{ijk}=a_{j,k}a_{j+k+l,i}$, because $a_{k+i+l,j}=0$. But $a_{j,k}a_{j+k+l,i}=a_{m-p,m+q}a_{m+(q-p),m-(p+q)}$ and $a_{m+(q-p),m-(p+q)}=0$, because $a_{m-s,m+s}=0$, $a_{m-s,m+s+1}\neq 0$ marks the last zero term in the m-family (when fixing $m-s$ and counting up the second index), but here $q-p \leq p+q$. It remains the third subcase where $k-r<j$, but then $r>p+q$. Thus $a_{j+k+l,i}=a_{m+q-p,m-r}=0$ and $a_{k,i}=a_{m+q,m-r}=0$ by the same reasoning as before. So the first case is settled.
Second case: $i=m-r$, $j=m+p$, and $k=m+q$ still with $i<j<k$, i.e. $q>p$. These conditions imply $a_{j,k}=0$, and we get

$$\begin{aligned} M_{ijk} &= a_{i,j}a_{i+j+l,k} + a_{k,i}a_{k+i+l,j} \\ &= a_{m-r,m+p}a_{m-(r-p),m+q} - a_{m-r,m+q}a_{m-(r-q),m+p}. \end{aligned}$$

Now we study the relative position of r to q: if first $r \geq q$, then $a_{m-r,m+q}=0$ and $a_{m-r,m+p}=0$. If $r<q$, then $a_{m-(r-q),m+p}=0$ and following the relative position of r to p, either $a_{m-(r-p),m+q}=0$ $(r<p)$ or $a_{m-r,m+p}=0$ $(r \geq p)$. In any case, all terms are zero. □

We now come to the last and main point of this section, namely the proof that the m-family is the only family in weight $l=-m \leq -5$ which satisfies the vanishing of

all Massey squares which cannot be compensated, i.e. of all Massey squares whose vanishing is necessary in order to have an true deformation.
Let therefore ω be a cocycle given by its coefficients $a_{i.j}$. By **2.1**, we have for $i+j \geq m+2$ the usual (or stable) cocycle identity $a_{i+1,j}+a_{i,j+1} = a_{i,j}$, and for $i+j = m, m+1$ just $a_{i+1,j} + a_{i,j+1} = 0$ while there is no equation for lower $i+j$. By **2.2**, we have $a_{i,j} = 0$ for $i+j = m+1$, compatible with the foregoing statements. By **2.0**, the coefficients $a_{1,2}, \ldots, a_{1,m-1}$ are zero, while by **2.3** $a_{1,m+1}, a_{1,m+2}, \ldots$ may be taken to be zero. Once again, we do not impose anything on $a_{1,m}$ in order to use the freedom of choice for a coboundary to take the first coefficients (from the diagonal) in the $(m+1)$st column to zero, according to **2.4**.
The Massey squares which can be compensated and thus do not impose conditions on ω are the M_{ijk} with $i+j = 2m+1$, $j+k = 2m+1$ or $k+i = 2m+1$.
Let us draw a diagram of the coefficients of ω:

	m-2	m-1	m	m+1	m+2
m-1	a				
m	a	b			
m+1	a-b	b	c		
m+2	a-2b	b-c	c	0	
m+3	a-3b+c	b-2c	c	0	e
m+4	a-4b+3c	b-3c	c	-e	e
m+5	a-5b+6c	b-4c	c+e	-2e	
m+6	a-6b+10c	b-5c-e	c+3e		
m+7	a-7b+15c+e	b-6c-4e			
m+8	a-8b+21c+5e				

Let us also expose some Massey squares M_{ijk} (such that no sum of pairs of indices gives $2m+1$):

$$\begin{aligned}
M_{m-1,m,m+3} &= a_{m-1,m}a_{m-1,m+3} + a_{m,m+3}a_{m+3,m-1} + a_{m+3,m-1}a_{m+2,m} \\
&= b(b-2c), \\
M_{m-1,m+1,m+3} &= a_{m-1,m+1}a_{m,m+3} + a_{m+1,m+3}a_{m+4,m-1} + a_{m+3,m-1}a_{m+2,m+1} \\
&= bc, \\
M_{m,m+2,m+3} &= a_{m,m+2}a_{m+2,m+3} + a_{m+2,m+3}a_{m+5,m} + a_{m+3,m}a_{m+3,m+1} \\
&= e(c-e), \\
M_{m-2,m,m+2} &= a_{m-2,m}a_{m-2,m+2} + a_{m,m+2}a_{m+2,m-2} \\
&= (a-c)(a-2b), \\
M_{m-2,m,m+4} &= a_{m-2,m}a_{m-2,m+4} + a_{m,m+4}a_{m+4,m-2} + a_{m+4,m-2}a_{m+2,m} \\
&= a(a-4b+3c), \\
M_{m-2,m,m+5} &= a_{m-2,m}a_{m-2,m+5} + a_{m,m+5}a_{m+5,m-2} + a_{m+5,m-2}a_{m+3,m} \\
&= (a-5b+6c)(a-e). \\
M_{m-1,m+1,m+3} &= bc = 0.
\end{aligned}$$

We now start a case study:

First case: $b = 0$, then by $M_{m-2,m,m+2} = 0$, either $a = 0$ or $a = c$. In the first subcase, $M_{m-2,m,m+5} = 0$ implies $ce = 0$, thus the only possibly non-zero parameter is c by $M_{m,m+2,m+3} = 0$. Note that a non-zero c corresponds to the m-family. In the second subcase, $a = c$ and then $M_{m-2,m,m+4} = 0$ implies $a = 0$. Finally $a = b = c = e = 0$ by $M_{m,m+2,m+3} = 0$.

Second case: $c = 0$, then by $M_{m,m+2,m+3} = 0$, $e = 0$, by $M_{m-1,m,m+3} = 0$, $b = 0$, and finally by $M_{m-2,m,m+4} = 0$, $a = 0$.

Now it is clear how to perform an induction step showing that the m-family is transmitted to a next two diagonals. For example using $M_{m-2,m,m+7}$:

$$M_{m-2,m,m+7} = a_{m-2,m+7}(a_{m-2,m} - a_{m,m+7} + a_{m,m+5}),$$

and by assumption $a_{m-2,m+7} \neq 0$, $a_{m-2,m} = 0$, and thus by $M_{m-2,m,m+7} = 0$, $a_{m,m+7} = a_{m,m+5} = c$ which is the induction step. This shows that starting from the $(m-2)$nd column, all coefficients are as in the m-family. In order to come to lower coefficients, take for example

$$M_{m-3,m+1,m+5} = a_{m-3,m+1}a_{m-2,m+5} + a_{m+1,m+5}a_{m+6,m-3} + a_{m+5,m-3}a_{m+2,m+1}.$$

Here, $a_{m+1,m+5} = a_{m+2,m+1} = 0$ and $a_{m-2,m+5} \neq 0$ by assumption, therefore $a_{m-3,m+1} = 0$ which transmits the m-family to the $(m-3)$rd column.
Finally, let us argue that the coefficient $a_{1,m}$ cannot be shown to be zero by the vanishing of Massey squares. Indeed, in order to involve $a_{1,m}$, the Massey square M_{ijk} (with ordered indices) must have either $i = 1$ or $i+j+l = 1$, $j+k+l = 1$ or $i+k+l = 1$. The first alternative is rather easily seen to be zero. Fix $i+j+l = 1$ for the second alternative. It describes a situation where the coefficient $a_{1,m}$ is multiplied by $a_{i,j}$ with $i+j = m+1$. This coefficient is zero.
To summarize, we have the following

Proposition 4.8. *The only non-zero cohomology class compatible with the vanishing of all Massey squares which cannot be compensated, is the m-family in weight $l = -m \leq -5$, with a possibly non-zero coefficient $a_{1,m}$.*

5. Deformations in zero and positive weights

a). **True deformations in weight $l = 0$.** In weight $l \geq 0$, a new phenomenon is happening: we have a relation between the Massey squares. Recall that the cocycle coefficients $a_{i,j}$ are supposed to be antisymmetric in i, j, and that $a_{i,i}$ is set to zero for all i.

Proposition 5.1. *Let i, j, k, be three integers, $i, k \geq 2$ and $j \geq 3$. We have the relation*

$$M_{ijk} + M_{i(j-1)(k+1)} + M_{(i+1)(j-1)k} \;=\; M_{i(j-1)k}.$$

Proof. We have by definition

$$\begin{aligned} M_{ijk} + M_{i(j-1)(k+1)} &= a_{i,j}a_{i+j+l,k} + a_{j,k}a_{j+k+l,i} + \\ a_{k,i}a_{i+k+l,j} + a_{i,j-1}a_{i+j-1+l,k+1} &+ a_{j-1,k+1}a_{j+k+l,i} + a_{k+1,i}a_{i+k+1+l,j-1}. \end{aligned}$$

We transform the terms $a_{j,k}a_{j+k+l,i} + a_{j-1,k+1}a_{j+k+l,i}$, using repeatedly the cocycle equation (2.6) to

$$a_{j+k+l,i}(a_{j,k} + a_{j-1,k+1}) \;=\; a_{j+k+l,i}a_{j-1,k} \;=\; a_{j-1,k}(a_{j+k-1+l,i} - a_{j+k-1+l,i+1}).$$

We transform the terms $a_{i,j}a_{i+j+l,k} + a_{i,j-1}a_{i+j-1+l,k+1}$, using the equations (2.6) to

$$a_{i,j}a_{i+j+l,k} + a_{i,j-1}a_{i+j-1+l,k} - a_{i,j-1}a_{i+j+l,k}.$$

We transform the terms $a_{k,i}a_{i+k+l,j} + a_{k+1,i}a_{i+k+1+l,j-1}$, using the equations (2.6) to

$$a_{k,i}a_{i+k+l,j-1} - a_{k,i}a_{i+k+1+l,j-1} + a_{k+1,i}a_{i+k+1+l,j-1}.$$

In these three transformations, the sum of the first term of the first, the second term of the second and the first term of the third give together

$$a_{j-1,k}a_{j+k-1+l,i} + a_{i,j-1}a_{i+j-1+l,k} + a_{k,i}a_{i+k+l,j-1} \;=\; M_{i(j-1)k}.$$

The remaining terms read

$$-a_{j-1,k}a_{j+k-1+l,i+1} + a_{i,j}a_{i+j+l,k} - a_{i,j-1}a_{i+j+l,k} - a_{k,i}a_{i+k+1+l,j-1} + a_{k+1,i}a_{i+k+1+l,j-1}.$$

Here, the second and third term give

$$a_{i,j}a_{i+j+l,k} - (a_{i,j}a_{i+j+l,k} + a_{i+1,j-1}a_{i+j+l,k}) = -a_{i+j+l,k}a_{i+1,j-1},$$

while the last two terms give

$$-(a_{k+1,i} + a_{k,i+1})a_{i+k+1+l,j-1} + a_{k+1,i}a_{i+k+1+l,j-1} = -a_{i+k+1+l,j-1}a_{k,i+1},$$

still using the equations (2.6).
In summary, the remaining terms give

$$-a_{j-1,k}a_{j+k-1+l,i+1} - a_{i+j+l,k}a_{i+1,j-1} - a_{i+k+1+l,j-1}a_{k,i+1} = -M_{(i+1)(j-1)k}.$$

This ends the proof of Propositin 5.1. □

Corollary 5.2.

$$M_{i(i+1)k} + M_{i(i+2)(k-1)} = M_{i(i+1)(k-1)}.$$

Observe that also repeated indices may give interesting relations: for example, for $i = 2$, $j = 4$ and $k = 4$, we get $M_{234} = M_{235}$. It is easily shown by these relations that the nullity of M_{23k} for all $k \geq 4$ is necessary for the nullity of all Massey squares, and that the nullity of M_{2rs} for all $r, s \geq 3$ is necessary and sufficient for the nullity of all Massey squares. We believe that the minimal set of Massey squares whose nullity implies the nullity of all Massey squares is somewhere in between these two sets, but we could not get hold on it.
Now, we will determine all square zero cocycles, i.e. all true deformations of $\mathfrak{m}_0$, in weight $l = 0$: first of all, the 2-family is such a cocycle. Then, let us suppose that ω is a non-trivial 2-cocycle which is independent of the 2-family and has zero Massey squares; as before, we think of ω as given by the coefficients $a_{i,j}$, and we will distribute letters to its initial terms: $a_{2,3} = a$, $a_{3,4} = b$, and so on.
Using equations (2.6), we establish the following diagram which is of course valid for all weights l; observe that the general expression for the coefficients in section **4.5**, proof of Proposition 6, leads for general coefficients $a_{2,3} =: u_2 = a, a_{3,4} =: u_3 = b, a_{4,5} =: u_4 = c$ and so on (by linearity) to the formula

$$a_{i,j} = \sum_{m=2}^{j-1} (-1)^{m-i} u_m \frac{(j-(m+1))!}{(m-i)!(j-2m+i-1)!}, \tag{5.8}$$

which may be used to compute the coefficients in the following diagram more easily (than by a recursive formula).

	2	3	4	5	6	7
3	a					
4	a	b				
5	a-b	b	c			
6	a-2b	b-c	c	d		
7	a-3b+c	b-2c	c-d	d	e	
8	a-4b+3c	b-3c+d	c-2d	d-e	e	f
9	a-5b+6c-d	b-4c+3d	c-3d+e	d-2e	e-f	f
10	a-6b+10c-4d	b-5c+6d-e	c-4d+3e	d-3e+f	e-2f	
11	a-7b+15c-10d+e	b-6c+10d-4e	c-5d+6e-f	d-4e+3f		
12	a-8b+21c-20d+5e	b-7c+15d-10e+f	c-6d+10e-4f			
13	a-9b+28c-35d+15e-f	b-8c+21d-20e+5f				
14	a-10b+36c-56d+35e-6f					

From now on, we consider weight $l = 0$.

Order the Massey squares by their *level*, i.e. we say that M_{ijk} has level $i + j + k$. Computing Massey squares and setting them equal to zero gives an infinite family of homogeneous quadratic equations for the infinite family of variables $a, b, c, d, \ldots$.
In level 9, the only Massey square is M_{234}, and its nullity gives

$$3b^2 - bc - 2ac = 0.$$

In level 10, the only Massey square is M_{235}, and its nullity gives the same equation. In level 11, there are Massey squares M_{236} and M_{245}, and their nullity gives (possibly by subtracting the previous equation) in both cases

$$2ad - 4bc - bd + 6c^2 - cd = 0.$$

In level 12, there are Massey squares M_{237}, M_{246} and M_{345}, and their nullity gives (possibly by subtracting the previous equations) in all cases

$$-3bd + 4c^2 - 3cd = 0.$$

Going higher in this hierarchy of equations and variables, there are at each new level some (possibly) linearly independent equations. Proposition 9 only tells us that the nullity of M_{2rs} with $2 < r < s$ is enough in order to have all Massey squares zero. We don't know which of these equations are in fact the independent ones.

In Massey square level 14, we arrive at 5 equations for the 5 variables a, b, c, d, e, which read (after subtracting at each step multiples of the previous equations):

$$\begin{aligned} 3b^2 - bc - 2ac &= 0 \\ 2ad - 4bc - bd + 6c^2 - cd &= 0 \\ -3bd + 4c^2 - 3cd &= 0 \\ e(-2a + 3b - d) + 5bd - 15cd + 10d^2 &= 0 \\ e(-6a + 15b - 4c - 11d) - 55cd + 50d^2 + 15bd &= 0 \end{aligned}$$

The discussion of these equations (either by hand or by a system computing a Gröbner basis for the homogeneous polynomials) gives as non-zero solutions the 2-family and one other family with coefficients $a = \frac{1}{6}$, $b = \frac{1}{60}$, $c = \frac{1}{420}$, etc. We describe this family from another point of view in subsection **5.3**, which will show that this family must verify all equations and not only the five equations we wrote down. These are the only square zero solutions in weight 0, and we have determined all true deformations in this case.

b). **True deformations in weight $l > 0$.** In the weight $l = 1$ case, we get from the same diagram as in weight $l = 0$ up to Massey level 15 (where we took only the equations of type M_{23k} in order to simplify) six homogeneous quadratic equations in six variables which read:

$$\begin{aligned} -3ac + 4b^2 - 3bc &= 0 \\ -5bc - 2bd + 10c^2 - 4cd + 3ad &= 0 \\ 5c^2 - 4bd + 2be + ec - 6cd &= 0 \\ e(-3a + 11b - 5d + 3c) - 6bd + 15c^2 - 39cd + 20d^2 &= 0 \\ e(-9a + 35b - 35d) + f(-4b + 2c + 2d) - 6bd + 30c^2 - 111cd + 90d^2 &= 0 \\ e(-18a + 75b + 11c - 186d + 35e - 6f) + f(3a + 20c - 24b + 16d) & \\ -4bd + 50c^2 - 234cd + 252d^2 &= 0 \end{aligned}$$

By a computation with MUPaD which determines a Gröbner basis for the homogeneous polynomials corresponding to these equations (actually we took here all equations of type M_{2rs}), one obtains as (non-zero) solutions the 2-family, a solution $a = 0, b = 0, c = 0, d = 0$ and $e = 1$, $f = \frac{35}{6}$, and a further solution $a = 1$, $b = \frac{1}{7}$, $c = \frac{1}{42}$, $d = \frac{1}{231}$, $e = \frac{5}{21 \cdot 286}$, $f = \frac{1}{21 \cdot 286}$. The solution with $e = 1$ and $f = \frac{35}{6}$ does not survive the next level of Massey squares.

But the solution starting with $a = 1$, $b = \frac{1}{7}$ continues with $g = \frac{1}{29172}$, $h = \frac{1}{138567}$, $i = \frac{1}{646646}$, $j = \frac{5}{14872858}$, $k = \frac{1}{13520780}$. We will describe this family from a different point of view in section **5.3**, and we will show there (implicitly) that this solution survives to infinity.

Proposition 5.3. *In weight $l = 1$, there are exactly two non-equivalent true deformations.*

The problem of determining the explicit square zero cocycles in each weight l case seems to be a rich problem. We tried to say something about the rank of the finite Jacobi matrix either associated to the set of equations of type $M_{2rs} = 0$ for all r, s, or to those of type M_{23r} for all r, when we truncate the number of variables and consider only those equations involving these variables. With this matrix, it is obvious that the set of solutions (of the truncated problem) is an algebraic variety of dimension greater or equal to 1 (because the equations are homogeneous), but we couldn't decide whether the dimension drops down to 1 in each weight. In fact, within the possibilities of our computer, we computed (using all equations of type $M_{2rs} = 0$) the dimension of this variety as far as possible for $l = 2$ and it remained 2. Is the set of solutions always a variety ? Is it always of finite dimension ? Can we give asymptotics or bounds or a formula for the dimension ?

c). **Identifying the cocycles and the deformed algebras in weight zero.** We now construct deformations from the previously determined weight l 2-cocycles given by their coefficients $a_{i,j}$ in the following way: using still the e_i for $i \geq 1$ as a basis, the deformed bracket reads

$$[e_i, e_j]_t \;=\; [e_i, e_j] \;+\; t a_{i,j} e_{i+j+l}.$$

It is clear that all square zero 2-cocycles give in this way true deformations of $\mathfrak{m}_0$ for which only the linear term is (possibly) non-zero; this means in particular that the bracket $[-,-]_t$ satisfies the Jacobi identity without adding terms containing higher powers in t.

The weight $l = 0$ case is the most interesting, because here deformations give automatically rise to $\mathbb{N}$-graded Lie algebras which must fit in the classification [1]. In this classification, the three $\mathbb{N}$-graded Lie algebras where e_1 has non-zero brackets with all other basis elements are

(1) $\mathfrak{m}_0$; brackets: $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$,
(2) $\mathfrak{m}_2$; brackets: $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+1}$ for all $j \geq 3$,
(3) L_1; brackets: $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \geq 1$.

The complete set of infinitesimal deformations of L_1 and the complete set of formal deformations of L_1 is given in [2]. Let us consider in this section the same problem for $\mathfrak{m}_0$ in weight $l = 0$.

Taking as 2-cocycle the 2-family, we get in weight $l = 0$ a Lie algebra $\mathfrak{m}_0^1(t)$ which must be $\mathbb{N}$-graded and which is easily seen to be generated by e_1 and e_2: indeed, $[e_1, e_i]_t = [e_1, e_i]$ for all $i \geq 2$. The complete relations for $\mathfrak{m}_0^1(t)$ are

$$\begin{cases} [e_1, e_i]_t &= e_{i+1} \quad \forall \quad i \geq 2 \\ [e_2, e_j]_t &= t e_{j+2} \quad \forall \quad j \geq 3 \end{cases}$$

Thus this family describes the deformation of $\mathfrak{m}_0$ to $\mathfrak{m}_2$.

Now there is also a cocycle describing the deformation of $\mathfrak{m}_0$ to L_1: observe that the generators e_i for $i \geq 1$ of $\mathfrak{m}_0$ do not satisfy the right relations, seen as elements of L_1. Therefore, one must first perform a change of base: let us define $\tilde{e}_1 = e_1$, $\tilde{e}_2 = e_2$, $\tilde{e}_3 = e_3$, $\tilde{e}_4 = \frac{1}{2} e_4$ and in general

$$\tilde{e}_i \;=\; \frac{1}{(i-2)!} e_i$$

for all $i \geq 5$. Then the relations are easily computed to be

$$[\tilde{e}_1, \tilde{e}_i] = (i-1)\tilde{e}_{i+1}.$$

The cocycle relation (2.6) for a cocycle given by coefficients $b_{i,j}$ transforms in the new basis to

$$(j-1) b_{j+1,k} \;+\; (k-1) b_{j,k+1} \;=\; (j+k+l-1) b_{j,k}.$$

It is easy to check that the 2-cochain given by the coefficients $b_{i,j} = (j - i)$ is indeed a 2-cocycle for $l = 0$. One also easily checks that the 2-cocycle $b_{i,j} = (j - i)$ is of

Massey square zero, i.e.

$$b_{i,j}b_{i+j,k} + b_{j,k}b_{j+k,i} + b_{k,i}b_{k+i,j} = 0.$$

It therefore determines a deformation $\mathfrak{m}_0^2(t)$ of $\mathfrak{m}_0$ in weight $l = 0$ to L_1, and we showed in the previous section that these are all possible deformations of $\mathfrak{m}_0$ in weight 0.
Let us finish with the identification of the deformations in weight $l = 1$. It is clear that the 2-family leads to a non-trivial true deformation. This is then a weight 1 variant of the Lie algebra $\mathfrak{m}_2$. The other cocycle, determined using MUPaD, is more interesting. Indeed, there is a general procedure of constructing positive weight, true deformations for $\mathfrak{m}_0$: consider the Lie algebra L_1, with its generators e_1, e_2, e_3, etc and its relations $[e_i, e_j] = (j-i)e_{i+j}$ for all $i, j \geq 1$. Define a Lie algebra $L_1\{2\}$ by generators e_1, e_3, e_4 etc (the suppression of e_2 is indicated by $\{2\}$ in the notation !) and the relations of L_1 for the remaining generators. Introduce a new basis $f_1 := e_1$, $f_2 := e_3$, $f_3 := e_4$ etc, and another new basis $g_1 := e_1$, $g_k := (k-1)!\ f_k$ for all $k > 1$. We compute the relations to

$$[g_1, g_k] = g_{k+1}, \qquad [g_k, g_{k+1}] = \frac{k!(k-1)!}{(2k+1)!} g_{2k+2}.$$

When interpreted as a deformation of $\mathfrak{m}_0$, one can then compute the coefficients $a_{i,j}$ of the corresponding 2-cocycle. One obtains $a_{2,3} = \frac{2!1!}{5!} = \frac{1}{60} \cdot 1$, $a_{3,4} = \frac{3!2!}{7!} = \frac{1}{60} \cdot \frac{1}{7}$, $a_{4,5} = \frac{4!3!}{9!} = \frac{1}{60} \cdot \frac{1}{42}$, $a_{5,6} = \frac{5!4!}{11!} = \frac{1}{60} \cdot \frac{1}{231}$, $a_{6,7} = \frac{6!5!}{13!} = \frac{1}{60} \cdot \frac{5}{21 \cdot 286}$, $a_{7,8} = \frac{7!6!}{15!} = \frac{1}{60} \cdot \frac{1}{21 \cdot 286}$, and so on. Thus, this deformation has as its infinitesimal cocycle the cocycle we determined using MUPaD before, up to a factor $\frac{1}{60}$. By construction, it is clear that it defines a cocycle and a true deformation, because $L_1\{2\}$ is a Lie algebra.
In the same way, one can define $L_1\{m\}$ by the span of the vectors e_1, e_{m+1}, e_{m+2}, etc for any $m > 2$, in other words, by the suppression of all basis vectors from e_2 up to and including e_{m+1}. This gives a non-trivial true deformation of weight $m-1$. Together with the deformation given by the 2-family, these two constitute two independent true deformations in any positive weight.

References

[1] Fialowski, Alice, "Classification of graded Lie algebras with two generators," Moscow Univ. Math. Bull. **38**, 2 (1983) 76–79

[2] Fialowski, Alice, "An example of formal deformations of Lie algebras," in *Deformation Theory of Algebras and Structures and Applications*, ed.: M. Hazewinkel and M. Gerstenhaber, 1988 Kluwer Academic Publishers

[3] Fialowski, Alice, Fuchs, Dmitry, "Construction of Miniversal Deformations of Lie Algebras," *J. Funct. Anal.*, **161** (1999) 76–110

[4] Fialowski, Alice, Millionschikov, Dmitri, "Cohomology of graded Lie algebras of maximal class," *J. Algebra*, **296**, no. 1 (2006) 157–176

[5] Fuchs, Dmitry B., *Cohomology of Infinite Dimensional Lie algebras*, Consultants Bureau, New York, 1987.

[6] Gontcharova, L. V., "Cohomology of Lie algebras of formal vector fields on the line" *Funct. Anal. Appl.*, **7**(2) (1973), 6–14

[7] Hartshorne, Robin, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977

[8] Shalev, Aner, Zelmanov, Efim I., "Narrow Lie algebras: A coclass theory and a characterization of the Witt algebra," *J. Algebra*, **189** (1997) 294–331

[9] Vergne, Michèle, "Cohomologie des algèbres de Lie nilpotentes," *Bull. Soc. Math. France*, **98** (1970) 81–116

Alice Fialowski, *email*: `fialowsk@cs.elte.hu`
Friedrich Wagemann, *email*: `wagemann@math.univ-nantes.fr`

Part III: Generalizations and Applications

APPEARED IN: *Commun. in Contemporary Math.*, **5** (2003), 921–945

Global Deformations of the Witt Algebra of Krichever-Novikov Type

Alice Fialowski
Institute of Mathematics
Eötvös Loránd University, Budapest

Martin Schlichenmaier
Laboratoire de Mathematiques
Université du Luxembourg

We dedicate this article to the memory of our good friend Peter Slodowy who passed away in 2002

Abstract By considering non-trivial global deformations of the Witt (and the Virasoro) algebra given by geometric constructions it is shown that, despite their infinitesimal and formal rigidity, they are globally not rigid. This shows the need of a clear indication of the type of deformations considered. The families appearing are constructed as families of algebras of Krichever-Novikov type. They show up in a natural way in the global operator approach to the quantization of two-dimensional conformal field theory. In addition, a proof of the infinitesimal and formal rigidity of the Witt algebra is presented.

1. Introduction

Deformations of mathematical structures are important in most part of mathematics and its applications. Deformation is one of the tools used to study a specific object, by deforming it into some families of "similar" structure objects. This way we get a richer picture about the original object itself.

But there is also another question approached via deformations. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (maybe up to certain equivalences) with the structure of a topological or even geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object the deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

In this respect, a clear success story is the classification of complex analytic structures on a fixed topological manifold. Also in algebraic geometry one has well-developed results in this direction. One of these results is that the local situation at a point $[C]$ of the moduli space is completely governed by the cohomological properties of the geometric object C. As typical example recall that for the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g over $\mathbb{C}$ (or equivalently, compact Riemann surfaces of genus g) the tangent space $T_{[C]}\mathcal{M}_g$ can be naturally identified with $\mathrm{H}^1(C, T_C)$, where T_C is the sheaf of holomorphic vector fields over C. This extends to higher dimension. In particular, it turns out that for compact complex manifolds M, the condition $\mathrm{H}^1(M, T_M) = \{0\}$ implies that M is rigid, [16, Thm. 4.4]. Rigidity means that any differentiable family $\pi : \mathcal{M} \to B \subseteq \mathbb{R}$, $0 \in B$ which contains M as the special member $M_0 := \pi^{-1}(0)$ is trivial in a neighbourhood of 0, i.e. for t small enough $M_t := \pi^{-1}(t) \cong M$. Even more generally, for M a compact complex manifold and $\mathrm{H}^1(M, T_M) \neq \{0\}$ there exists a versal family which can be realized locally as a family

1991 Mathematics Subject Classification. Primary: 17B66; Secondary: 17B56, 17B65, 17B68, 14D15, 14H52, 30F30, 81T40

Key words and phrases. Deformations of algebras, rigidity, Virasoro algebra, Krichever-Novikov algebras, elliptic curves, Lie algebra cohomology, conformal field theory

The research of the first author was partially supported by the grants OTKA T030823, T29525. The last version of the work was completed during her stay at the Max-Planck-Institut für Mathematik Bonn.

over a certain subspace of $\mathrm{H}^1(M, T_M)$ such that every appearing deformation family is "contained" in this versal family (see also [18] for definitions, results, and further references).

These positive results lead to the impression that the vanishing of the relevant cohomology spaces will imply rigidity with respect to deformations also in the case of other structures.

The goal of this article is to shed some light on this in the context of deformations of infinite-dimensional Lie algebras. We consider the case of the Witt algebra $\mathcal{W}$ (see its definition further down). The cohomology "responsible" for deformations is $\mathrm{H}^2(\mathcal{W}, \mathcal{W})$. It is known that $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = \{0\}$ (see Section 3). Hence, guided by the experience in the theory of deformations of complex manifolds, one might think that $\mathcal{W}$ is rigid in the sense that all families containing $\mathcal{W}$ as a special element will be trivial. But this is not the case as we will show. Certain natural families of Krichever-Novikov type algebras of geometric origin (see Section 4 for their definition) will appear which contain $\mathcal{W}$ as special element. But none of the other elements will be isomorphic to $\mathcal{W}$. In fact, from $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = \{0\}$ it follows that the Witt algebra is infinitesimally and formally rigid. But this condition does not imply that there are no non-trivial global deformation families. The main point to learn is that it is necessary to distinguish clearly the formal and the global deformation situation. The formal rigidity of the Witt algebra indeed follows from $\mathrm{H}^2(\mathcal{W}, \mathcal{W}) = \{0\}$, but no statement like that about global deformations.

How intricate the situation is, can be seen from the fact that for the subalgebra L_1 of $\mathcal{W}$, consisting of those vector fields vanishing at least of order two at zero, there exists a versal formal family consisting of three different families parameterized over a collection of three curves in $\mathrm{H}^2(L_1, L_1)$. Suitably adjusted each family corresponds to a scalar multiple of the same cohomology class ω [6],[7],[9] which gives their infinitesimal deformations. It turns out that $\mathrm{H}^2(L_1, L_1)$ is 3-dimensional, but only the infinitesimal deformations corresponding to scalar multiples of ω can be extended to formal deformations.

The results of this article will show that the theory of deformations of infinite-dimensional Lie algebras is still not in satisfactory shape. Hopefully, the appearing features will be of help in a further understanding.

Clearly, what will be done here, can also be done in the case of associative algebras. In particular, there will be global deformations of the associative algebra of Laurent polynomials of Krichever-Novikov type obtained by the same process as the one presented here.

First, let us introduce the basic definitions. Consider the complexification of the Lie algebra of polynomial vector fields on the circle with generators

$$l_n := \exp(in\varphi)\frac{d}{d\varphi}, \qquad n \in \mathbb{Z},$$

where φ is the angular parameter. The bracket operation in this Lie algebra is

$$[l_n, l_m] = (m - n)l_{n+m}.$$

We call it the *Witt algebra* and denote it by $\mathcal{W}$. Equivalently, the Witt algebra can be described as the Lie algebra of meromorphic vector fields on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are holomorphic outside the points 0 and ∞. In this presentation $l_n = z^{n+1}\frac{d}{dz}$, where z is the quasi-global coordinate on $\mathbb{P}^1(\mathbb{C})$.

The Lie algebra $\mathcal{W}$ is infinite dimensional and graded with the standard grading $\deg l_n = n$. By taking formal vector fields with the projective limit topology we get the completed topological Witt algebra $\mathcal{W}_t op$. Throughout this paper we will consider its everywhere dense subalgebra $\mathcal{W}$.

It is well-known that $\mathcal{W}$ (up to equivalence and rescaling) has a unique nontrivial one-dimensional central extension, the *Virasoro algebra* $\mathcal{V}$. It is generated by l_n $(n \in \mathbb{Z})$ and the central element c, and its bracket operation is defined by

$$[l_n, l_m] = (m - n)l_{n+m} + \tfrac{1}{12}(m^3 - m)\delta_{n,-m}\, c, \qquad [l_n, c] = 0. \tag{1.9}$$

In Section 2 we recall the different concepts of deformation. There is a lot of confusion in the literature in the notion of a deformation. Several different (inequivalent) approaches exist. One of the aims of this paper is to clarify the difference between deformations of geometric origin and so called formal deformations. Formal deformation theory has the advantage of using cohomology. It is also complete in the sense that under some natural cohomology assumptions there exists a versal formal deformation, which induces all other deformations.
In this context see Theorem 2.7 and Corollary 2.8 which are quoted from [6],[9]. Formal deformations are deformations with a complete local algebra as base. A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the base of deformation – which is a commutative algebra of functions – with a smooth manifold, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations. As already indicated above, in infinite dimension, there is no tight relation between global and formal deformations, as we will show in this paper.

In Section 3, we supply a detailed proof of the infinitesimal and formal rigidity of the Witt algebra $\mathcal{W}$, by showing that $\mathrm{H}^2(\mathcal{W},\mathcal{W}) = 0$.

In Section 4 we introduce and recall the necessary properties of the Krichever-Novikov type vector field algebras. They are generalizations of the Witt algebra (in its presentation as vector fields on $\mathbb{P}^1(\mathbb{C})$) to higher genus smooth projective curves.

In Section 5 we construct global deformations of the Witt algebra by considering certain families of algebras for the genus one case (i.e. the elliptic curve case) and let the elliptic curve degenerate to a singular cubic. The two points, where poles are allowed, are the zero element of the elliptic curve (with respect to its additive structure) and a 2-torsion point. In this way we obtain families parameterized over the affine line with the peculiar behaviour that every family is a global deformation of the Witt algebra, i.e. $\mathcal{W}$ is a special member, whereas all other members are mutually isomorphic but not isomorphic to $\mathcal{W}$, see Theorem 5.3. Globally these families are non-trivial, but infinitesimally and formally they are trivial. The construction can be extended to the centrally extended algebras, yielding a global deformation of the Virasoro algebra. Finally, we consider the subalgebra L_1 of $\mathcal{W}$ corresponding to the vector fields vanishing at least with order two at 0. This algebra is formally not rigid, and its formal versal deformation has been determined [6, 8]. We identify one of the appearing three families in our geometric context.

The results obtained do not have only relevance to the deformation theory of algebras but also to the theory of two-dimensional conformal fields and their quantization. It is well-known that the Witt algebra, the Virasoro algebra, and their representations are of fundamental importance for the local description of conformal field theory on the Riemann sphere (i.e. for genus zero), see [1]. Krichever and Novikov [17] proposed in the case of higher genus Riemann surfaces the use of global operator fields which are given with the help of the Lie algebra of vector fields of Krichever-Novikov type, certain related algebras, and their representations (Section 4).
In the process of quantization of the conformal fields one has to consider families of algebras and representations over the moduli space of compact Riemann surfaces (or equivalently, of smooth projective curves over $\mathbb{C}$) of genus g with N marked points. See [28] for a global operator version, and [29] for a sheaf version. In passing to the boundary of the moduli space one obtains the limit objects which are defined over the normalization of curves of lower genus. Assuming good behaviour of the examined objects under deformation also in the limit (e.g. "factorization"), the degeneration is an important technique to obtain via induction from results for lower genus results for all genera. See Tsuchiya, Ueno, and Yamada's proof of the Verlinde formula [29] as an application of this principle.
By a maximal degeneration a collection of $\mathbb{P}^1(\mathbb{C})$'s will appear. For the vector field algebras (with or without central extension) one obtains families of algebras which are

related to the Witt or Virasoro algebra or certain subalgebras respectively. Indeed, the examples considered in this article are exactly of this type. They appear as families which are naturally defined over the moduli space of complex one-dimensional tori (i.e. of elliptic curves) with two marked points. The full geometric picture behind it was discussed in [25]. In special cases the Witt and Virasoro algebra appear as degenerations of the Krichever-Novikov algebras. Considered from the opposite point of view, in the sense of this article, the Krichever-Novikov algebras are global deformations of the Witt and Virasoro algebra. Nevertheless, as we show here, the structure of these algebras are not determined by the Witt algebra, despite the formal rigidity of the latter.

2. Deformations and formal deformations

a). **Intuitively.** Let us start with the intuitive definition. Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$. A *deformation* of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket

$$\mu_t = \mu_0 + t\phi_1 + t^2\phi_2 + ... \tag{2.10}$$

where the ϕ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\bigwedge^2 \mathcal{L}, \mathcal{L}) = C^2(\mathcal{L}, \mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. (see [5, 11]). Two deformations $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\hat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + ...$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C^1(\mathcal{L}, \mathcal{L})$, such that

$$\mu'_t(x, y) = \hat{\psi}_t^{-1}(\mu_t(\hat{\psi}_t(x), \hat{\psi}_t(y))). \tag{2.11}$$

The Jacobi identity for the algebra $\mathcal{L}_t$ implies that the 2-cochain ϕ_1 is indeed a cocycle, i.e. it fulfills $d_2\phi_1 = 0$ with respect to the Lie algebra cochain complex of $\mathcal{L}$ with values in $\mathcal{L}$ (see [10] for the definitions). If ϕ_1 vanishes identically, the first nonvanishing ϕ_i will be a cocycle. If μ'_t is an equivalent deformation (with cochains ϕ'_i) then

$$\phi'_1 - \phi_1 = d_1\psi_1. \tag{2.12}$$

Hence every equivalence class of deformations defines uniquely an element of $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$. This class is called the *differential* of the deformation. The differential of a family which is equivalent to a trivial family will be the zero cohomology class.

b). **Global deformations.** Consider now a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra A over $\mathbb{K}$ with identity as base of a deformation.
In the following we will assume that A is a commutative algebra over the field $\mathbb{K}$ of characteristic zero which admits an augmentation $\epsilon : A \to \mathbb{K}$. This says that ϵ is a $\mathbb{K}$-algebra homomorphism, e.g. $\epsilon(1_A) = 1$. The ideal $m_\epsilon := \mathrm{Ker}\,\epsilon$ is a maximal ideal of A. Vice versa, given a maximal ideal m of A with $A/m \cong \mathbb{K}$, then the natural factorization map defines an augmentation.
In case that A is a finitely generated $\mathbb{K}$-algebra over an algebraically closed field $\mathbb{K}$, $A/m \cong \mathbb{K}$ is true for every maximal ideal m. Hence in this case every such A admits at least one augmentation and all maximal ideals are coming from augmentations.
Let us consider a Lie algebra $\mathcal{L}$ over the field $\mathbb{K}$, ϵ a fixed augmentation of A, and $m = \mathrm{Ker}\,\epsilon$ the associated maximal ideal.

Definition 1. A *global deformation* λ of $\mathcal{L}$ with the base (A, m) or simply with the base A, is a Lie A-algebra structure on the tensor product $A \bigotimes_{\mathbb{K}} \mathcal{L}$ with bracket $[.,.]_\lambda$ such that

$$\epsilon \bigotimes \mathrm{id} : A \bigotimes \mathcal{L} \to \mathbb{K} \bigotimes \mathcal{L} = \mathcal{L} \tag{2.13}$$

is a Lie algebra homomorphism (see [9]). Specifically, it means that for all $a, b \in A$ and $x, y \in \mathcal{L}$,

(1) $[a \bigotimes x, b \bigotimes y]_\lambda = (ab \bigotimes \mathrm{id})[1 \bigotimes x, 1 \bigotimes y]_\lambda$,

(2) $[.,.]_\lambda$ is skew-symmetric and satisfies the Jacobi identity,

(3) $\epsilon \bigotimes \mathrm{id}\,([1 \bigotimes x, 1 \bigotimes y]_\lambda) = 1 \bigotimes [x,y]$.

By Condition (1), to describe a deformation it is enough to specify the elements $[1 \bigotimes x, 1 \bigotimes y]_\lambda$ for all $x, y \in \mathcal{L}$. By condition (3) it follows that for them the Lie product has the form

$$[1 \bigotimes x, 1 \bigotimes y]_\lambda = 1 \bigotimes [x,y] + \textstyle\sum_i a_i \bigotimes z_i, \tag{2.14}$$

with $a_i \in m$, $z_i \in \mathcal{L}$.
A deformation is called *trivial* if $A \bigotimes_{\mathbb{K}} \mathcal{L}$ carries the trivially extended Lie structure, i.e. (2.14) reads as $[1 \bigotimes x, 1 \bigotimes y]_\lambda = 1 \bigotimes [x,y]$. Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \bigotimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\epsilon \bigotimes \mathrm{id}$.
We say that a deformation is *local* if A is a local $\mathbb{K}$-algebra with unique maximal ideal m_A. By assumption, $m_A = \operatorname{Ker} \epsilon$ and $A/m_A \cong \mathbb{K}$. In case that in addition ${m_A}^2 = 0$, the deformation is called *infinitesimal.*

Example 2.1. If $A = \mathbb{K}[t]$, this is the same as an algebraic 1-parameter deformation of $\mathcal{L}$. In this case we sometimes use simply the expression "deformation over the affine line." This can be extended to the case where A is the algebra of regular functions on an affine variety X. In this way we obtain algebraic deformations over an affine variety. These deformations are non-local, and will be the objects of our study in Section 5.

Let A' be another commutative algebra over $\mathbb{K}$ with a fixed augmentation $\epsilon' : A' \to \mathbb{K}$, and let $\phi : A \to A'$ be an algebra homomorphism with $\phi(1) = 1$ and $\epsilon' \circ \phi = \epsilon$. If a deformation λ of $\mathcal{L}$ with base $(A, \operatorname{Ker} \epsilon = m)$ is given, then the *push-out* $\lambda' = \phi_* \lambda$ is the deformation of $\mathcal{L}$ with base $(A', \operatorname{Ker} \epsilon' = m')$, which is the Lie algebra structure

$$[a'_1 \bigotimes_A (a_1 \bigotimes l_1), a'_2 \bigotimes_A (a_2 \bigotimes l_2)]_{\lambda'} := a'_1 a'_2 \bigotimes_A [a_1 \bigotimes l_1, a_2 \bigotimes l_2]_\lambda$$

$(a'_1, a'_2 \in A', a_1, a_2 \in A, l_1, l_2 \in \mathcal{L})$ on $A' \bigotimes \mathcal{L} = (A' \bigotimes_A A) \bigotimes \mathcal{L} = A' \bigotimes_A (A \bigotimes \mathcal{L})$. Here A' is regarded as an A-module with the structure $aa' = a'\phi(a)$.

Remark 2.2. For non-local algebras there exist more than one maximal ideal, and hence in general many different augmentations ϵ. Let $\mathcal{L}$ be a $\mathbb{K}$-vector space and assume that there exists a Lie A-algebra structure $[.,.]_A$ on $A \bigotimes_{\mathbb{K}} \mathcal{L}$. Given an augmentation $\epsilon : A \to \mathbb{K}$ with associated maximal ideal $m_\epsilon = \operatorname{Ker} \epsilon$, one obtains a Lie $\mathbb{K}$-algebra structure $\mathcal{L}^\epsilon = (\mathcal{L}, [.,.]_\epsilon)$ on the vector space $\mathcal{L}$ by

$$\epsilon \bigotimes \mathrm{id}\,([1 \bigotimes x, 1 \bigotimes y]_A) = 1 \bigotimes [x,y]_\epsilon. \tag{2.15}$$

Comparing this with Definition 1 we see that by construction the Lie A-algebra $A \bigotimes_{\mathbb{K}} \mathcal{L}$ will be a global deformation of the Lie $\mathbb{K}$-algebra $\mathcal{L}^\epsilon$. On the level of structure constants the described construction corresponds simply to the effect of "reducing the structure constants of the algebra modulo m_ϵ." In other words, for $x, y, z \in \mathcal{L}$ basis elements, let the Lie A-algebra structure be given by

$$[1 \otimes x, 1 \otimes y]_A = \sum_z C^z_{x,y} (1 \otimes z), \qquad C^z_{x,y} \in A. \tag{2.16}$$

Then $\mathcal{L}^\epsilon$ is defined via

$$[x,y]_\epsilon := \sum_z (C^z_{x,y} \bmod m_\epsilon)\, z. \tag{2.17}$$

In general, the algebras $\mathcal{L}^\epsilon$ will be different for different ϵ. The Lie A-algebra $A \bigotimes_{\mathbb{K}} \mathcal{L}$ will be a deformation of different Lie $\mathbb{K}$-algebras L^ϵ.

Example 2.3. For a deformation of the Lie algebra $\mathcal{L} = \mathcal{L}_0$ over the affine line, the Lie structure $\mathcal{L}_\alpha$ in the fiber over the point $\alpha \in \mathbb{K}$ is given by considering the augmentation corresponding to the maximal ideal $m_\alpha = (t - \alpha)$. This explains the picture in the geometric interpretation of the deformation.

c). **Formal deformations.** Let A be a complete local algebra over $\mathbb{K}$, so $A = \varprojlim_{n\to\infty}(A/m^n)$, where m is the maximal ideal of A and we assume that $A/m \cong \mathbb{K}$.

Definition 2. A *formal deformation* of $\mathcal{L}$ with base A is a Lie algebra structure on the completed tensor product $A\widehat{\otimes}\mathcal{L} = \varprojlim_{n\to\infty}((A/m^n)\bigotimes\mathcal{L})$ such that

$$\epsilon\widehat{\otimes}\mathrm{id} : A\widehat{\otimes}\mathcal{L} \to \mathbb{K}\bigotimes\mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Example 2.4. If $A = \mathbb{K}[[t]]$, then a formal deformation of $\mathcal{L}$ with base A is the same as a formal 1-parameter deformation of $\mathcal{L}$ (see [11]).

There is an analogous definition for equivalence of deformations parameterized by a complete local algebra.

d). **Infinitesimal and versal formal deformations.** In the following let the base of the deformation be a local $\mathbb{K}$-algebra (A, m) with $A/m \cong \mathbb{K}$. In addition we assume that $\dim(m^k/m^{k+1}) < \infty$ for all k.

Proposition 2.5. *([9]) With the assumption* $\dim \mathrm{H}^2(\mathcal{L};\mathcal{L}) < \infty$*, there exists a universal infinitesimal deformation* $\eta_{\mathcal{L}}$ *of the Lie algebra* $\mathcal{L}$ *with base* $B = \mathbb{K}\oplus H^2(\mathcal{L};\mathcal{L})'$*, where the second summand is the dual of* $\mathrm{H}^2(\mathcal{L};\mathcal{L})$ *equipped with zero multiplication, i.e.*

$$(\alpha_1, h_1)\cdot(\alpha_2, h_2) = (\alpha_1\alpha_2, \alpha_1 h_2 + \alpha_2 h_1).$$

This means that for any infinitesimal deformation λ of the Lie algebra $\mathcal{L}$ with finite dimensional (local) algebra base A there exists a unique homomorphism $\phi : \mathbb{K}\oplus H^2(\mathcal{L};\mathcal{L})' \to A$ such that λ is equivalent to the push-out $\phi_*\eta_{\mathcal{L}}$.

Although in general it is impossible to construct a universal formal deformation, there is a so-called versal element.

Definition 3. A formal deformation η of $\mathcal{L}$ parameterized by a complete local algebra B is called *versal* if for any deformation λ, parameterized by a complete local algebra (A, m_A), there is a morphism $f : B \to A$ such that

1) The push-out $f_*\eta$ is equivalent to λ.
2) If A satisfies ${m_A}^2 = 0$, then f is unique (see [5, 9]).

Remark 2.6. A versal formal deformation is sometimes called miniversal.

Theorem 2.7. *([6],[9, Thm. 4.6]) Let the space* $\mathrm{H}^2(\mathcal{L};\mathcal{L})$ *be finite dimensional.*
(a) There exists a versal formal deformation of $\mathcal{L}$*.*
(b) The base of the versal formal deformation is formally embedded into $\mathrm{H}^2(\mathcal{L};\mathcal{L})$*, i.e. it can be described in* $\mathrm{H}^2(\mathcal{L};\mathcal{L})$ *by a finite system of formal equations.*

A Lie algebra $\mathcal{L}$ is called (infinitesimally, formally, or globally) rigid if every (infinitesimal, formal, global) family is equivalent to a trivial one. Assume $\mathrm{H}^2(\mathcal{L};\mathcal{L}) < \infty$ in the following. By Proposition 2.5 the elements of $H^2(\mathcal{L};\mathcal{L})$ correspond bijectively to the equivalence classes of infinitesimal deformations, as equivalent deformations up to order 1 differ from each other only in a coboundary. Together with Theorem 2.7, Part (b), follows

Corollary 2.8.
(a) $\mathcal{L}$ *is infinitesimally rigid if and only if* $\mathrm{H}^2(\mathcal{L};\mathcal{L}) = \{0\}$*.*
(b) $H^2(\mathcal{L};\mathcal{L}) = \{0\}$ *implies that* $\mathcal{L}$ *is formally rigid.*

Let us stress the fact, that $\mathrm{H}^2(\mathcal{L};\mathcal{L}) = \{0\}$ does not imply that every global deformation will be equivalent to a trivial one. Hence, $\mathcal{L}$ is in this case not necessarily globally rigid. In Section 5 we will see plenty of nontrivial global deformations of the Witt algebra $\mathcal{W}$. Hence, the Witt algebra is not globally rigid. In the next section we will present the proof of $\mathrm{H}^2(\mathcal{W};\mathcal{W}) = \{0\}$, which implies infinitesimal and formal rigidity of $\mathcal{W}$.

3. Formal rigidity of the Witt and Virasoro algebras

As we pointed out in the Introduction, in formal deformation theory cohomology is an important tool.

The Lie algebras considered in this paper are infinite dimensional. Such Lie algebras possess a topology with respect to which all algebraic operations are continuous. In this situation, in a cochain complex it is natural to distinguish the sub-complex formed by the continuous cohomology of the Lie algebra (see [3]).

It is known (see [7]) that the Witt and the Virasoro algebra are formally rigid in the sense introduced in Section 2. The statement follows from a general result of Tsujishita [30], combined with results of Goncharova [14]. The goal of this section is to explain the relation in more detail.

First we recall the result of Tsujishita. Recall that in this article the Witt algebra $\mathcal{W}$ is defined to be the complexification of the Lie algebra of polynomial vector fields on S^1. They constitute a dense subalgebra of the algebra $\operatorname{Vect} S^1$ of all smooth vector fields $\operatorname{Vect} S^1$. The results of Tsujishita concerns the continuous cohomology of $\operatorname{Vect} S^1$ with values in formal tensor fields.

In fact he deals with the cohomology of the algebra of vector fields on a general smooth compact manifold, but we only need his result in case of the unit circle S^1. To formulate his results we have to introduce the space $Y(S^1)$ which is defined as follows. Let us consider the trivial principal $U(1)$-bundle $u(S^1)$, associated with the complexification of the real tangent bundle of S^1, and let $U(S^1)$ be its total space. Denote by $x(S^1)$ the trivial principal bundle $S^1 \times S^3 \to S^1$ with structural group $SU(2)$ and base S^1 and let ΩS^3 be the loop space of S^3. The space of sections $\operatorname{Sec} x(S^1)$ of the bundle $x(S^1)$ is the space $\operatorname{Map}(S^1, S^3) = S^3 \times \Omega S^3$. Consider $u(S^1)$ as a subbundle of $x(S^1)$. Now $Y(S^1)$ is the space

$$Y(S^1) := \{(y, s) \in S^1 \times \operatorname{Sec} x(S^1) \mid s(y) \in U(S^1)\}. \tag{3.18}$$

The space $Y(S^1)$ is homeomorphic to $S^1 \times S^1 \times \Omega S^3$, as can be seen as follows. We note that $s(y) \in y \times U(S^1)$, so we can write the section in the form $s(u) = (u, f(u))$, $u \in S^1$, where $f : S^1 \to S^3$, $f(u) \in U(1) \subset S^3$. Now let h be f right translated by $f(1)^{-1}$, i.e. $h(u) = f(u)f(1)^{-1}$. Then h takes 1 to 1 in S^3 and we get the required mapping from $Y(S^1)$ to $S^1 \times S^1 \times \Omega S^3$. On the other hand, take $y \in S^1$, $z \in S^1 = U(1)$ and a loop $h \in \Omega S^3$ such that $h : S^1 \to S^3$, $h(1) = 1$. Then the section $s(y) = (y, h(y)[h(y)^{-1} z])$ defines an element of $Y(S^1)$.

Theorem 3.1. *(Tsujishita* [30]*, Reshetnikov* [19]*)*
The cohomology ring $\mathrm{H}^*(\operatorname{Vect} S^1; C^\infty(S^1))$ *is isomorphic to* $\mathrm{H}^*(Y(S^1), \mathbb{R})$.

The real (topological) cohomology ring $\mathrm{H}^*(Y(S^1); \mathbb{R})$ of the space $Y(S^1)$ is known to be the free skew-symmetric $\mathbb{R}$-algebra $S(t, \theta, \xi)$, where $\deg t = \deg \theta = 1, \deg \xi = 2$. Hence $\mathrm{H}^*(Y(S^1); \mathbb{R}) \cong S(t, \theta, \xi)$ as graded algebra.

Theorem 3.2. *(Tsujishita* [30]*) For an arbitrary tensor* $\mathfrak{gl}(n, \mathbb{R})$*-module* A *and the space* $\mathcal{A}$ *of the corresponding formal tensor fields,* $\mathrm{H}^*(\operatorname{Vect} S^1; \mathcal{A})$ *is isomorphic to the tensor product of the ring* $\mathrm{H}^*(Y(S^1); \mathbb{R})$ *and* $\operatorname{Inv}_{\mathfrak{gl}(n,\mathbb{R})}(\mathrm{H}^*(L_1) \bigotimes A)$*, where* L_1 *denotes the subalgebra of* $\mathcal{W}$ *with basis* $(l_1, l_2, ...)$.

See the book of Fuchs [10] concerning this form of the theorem and for related results.

Hence, for computing the cohomology ring, we need to know the cohomology (with trivial coefficients) of the Lie algebra L_1. This is computed by Goncharova [14]. She computed the cohomology spaces for all Lie algebras L_k with basis $(l_k, l_{k+1}, ...)$, but we will only state the result we need now. We point out that her computation is carried-out for graded cohomology.

Let $\mathrm{H}^q_{(s)}$ be the s-homogeneous part of the cohomology space H^q where the grading is induced by the grading of $\mathcal{W}$, i.e. by $\deg l_n = n$.

Theorem 3.3. *(Goncharova [14]) For $q \geq 0$, the dimension of the graded cohomology spaces is:*

$$\dim \mathrm{H}^q_{(s)}(L_1) = \begin{cases} 1, & s = \frac{3q^2 \pm q}{2}, \\ 0, & s \neq \frac{3q^2 \pm q}{2}. \end{cases} \tag{3.19}$$

For the manifold S^1, all $\mathfrak{gl}(1,\mathbb{R})$-modules of formal tensor fields are of the form $C^\infty(S^1)d\varphi^s$ for some integer s, where φ is the angular coordinate on the circle. Using Goncharova's and Tsujishita's result we obtain

Theorem 3.4. *For $q \geq 0$*

$$\mathrm{H}^q(\mathrm{Vect}\, S^1; C^\infty(S^1)d\varphi^s) = \begin{cases} \mathrm{H}^{q-r}(Y(S^1);\mathbb{R}), & s = \frac{3r^2 \pm r}{2}, \\ \{0\}, & s \neq \frac{3r^2 \pm r}{2}. \end{cases} \tag{3.20}$$

In particular,

Corollary 3.5. *In case $s = -1$, we have*

$$\mathrm{H}^*(\mathrm{Vect}\, S^1; \mathrm{Vect}\, S^1) = 0.$$

Especially, $\mathrm{H}^2(\mathrm{Vect}\, S^1; \mathrm{Vect}\, S^1) = 0$*, so the algebra* $\mathrm{Vect}\, S^1$ *is formally rigid. Consequently, for the algebra of complexified vector fields* $\mathrm{Vect}\, S^1 \bigotimes \mathbb{C}$ *we have* $\mathrm{H}^2(\mathrm{Vect}\, S^1 \bigotimes \mathbb{C}; \mathrm{Vect}\, S^1 \bigotimes \mathbb{C}) = 0$*, and hence* $\mathrm{Vect}\, S^1 \bigotimes \mathbb{C}$ *is formally rigid as well.*

Corollary 3.6.
(a) For the Witt algebra $\mathcal{W}$ we have $\mathrm{H}^2(\mathcal{W};\mathcal{W}) = 0$, hence the Witt algebra is formally rigid.
(b) For the Virasoro algebra $\mathcal{V}$ we have $\mathrm{H}^2(\mathcal{V};\mathcal{V}) = 0$, hence the Virasoro algebra is formally rigid

Proof. The algebra $\mathcal{W}$ is the subalgebra of complexified polynomial vector fields of $\mathrm{Vect}\, S^1 \bigotimes \mathbb{C}$. By density arguments $\mathrm{H}^2(\mathcal{W};\mathcal{W}) = 0$ in the graded sense and the formal rigidity follows from Corollary 2.8. The algebra $\mathcal{V}$ is a one-dimensional central extension of $\mathcal{W}$. Using the Serre-Hochschild spectral sequence we obtain that $\mathcal{V}$ as a $\mathcal{V}$-module is an extension of $\mathcal{W}$ as a $\mathcal{W}$-module. Statement (b) then follows from the long exact cohomology sequence. □

4. Krichever-Novikov algebras

a). **The algebras with their almost-grading.** Algebras of Krichever-Novikov types are generalizations of the Virasoro algebra and all its related algebras. In this section we only recall the definitions and facts needed here. Let M be a compact Riemann surface of genus g, or in terms of algebraic geometry, a smooth projective curve over $\mathbb{C}$. Let $N, K \in \mathbb{N}$ with $N \geq 2$ and $1 \leq K < N$ be numbers. Fix

$$I = (P_1, \ldots, P_K), \quad \text{and} \quad O = (Q_1, \ldots, Q_{N-K})$$

disjoint ordered tuples of distinct points ("marked points", "punctures") on the curve. In particular, we assume $P_i \neq Q_j$ for every pair (i,j). The points in I are called the *in-points*, the points in O the *out-points*. Sometimes we consider I and O simply as sets and $A = I \cup O$ as a set.
Denote by $\mathcal{L}$ the Lie algebra consisting of those meromorphic sections of the holomorphic tangent line bundle which are holomorphic outside of A, equipped with the Lie bracket $[.,.]$ of vector fields. Its local form is

$$[e,f]_| = [e(z)\frac{d}{dz}, f(z)\frac{d}{dz}] := \left(e(z)\frac{df}{dz}(z) - f(z)\frac{de}{dz}(z)\right)\frac{d}{dz}\ . \tag{4.21}$$

To avoid cumbersome notation we will use the same symbol for the section and its representing function.
For the Riemann sphere ($g = 0$) with quasi-global coordinate z, $I = \{0\}$ and $O = \{\infty\}$, the introduced vector field algebra is the Witt algebra. We denote for short this situation as the *classical situation*.

For infinite dimensional algebras and modules and their representation theory a graded structure is usually of importance to obtain structure results. The Witt algebra is a graded Lie algebra. In our more general context the algebras will almost never be graded. But it was observed by Krichever and Novikov in the two-point case that a weaker concept, an almost-graded structure, will be enough to develop an interesting theory of representations (Verma modules, etc.).

Definition 4. Let $\mathcal{A}$ be an (associative or Lie) algebra admitting a direct decomposition as vector space $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}_n$. The algebra $\mathcal{A}$ is called an *almost-graded* algebra if (1) $\dim \mathcal{A}_n < \infty$ and (2) there are constants R and S with

$$\mathcal{A}_n \cdot \mathcal{A}_m \quad \subseteq \bigoplus_{h=n+m+R}^{n+m+S} \mathcal{A}_h, \qquad \forall n,m \in \mathbb{Z} \; . \tag{4.22}$$

The elements of $\mathcal{A}_n$ are called *homogeneous elements of degree n.*

For the 2-point situation for M a higher genus Riemann surface and $I = \{P\}$, $O = \{Q\}$ with $P,Q \in M$, Krichever and Novikov [17] introduced an almost-graded structure of the vector field algebras $\mathcal{L}$ by exhibiting a special basis and defining their elements to be the homogeneous elements. In [21, 22, 23, 24] its multi-point generalization was given, again by exhibiting a special basis. Essentially, this is done by fixing their order at the points in I and O in a complementary way. For every $n \in \mathbb{Z}$, and $p = 1,\dots,K$ a certain element $e_{n,p} \in \mathcal{L}$ is exhibited. The $e_{n,p}$ for $p = 1,\dots,K$ are a basis of a subspace $\mathcal{L}_n$ and it is shown that $\mathcal{L} = \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_n$.

Proposition 4.1. [21, 24] *With respect to the above introduced grading the Lie algebras $\mathcal{L}$ are almost-graded. The almost-grading depends on the splitting $A = I \cup O$.*

In the following we will have an explicit description of the basis elements for certain genus zero and one situation. Hence, we will not recall their general definition.

b). **Central extensions.** To obtain the equivalent of the Virasoro algebra we have to consider central extensions of the algebras. Central extensions are given by elements of $\mathrm{H}^2(\mathcal{L};\mathbb{C})$. The usual definition of the Virasoro cocycle is not coordinate independent. We have to introduce a projective connection R.

Definition 5. Let $(U_\alpha, z_\alpha)_{\alpha\in J}$ be a covering of the Riemann surface by holomorphic coordinates, with transition functions $z_\beta = f_{\beta\alpha}(z_\alpha)$. A system of local (holomorphic, meromorphic) functions $R = (R_\alpha(z_\alpha))$ is called a (holomorphic, meromorphic) projective connection if it transforms as

$$R_\beta(z_\beta)\cdot(f'_{\beta,\alpha})^2 = R_\alpha(z_\alpha) + S(f_{\beta,\alpha}), \qquad \text{with} \quad S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2, \tag{4.23}$$

the Schwartzian derivative. Here $'$ denotes differentiation with respect to the coordinate z_α.

Every Riemann surface admits a holomorphic projective connection R [15]. From (4.23) it follows that the difference of two projective connections will be a quadratic differential. Hence, after fixing one projective connection all others are obtained by adding quadratic differentials.
For the vector field algebra $\mathcal{L}$ the 2-cocycle

$$\gamma_{S,R}(e,f) := \frac{1}{24\pi\mathrm{i}} \int_{C_S} \left(\frac{1}{2}(e'''f - ef''') - R\cdot(e'f - ef')\right) dz \tag{4.24}$$

defines a central extension. Here C_S is a cycle separating the in-points from the out-points. In particular, C_S can be taken to be $C_S = \sum_{i=1}^{K} C_i$ where the C_i are deformed circles around the points in I. Recall that we use the same letter for the vector field and its local representing function.

Theorem 4.2. [26, 21]
(a) The cocycle class of $\gamma_{S,R}$ does not depend on the chosen connection R.
(b) The cocycle $\gamma_{S,R}$ is cohomologically non-trivial.
(c) The cocycle $\gamma_{S,R}$ is local, i.e. there exists an $M \in \mathbb{Z}$ such that

$$\forall n, m: \quad \gamma(\mathcal{L}_n, \mathcal{L}_m) \neq 0 \implies M \leq n+m \leq 0.$$

(d) Every local cocycle for $\mathcal{L}$ is either a coboundary or a scalar multiple of (4.24) with R a meromorphic projective connection which is holomorphic outside A.

The central extension $\widehat{\mathcal{L}}$ can be given via $\widehat{\mathcal{L}} = \mathbb{C} \oplus \mathcal{L}$ with Lie structure (using the notation $\widehat{e} = (0, e)$, $c = (1, 0)$)

$$[\widehat{e}, \widehat{f}] = \widehat{[e, f]} + \gamma_{S,R}\, c, \quad [c, \mathcal{L}] = 0. \tag{4.25}$$

Using the locality, by defining $\deg c := 0$, the almost-grading can be extended to the central extension $\widehat{\mathcal{L}}$.
Note that Theorem 4.2 does not claim that there is only one non-trivial cocycle class (which in general is not true). It only says that there is, up to multiplication with a scalar, only one class such that it contains cocycles which are local with respect to the almost-grading. Recall that the almost-grading is given by the splitting of A into $I \cup O$.

5. The algebra for the elliptic curve case

a). **The family of elliptic curves.** We consider the genus one case, i.e. the case of one-dimensional complex tori or equivalently the elliptic curve case. We have degenerations in mind. Hence it is more convenient to use the purely algebraic picture. Recall that the elliptic curves can be given in the projective plane by

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3, \quad g_2, g_3 \in \mathbb{C}, \quad \text{with } \Delta := {g_2}^3 - 27{g_3}^2 \neq 0. \tag{5.26}$$

The condition $\Delta \neq 0$ assures that the curve will be nonsingular. Instead of (5.26) we can use the description

$$Y^2 Z = 4(X - e_1 Z)(X - e_2 Z)(X - e_3 Z) \tag{5.27}$$

with

$$e_1 + e_2 + e_3 = 0, \quad \text{and} \quad \Delta = 16(e_1 - e_2)^2 (e_1 - e_3)^2 (e_2 - e_3)^2 \neq 0. \tag{5.28}$$

These presentations are related via

$$g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3), \quad g_3 = 4(e_1 e_2 e_3). \tag{5.29}$$

The elliptic modular parameter classifying the elliptic curves up to isomorphism is given as

$$j = 1728\, \frac{g_2^3}{\Delta}. \tag{5.30}$$

We set

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0, \quad e_i \neq e_j \text{ for } i \neq j\}. \tag{5.31}$$

In the product $B \times \mathbb{P}^2$ we consider the family of elliptic curves $\mathcal{E}$ over B defined via (5.27). The family can be extended to

$$\widehat{B} := \{e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0\}. \tag{5.32}$$

The fibers above $\widehat{B} \setminus B$ are singular cubic curves. Resolving the one linear relation in $\widehat{B}$ via $e_3 = -(e_1 + e_2)$ we obtain a family over $\mathbb{C}^2$.
Consider the complex lines in $\mathbb{C}^2$

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\}, \quad s \in \mathbb{C}, \qquad D_\infty := \{(0, e_2) \in \mathbb{C}^2\}. \tag{5.33}$$

Set also

$$D_s^* = D_s \setminus \{(0,0)\} \tag{5.34}$$

for the punctured line. Now

$$B \cong \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2}). \tag{5.35}$$

Note that above D^*_1 we have $e_1 = e_2 \neq e_3$, above $D^*_{-1/2}$ we have $e_2 = e_3 \neq e_1$, and above D^*_{-2} we have $e_1 = e_3 \neq e_2$. In all these cases we obtain the nodal cubic. The nodal cubic E_N can be given as

$$Y^2Z = 4(X - eZ)^2(X + 2eZ) \tag{5.36}$$

where e denotes the value of the coinciding $e_i = e_j$ ($-2e$ is then necessarily the remaining one). The singular point is the point $(e : 0 : 1)$. It is a node. It is up to isomorphism the only singular cubic which is stable in the sense of Mumford-Deligne. Above the unique common intersection point $(0, 0)$ of all D_s there is the cuspidal cubic E_C

$$Y^2Z = 4X^3. \tag{5.37}$$

The singular point is $(0 : 0 : 1)$. The curve is not stable in the sense of Mumford-Deligne. In both cases the complex projective line is the desingularisation.
In all cases (non-singular or singular) the point $\infty = (0 : 1 : 0)$ lies on the curves. It is the only intersection with the line at infinity, and is a non-singular point. In passing to an affine chart in the following we will loose nothing.
For the curves above the points in D^*_s we calculate $e_2 = se_1$ and $e_3 = -(1 + s)e_1$ (resp. $e_3 = -e_2$ if $s = \infty$). Due to the homogeneity, the modular parameter j for the curves above D^*_s will be constant along the line. In particular, the curves in the family lying above D^*_s will be isomorphic. For completeness let us write down

$$j(s) = 1728\,\frac{4(1 + s + s^2)^3}{(1 - s)^2(2 + s)^2(1 + 2s)^2}, \quad j(\infty) = 1728. \tag{5.38}$$

b). **The family of vector field algebras.** We have to introduce the points where poles are allowed. For our purpose it is enough to consider two marked points. More marked points are considered in [25, 20]. We will always put one marking to $\infty = (0 : 1 : 0)$ and the other one to the point with the affine coordinate $(e_1, 0)$. These markings define two sections of the family $\mathcal{E}$ over $\widehat{B} \cong \mathbb{C}^2$. With respect to the group structure on the elliptic curve given by ∞ as the neutral element (the first marking) the second marking chooses a two-torsion point. All other choices of two-torsion points will yield isomorphic situations.
In [25] for this situation (and for a three-point situation) a basis of the Krichever-Novikov type vector field algebras were given.

Theorem 5.1. *For any elliptic curve $E_{(e_1,e_2)}$ over $(e_1, e_2) \in \mathbb{C}^2 \setminus (D^*_1 \cup D^*_{-1/2} \cup D^*_{-2})$ the Lie algebra $\mathcal{L}^{(e_1,e_2)}$ of vector fields on $E_{(e_1,e_2)}$ has a basis $\{V_n,\ n \in \mathbb{Z}\}$ such that the Lie algebra structure is given as*

$$[V_n, V_m] = \begin{cases} (m - n)V_{n+m}, & n, m \text{ odd}, \\ (m - n)\big(V_{n+m} + 3e_1V_{n+m-2} \\ \qquad +(e_1 - e_2)(e_1 - e_3)V_{n+m-4}\big), & n, m \text{ even}, \\ (m - n)V_{n+m} + (m - n - 1)3e_1V_{n+m-2} \\ \qquad +(m - n - 2)(e_1 - e_2)(e_1 - e_3)V_{n+m-4}, & n \text{ odd},\ m \text{ even}. \end{cases} \tag{5.39}$$

By defining $\deg(V_n) := n$, *we obtain an almost-grading.*

Proof. This is proved in [25, Prop.3,Prop.4]. Our generators are

$$V_{2k+1} := (X - e_1)^kY\frac{d}{dX}, \quad V_{2k} := 1/2f(X)(X - e_1)^{k-2}\frac{d}{dX}, \tag{5.40}$$

with $f(X) = 4(X - e_1)(X - e_2)(X - e_3)$. Note that here V_n is the V_{n-1} given in [25]. □

The algebras of Theorem 5.1 defined with the structure (5.39) make sense also for the points $(e_1, e_2) \in D_1 \cup D_{-1/2} \cup D_{-2}$. Altogether this defines a two-dimensional family of Lie algebras parameterized over $\mathbb{C}^2$. In particular, note that we obtain for $(e_1, e_2) = 0$ the Witt algebra.

Let us remark that this two-dimensional family of geometric origin can also be written just with the symbols p and q instead of $3e_1$ and $(e_1 - e_2)(e_1 - e_3)$. In this form it was algebraically found by Deck [2], (see also Ruffing, Deck and Schlichenmaier [20]) as a two-dimensional family of Lie algebra. Guerrini [12, 13] related it later (again in a purely algebraic manner) to deformations of the Witt algebra over certain spaces of polynomials. Due to its geometric interpretation we prefer to use the parameterization (5.39). Further higher-dimensional families of geometric origin can be obtained if we consider the multi-point situation for the elliptic curve and degenerate the curve to the cuspidal cubic and let the marked points (beside the point at ∞) move to the singularity. But no new effects will appear.

We consider now the family of algebras obtained by taking as base variety the line D_s (for any s). First consider $s \neq \infty$. We calculate $(e_1 - e_2)(e_1 - e_3) = e_1^2(1-s)(2+s)$ and can rewrite for these curves (5.39) as

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + 3e_1 V_{n+m-2} \\ \qquad + e_1^2(1-s)(2+s)V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1 V_{n+m-2} \\ \qquad + (m-n-2)e_1^2(1-s)(2+s)V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.41}$$

For D_∞ we have $e_3 = -e_2$ and $e_1 = 0$ and obtain

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} - e_2^2 V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} - (m-n-2)e_2^2 V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.42}$$

If we take $V_n^* = (\sqrt{e_1})^{-n} V_n$ (for $s \neq \infty$) as generators, we obtain for $e_1 \neq 0$ always the algebra with $e_1 = 1$ in our structure equations. For $s = \infty$ a rescaling with $(\sqrt{e_2})^{-n} V_n$ will do the same (for $e_2 \neq 0$).

Hence we see that for fixed s in all cases the algebras will be isomorphic above every point in D_s as long as we are not above $(0, 0)$.

Proposition 5.2. *For $(e_1, e_2) \neq (0, 0)$ the algebras $\mathcal{L}^{(e_1, e_2)}$ are not isomorphic to the Witt algebra.*

Proof. Assume that we have a Lie isomorphism $\Phi : \mathcal{W} = \mathcal{L}^{(0,0)} \to \mathcal{L}^{(e_1, e_2)}$. Denote the generators of the Witt algebra by $\{l_n, \, n \in \mathbb{Z}\}$. In particular, we have $[l_0, l_n] = n l_n$ for every n. We assign to every l_n, numbers $m(n) \leq M(n)$ such that $\Phi(l_n) = \sum_{k=m(n)}^{M(n)} \alpha_k(n) V_k$ with $\alpha_{m(n)}(n), \alpha_{M(n)}(n) \neq 0$. From the relation in the Witt algebra we obtain

$$[\Phi(l_0), \Phi(l_n)] = \sum_{k=m(0)}^{M(0)} \sum_{l=m(n)}^{M(n)} \alpha_k(0)\alpha_l(n)[V_k, V_l] = n \cdot \sum_{l=m(n)}^{M(n)} \alpha_l(n) V_l.$$

We can choose n in such a way that the structure constants in the expression of $[V_k, V_l]$ at the boundary terms will not vanish. Using the almost-graded structure we obtain $M(0) + M(n) = M(n)$ which implies $M(0) = 0$, and $m(0) + m(n) - 4 = m(n)$ or $m(0) + m(n) - 2 = m(n)$ (for $s = 1$ or $s = -2$) which implies $2 \leq m(0) \leq M(0) = 0$ which is a contradiction. □

It is necessary to stress the fact, that in our approach the elements of the algebras are only finite linear combinations of the basis elements V_n.

In particular, we obtain a family of algebras over the base D_s, which is always the affine line. In this family the algebra over the point $(0, 0)$ is the Witt algebra and the isomorphism type above all other points will be the same but different from the special element, the Witt algebra. This is a phenomena also appearing in algebraic geometry. There it is related to non-stable singular curves (which is for genus one only the cuspidal cubic). Note that it is necessary to consider the two-dimensional family introduced above to "see the full behaviour" of the cuspidal cubic E_C.

Let us collect the facts:

Theorem 5.3. *For every $s \in \mathbb{C} \cup \{\infty\}$ the families of Lie algebras defined via the structure equations (5.41) for $s \neq \infty$ and (5.42) for $s = \infty$ define global deformations $\mathcal{W}_t^{(s)}$ of the Witt algebra $\mathcal{W}$ over the affine line $\mathbb{C}[t]$. Here t corresponds to the parameter e_1 and e_2 respectively. The Lie algebra above $t = 0$ corresponds always to the Witt algebra, the algebras above $t \neq 0$ belong (if s is fixed) to the same isomorphism type, but are not isomorphic to the Witt algebra.*

If we denote by $g(s) := (1-s)(2+s)$ the polynomial appearing in the structure equations (5.41), we see that the algebras over D_s will be isomorphic to the algebras over D_t if $g(s) = g(t)$. This is the case if and only if $t = -1-s$. Under this map the lines D_∞ and $D_{-1/2}$ remain fixed. Geometrically this corresponds to interchanging the role of e_2 and e_3.

c). **The degenerations and the three-point algebras for genus zero.** Next we want to identify the algebras corresponding to the singular cubic situation. We have three different possibilities:
(I) All three e_1, e_2 and e_3 come together. This implies necessarily that $e_1 = e_2 = e_3 = 0$. We obtain the cuspidal cubic. The pole at $(e_1, 0)$ moves to the singular point $(0,0)$. This appears if we approach in our two-dimensional family the point $(0,0)$.
(II) If 2 but not 3 of the e_i come together, we obtain the nodal cubic and we have to distinguish 2 subcases with respect to the marked point:
(IIa) $e_1 \neq e_2 = e_3$, then the point of a possible pole will remain non-singular. This appears if we approach a point of $D^*_{-1/2}$.
(IIb) Either $e_1 = e_2 \neq e_3$ or $e_1 = e_3 \neq e_2$, then the singular point (the node) will become a possible pole. This situation occurs if we approach points from $D^*_1 \cup D^*_{-2}$.
In the cases (IIa) and (IIb) we obtain the algebras by specializing the value of s in (5.41).
We want to identify these exceptional algebras above D_s for $s = 1, -1/2$ and -2.
First, clearly above $(0,0)$ there is always the Witt algebra corresponding to meromorphic vector fields on the complex line holomorphic outside $\{0\}$ and $\{\infty\}$. This corresponds to situation (I).
Next we consider the geometric situation $M = \mathbb{P}^1(\mathbb{C})$, $I = \{\alpha, -\alpha\}$ and $O = \{\infty\}$, $\alpha \neq 0$. As shown in [25], a basis of the corresponding Krichever-Novikov algebra is given by

$$V_{2k} := z(z-\alpha)^k(z+\alpha)^k \frac{d}{dz}, \qquad V_{2k+1} := (z-\alpha)^{k+1}(z+\alpha)^{k+1}\frac{d}{dz}, \qquad k \in \mathbb{Z}. \tag{5.43}$$

Here z is the quasi-global coordinate on $\mathbb{P}^1(\mathbb{C})$. The grading is given by $\deg(V_n) := n$. One calculates

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} + \alpha^2 V_{n+m-2}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)\alpha^2 V_{n+m-2}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.44}$$

If we set $\alpha = \sqrt{e_1}$ we get exactly the structure for the algebras obtained in the degeneration (IIb). Hence,

Proposition 5.4. *The algebras $\mathcal{L}^\lambda$ for $\lambda \in D^*_1 \cup D^*_{-2}$ are isomorphic to the algebra of meromorphic vector fields on $\mathbb{P}^1$ which are holomorphic outside $\{\infty, \alpha, -\alpha\}$.*

Finally, we consider the subalgebra of the Witt algebra defined by the basis elements

$$\begin{aligned} V_{2k} &= z^{2k-3}(z^2-\alpha^2)^2\frac{d}{dz} = l_{2k} - 2\alpha^2 l_{2k-2} + \alpha^4 l_{2k-4}, \\ V_{2k+1} &= z^{2k}(z^2-\alpha^2)\frac{d}{dz} = l_{2k+1} - \alpha^2 l_{2k-1}. \end{aligned} \tag{5.45}$$

One calculates

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)\big(V_{n+m} - 2\alpha V_{n+m-2} + \alpha^2 V_{n+m-4}\big), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)(-2\alpha)V_{n+m-2} & \\ \qquad +(m-n-2)\alpha^2 V_{n+m-4}, & n \text{ odd}, \ m \text{ even}. \end{cases} \tag{5.46}$$

This is the algebra obtained by the degeneration (IIa) if we set $\alpha = \mathrm{i}\sqrt{3e_1/2}$. Hence,

Proposition 5.5. *The algebras $\mathcal{L}^\lambda$ for $\lambda \in D^*_{-1/2}$ are isomorphic to the subalgebra of the Witt algebra generated by the above basis elements.*

This subalgebra can be described as the subalgebra of meromorphic vector fields vanishing at α and $-\alpha$, with possible poles at 0 and ∞ and such that in the representation of $V(z) = f(z)(z^2 - \alpha^2)\frac{d}{dz}$ the function f fulfills $f(\alpha) = f(-\alpha)$.
Clearly, as explained above, as long as $\alpha \neq 0$, by rescaling, the case $\alpha = 1$ can be obtained. Hence for $\alpha \neq 0$ the algebras are all isomorphic.
If we choose a line E in $\mathbb{C}^2$ not passing through the origin, by restricting our two-dimensional family to those algebras above E we obtain a family of algebras over an affine line. A generic line will meet D_1, D_{-2} and $D_{-1/2}$. In this way we obtain global deformations of these special algebras.

d). **Geometric interpretation of the deformation results.** The identification of the algebras obtained in the last subsection is not a pure coincidence. There is a geometric scheme behind, which was elaborated in [25]. To put the results in the right context we want to indicate the relation. In both cases of the singular cubic the desingularisation (which will be also the normalization) will be the projective line. The vector fields given in (5.40) make sense also for the degenerate cases. Vector fields on the singular cubic will correspond to vector fields on the normalization which have at the points lying above the singular points an additional zero.
In case of the cuspidal degeneration the possible pole moves to the singular point. Hence we will obtain the full Witt algebra. In the case of the nodal cubics we have to distinguish the two cases. If $(e_1, 0)$ will not be the singular point, one obtains the subalgebra of the Witt algebra consisting of vector fields which have a zero at α and $-\alpha$ (where α is the point lying above the singular point) and fulfill the additional condition on f (see above). If $(e_1, 0)$ becomes a singular point, a pole at $(e_1, 0)$ will produce poles at the two points α and $-\alpha$ lying above $(e_1, 0)$. Hence we end up with the 3-point algebra for genus zero.

e). **Cohomology classes of the deformations.** Let $\mathcal{W}_t$ be a one-parameter deformation of the Witt algebra $\mathcal{W}$ with Lie structure

$$[x, y]_t = [x, y] + t^k \omega_0(x, y) + t^{k+1}\omega_1(x, y) + \cdots , \tag{5.47}$$

such that $\omega = \omega_0$ is a non-vanishing bilinear form. The form ω will be a 2-cocycle in $C^2(\mathcal{W}, \mathcal{W})$. The element $[\omega] \in \mathrm{H}^2(\mathcal{W}, \mathcal{W})$ will be the cohomology class characterizing the infinitesimal family. Recall that a class ω is per definition a coboundary if

$$\omega(x, y) = (d_1\Phi)(x, y) := \Phi([x, y]) - [\Phi(x), y] - [x, \Phi(y)] \tag{5.48}$$

for a linear map $\Phi : \mathcal{W} \to \mathcal{W}$. For the global deformation families $\mathcal{W}_t^{(s)}$ appearing in Theorem 5.3 we obtain with respect to the parameterization by e_1 and e_2 respectively, as first nontrivial contribution the following two cocycles.

$$\omega(l_n,l_m)=\begin{cases}0, & n,m \text{ odd},\\ (m-n)3\,l_{n+m-2}, & n,m \text{ even}\\ (m-n-1)3\,l_{n+m-2}, & n \text{ odd },m \text{ even},\end{cases}$$
and (5.49)
$$\omega(l_n,l_m)=\begin{cases}0, & n,m \text{ odd},\\ -(m-n)l_{n+m-4}, & n,m \text{ even},\\ -(m-n-1)l_{n+m-4}, & n \text{ odd },m \text{ even}.\end{cases}$$

From Section 3 we know that the Witt algebra is infinitesimally and formally rigid. Hence, the cohomology classes $[\omega]$ in $\mathrm{H}^2(\mathcal{W},\mathcal{W})$ must vanish. For illustration we will verify this directly. By a suitable ansatz one easily finds that $\omega=d_1\Phi$ for

$$\Phi(l_n)=\begin{cases}-3\,l_{n-2}, & n \text{ even},\\ -3/2\,l_{n-2}, & n \text{ odd},\end{cases}\quad \text{resp.}\quad \Phi(l_n)=\begin{cases}l_{n-4}, & n \text{ even},\\ 1/2\,l_{n-4}, & n \text{ odd}.\end{cases} \tag{5.50}$$

From the formal rigidity of $\mathcal{W}$ we can conclude that the family $\mathcal{W}_t^s$ considered as a formal family over $\mathbb{C}[[t]]$ is equivalent to the trivial family. Hence on the formal level there is an isomorphism φ given by

$$\varphi_t(l_n)=V_n+\sum_{k=1}^{\infty}\alpha_k t^k V_{n-k}. \tag{5.51}$$

Here t is a formal variable. The formal sum (5.51) will not terminate, and even if we specialize t to a non-zero number, the element $\varphi_t(l_n)$ will not live in our Krichever-Novikov algebra.

f). **Families of the centrally extended algebras.** In all families considered above it is quite easy to incorporate a central term as defined via the local cocycle (4.24). In the genus one case with respect to the standard flat coordinates $(z-a)$ the projective connection $R\equiv 0$ will do. The difference of two projective connections will be a quadratic differential. Hence, we obtain that any local cocycle is either a coboundary or can be obtained as a scalar multiple of (4.24) with a suitable meromorphic quadratic differential R which has only poles at A. The integral (4.24) is written in the complex analytic picture. But the integration over a separating cycle can be given by integration over circles around the points where poles are allowed. Hence, it is given as sum of residues and the cocycle makes perfect sense in the purely algebraic picture. For the explicit calculation of the residue it is useful to use the fact that for tori $T=\mathbb{C}/\Lambda$ with lattice $\Lambda=\langle 1,\tau\rangle_{\mathbb{Z}}$, $\mathrm{Im}\,\tau>0$, the complex analytic picture is isomorphic to the algebraic elliptic curve picture via

$$\bar{z}=z \bmod \Lambda\to\begin{cases}(\wp(z):\wp'(z):1), & \bar{z}\neq 0,\\ (0:1:0)=\infty, & \bar{z}=0.\end{cases} \tag{5.52}$$

Here $\wp$ denotes the Weierstraß $\wp$-function. Recall that the points where poles are allowed in the algebraic picture are ∞ and $(e_1:0:1)$. They correspond to $\bar{0}$ and $\overline{\frac{1}{2}}$ respectively in the analytic picture. The point $\overline{\frac{1}{2}}$ is a 2-torsion point. Replacing $\frac{1}{2}$ by any one of the other 2-torsion points $\frac{\tau}{2}$ and $\frac{\tau+1}{2}$ respectively, yields isomorphic structures.

Let $\mathcal{W}_t^{(s)}$ be any of the above considered families parameterized over $D_s^*\cong\mathbb{C}\setminus\{0\}$ with parameter t such that $t=0$ corresponds to $(0,0)$. Then $\widehat{\mathcal{W}_t^{(s)}}=\mathbb{C}\oplus\mathcal{W}_t^{(s)}$ with $\widehat{x}=(0,x)$, $c=(1,0)$ with

$$[\widehat{x},\widehat{y}]=\widehat{[x,y]}+\widetilde{c}(t)\gamma_{S,R_t}(x,y)\cdot c,\qquad [\widehat{x},c]=0, \tag{5.53}$$

for $\widetilde{c}:\mathbb{C}\to\mathbb{C}$ a non-vanishing algebraic function, and R_t a family of quadratic differentials varying algebraically with respect to t, will define a family of centrally extended algebras.

As shown above, the non-extended algebras for fixed s are mutually isomorphic. Recall from Section b) that choosing a different R_t gives only a different cocycle in the same cohomology class and that $\widetilde{c}(t)$ as long as $\widetilde{c}(t) \neq 0$ is just a rescaling of the central element. Hence, we obtain that also the centrally extended algebras are mutually isomorphic over D_s^*.
The cocycle (4.24) expressed as residue and calculated at the point ∞ makes perfect sense also for $t = 0$. For $t = 0$ it will yield the Virasoro cocycle. In this way we obtain a nontrivial deformation family for the Virasoro algebra. Clearly, to obtain examples we might directly take $c \equiv 1$ and $R_t \equiv 0$.

Remark 5.6. A typical appearing in 2-dimensional conformal field theory of centrally extended vector field algebras is via the Sugawara representation, i.e. by the modes of the energy-momentum tensor in representations of affine algebras (gauge algebras) or of algebras of $b - c$ systems. The classical constructions extend also to the higher genus multi-point situation, [27, 21], i.e. if the 2-dimensional conformal field theory is considered for higher genus Riemann surfaces. If one studies families of such systems varying with the moduli parameters, corresponding to deformation of the complex structure and moving the "insertions points", one obtains in a natural way families of centrally extended algebras. In these cases $\widetilde{c}(t)$ and R_t might vary. In [2] and [20] for $b - c$-systems explicit formulas for the central term are given.

g). **Deformations of the Lie algebra L_1.** Let L_1 be the subalgebra of the Witt algebra consisting of those vector fields which vanish of order ≥ 2 at 0, i.e. $L_1 = \langle l_n \mid n \geq 1 \rangle$. It was shown by Fialowski in [4] that this algebra is not formally rigid, and that there are three independent formal one-parameter deformations. They correspond to pairwise non-equivalent deformations. Indeed any formal one-parameter deformation of L_1 can be reduced by a formal parameter change to one of these deformations (see also [8]):

$$\begin{aligned} &[l_n, l_m]_t^{(1)} := (m-n)(l_{n+m} + t l_{n+m-1}); \\ &[l_n, l_m]_t^{(2)} := \begin{cases} (m-n)l_{n+m}, & n \neq 1, m \neq 1 \\ (m-1)l_{m+1} + tml_m, & n = 1, m \neq 1; \end{cases} \\ &[l_n, l_m]_t^{(3)} := \begin{cases} (m-n)l_{n+m}, & n \neq 2, m \neq 2 \\ (m-2)l_{m+2} + tml_m, & n = 2, m \neq 2. \end{cases} \end{aligned} \tag{5.54}$$

The cocycles representing the infinitesimal deformations are given by

$$\begin{aligned} &\beta^{(1)}(l_n, l_m) := (m-n)l_{n+m-1}; \\ &\beta^{(2)}(l_n, l_m) := \begin{cases} ml_m, & n = 1, m \neq 1 \\ 0, & n \neq 1, m \neq 1; \end{cases} \\ &\beta^{(3)}(l_n, l_m) := \begin{cases} ml_m, & n = 2, m \neq 2 \\ 0, & n \neq 2, m \neq 2. \end{cases} \end{aligned} \tag{5.55}$$

It is shown in the above cited article that the cohomology classes $[\beta^{(1)}] = [\beta^{(2)}] = 0$ and $[\beta^{(3)}] \neq 0$. To avoid misinterpretations let us point out that these infinitesimal classes are not invariant under formal equivalence of formal deformations. Take for the first two families the Lie algebra 1-cocycles $\gamma^{(i)}$ with $\beta^{(i)} = d_1\gamma^{(i)}$ $(i = 1, 2)$. In [6] it is shown that by the formal isomorphisms $\phi_t^{(i)}(x) = x + t\gamma^{(i)}(x)$ each of these two families is equivalent to a corresponding formal family for which the infinitesimal class is a non-vanishing scalar multiple of $[\beta^{(3)}]$.
In our geometric situation we consider the algebra $W_{1,\alpha^2} := \langle V_n \mid n \geq 1 \rangle$ with the structure equations (5.44). If we vary α we obtain a family $\mathcal{W}_{1,\alpha^2}$. These algebras correspond to the algebra of vector fields on $\mathbb{P}^1(\mathbb{C})$ which might have a pole at the point ∞ and zeros of order at least 1 at the points α and $-\alpha$. By Proposition 5.4 we know that as long as $\alpha \neq 0$ they are isomorphic to the corresponding subalgebra of

the vector field algebra $\mathcal{L}^\lambda$ for $\lambda \in D_1^* \cup D_{-2}^*$. As long as $\alpha \neq 0$ we can rescale and obtain that all these subalgebras belong to the same isomorphism type.

Proposition 5.7. *The algebras $\mathcal{W}_{1,\alpha^2}$ for $\alpha \neq 0$ are not isomorphic to the algebra L_1.*

Proof. The proof again uses the almost-graded (respectively, graded) structure as in the proof of Proposition 5.2. By the absence of l_0 some additional steps are needed. We will only sketch them. Assume that there is an isomorphism $\phi : L_1 \to \mathcal{W}_{1,\alpha^2}$. From the structure (5.44) we conclude that $M(n) = nM(1)$ (notation as in the above-mentioned proof) with $M(1) \in \mathbb{N}$. If we assume $M(1) > 1$, the basis element V_1 will not be in $\phi(L_1)$. So ϕ cannot be an isomorphism. Hence, $M(1) = 1$. Now $\phi(l_1) = \alpha_1 V_1$, $\phi(l_2) = \alpha_2 V_2 + \alpha_{2,1} V_1$ and in further consequence from $[l_1, l_2] = 2l_3$ and $[l_1, l_3] = 3l_4$ it follows that $\phi(l_3) = \alpha_3 V_3$ and $\phi(l_4) = \alpha_4 V_4$. The relations $[l_1, l_4] = 3l_5$ and $[l_3, l_2] = -l_5$ in L_1 lead under ϕ to two relations in $\mathcal{W}_{1,\alpha^2}$ which are in contradiction. Hence there is no such ϕ.
Note that an alternative way to see the statement is to use Proposition 5.9 further down. □

In this way we obtain a non-trivial global deformation family $\mathcal{W}_{1,t}$ for the algebra L_1. For its 2-cocycle we calculate

$$\omega(l_n, l_m) = \begin{cases} 0, & n, m \text{ odd}, \\ (m-n) l_{n+m-2}, & n, m \text{ even}, \\ (m-n-1) l_{n+m-2}, & n \text{ odd},\ m \text{ even}. \end{cases} \tag{5.56}$$

Again with a suitable ansatz we find with

$$\Phi(l_m) := \begin{cases} -\frac{m+8}{6}\, l_{m-2}, & m \text{ even},\ m \geq 4, \\ -\frac{m+5}{6}\, l_{m-2}, & m \text{ odd},\ m \geq 3, \\ 0, & m = 1, m = 2. \end{cases} \tag{5.57}$$

that $\omega - d_1\Phi = \frac{1}{3}\beta^{(3)}$.
From the structure equations (5.44) we can immediately verify

Lemma 5.8. *For $\alpha \neq 0$ the commutator ideal calculates to*

$$[W_{1,\alpha^2}, W_{1,\alpha^2}] = \langle V_n \mid n \geq 3 \rangle, \tag{5.58}$$

and we have

$$\dim W_{1,\alpha^2} / [W_{1,\alpha^2}, W_{1,\alpha^2}] = 2. \tag{5.59}$$

Proposition 5.9. *The family $\mathcal{W}_{1,\alpha^2}$ is formally equivalent to the first family $[.,.]_t^1$ in (5.54).*

Proof. Clearly $\mathcal{W}_{1,\alpha^2}$ defines a formal family. From the above calculation we obtain $[\omega] = 1/3[\beta^{(3)}]$. But $[\beta^{(3)}] \neq 0$, hence considered as formal family it is also non-trivial. By the results about the versal family of L_1 it has to be equivalent to one of the three families of (5.54). Only the first family $L_1^{(1)}$ has $\dim L_1^{(1)}/[L_1^{(1)}, L_1^{(1)}] = 2$, for the other two this dimension equals one. By Lemma 5.8 we obtain the claim. □

References

[1] Belavin, A.A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. B **241**, 333-380 (1084)

[2] Deck, Th.: Deformations from Virasoro to Krichever-Novikov algebras. Phys. Lett. B **251**, 535–540 (1990)

[3] Feigin, B. L., Fuchs, D. B.: Cohomologies of Lie Groups and Lie Algebras. In: Onishchik, A.L., and Vinberg, E.B. (eds) *Lie groups and Lie algebras II.* Encyclopaedia of Math. Sciences, Vol. 21, pp. 125–215, Springer, New-York, Berlin, Heidelberg, Tokyo, 2000

[4] Fialowski, A.: Deformations of the Lie algebra of vector fields on the line. Uspkhi Mat. Nauk, **38** 201-202 (1983); English translation: Russian Math. Surveys, **38**, No. 1, 185–186 (1983)

[5] Fialowski, A.: Deformations of Lie algebras. Math. USSR Sbornik **55**, 467-472 (1986)

[6] Fialowski, A.: An example of formal deformations of Lie algebras. In: *Proceedings of NATO Conference on Deformation Theory of Algebras and Applications, Il Ciocco, Italy, 1986,* pp. 375–401, Kluwer, Dordrecht, 1988

[7] Fialowski, A.: Deformations of some infinite-dimensional Lie algebras. J. Math. Phys. **31**, 1340-1343 (1990)

[8] Fialowski, A., and Fuchs, D.: Singular deformations of Lie algebras. Example: Deformations of the Lie algebra L_1. In: *Topics in singularity theory, V.I. Arnold's 60th anniversary collection*, Transl. Ser. 2, Am. Math. Soc. 180(34), pp.77-92, 1997

[9] Fialowski, A., and Fuchs, D.: Construction of miniversal deformations of Lie algebras. J. Funct. Anal. **161**, 76-110 (1999)

[10] Fuchs, D.: *Cohomology of infinite-dimensional Lie algebras*, Consultants Bureau, N.Y., London, 1986.

[11] Gerstenhaber, M.: On the deformation of rings and algebras I,II,III Ann. Math. **79**, 59-10 (1964), **84**, 1-19 (1966), **88**, 1-34 (1968)

[12] Guerrini, L.: Formal and analytic deformations of the Witt algebra. Lett. Math. Phys. **46**, 121-129 (1998)

[13] Guerrini, L.: Formal and analytic rigidity of the Witt algebra. Rev. Math. Phys. **11**, 303-320 (1999)

[14] Goncharova, I.V.: Cohomology of Lie algebras of formal vector fields on the line. Funct. Anal. Appl. **7**, No.2, 6-14 (1973)

[15] Gunning, R.C.: *Lectures on Riemann surfaces*, Princeton Mathematical Notes. Princeton. N. J., Princeton University Press, 1966

[16] Kodaira, K.: *Complex manifolds and deformation of complex structures*, Springer, New-York, Berlin, Heidelberg, Tokyo, 1986

[17] Krichever I.M., and Novikov S.P.: Algebras of Virasoro type, Riemann surfaces and structures of the theory of solitons, Funktional Anal. i. Prilozhen. **21**, 46–63 (1987); Virasoro type algebras, Riemann surfaces and strings in Minkowski space. Funktional Anal. i. Prilozhen. **21**, 47–61 (1987); Algebras of Virasoro type, energy-momentum tensors and decompositions of operators on Riemann surfaces, Funktional Anal. i. Prilozhen. **23**, 46–63 (1989)

[18] Palamodov, V.P.: Deformations of complex structures. In: Gindikin, S.G., Khenkin, G.M. (eds) *Several complex variables IV.* Encyclopaedia of Math. Sciences, Vol. 10, pp. 105-194, Springer, New-York, Berlin, Heidelberg, Tokyo, 1990

[19] Reshetnikov, V.N.: On the cohomology of the Lie algebra of vector fields on a circle. Usp. Mat. Nauk **26**, 231-232 (1971)

[20] Ruffing, A., Deck, Th., and Schlichenmaier, M.: String branchings on complex tori and algebraic representations of generalized Krichever-Novikov algebras. Lett. Math. Phys. **26**), 23-32 (1992)

[21] Schlichenmaier, M.: *Verallgemeinerte Krichever - Novikov Algebren und deren Darstellungen*, University of Mannheim, June 1990.

[22] Schlichenmaier, M.: Krichever-Novikov algebras for more than two points. Lett. Math. Phys. **19**, 151-165 (1990)

[23] Schlichenmaier, M.: Krichever-Novikov algebras for more than two points: explicit generators. Lett. Math. Phys. **19**, 327-336 (1990)

[24] Schlichenmaier, M.: Central extensions and semi-infinite wedge representations of Krichever-Novikov algebras for more than two points, Lett. Math. Phys. **20**, 33-46 (1990)

[25] Schlichenmaier, M.: *Degenerations of generalized Krichever-Novikov algebras on tori*, Jour. Math. Phys. **34**, 3809-3824 (1993)

[26] Schlichenmaier, M.: Local cocycles and central extensions for multi-point algebras of Krichever-Novikov type. math/0112116, Jour. Reine u. Angewandte Math., **559** (2003), 53–94.

[27] Schlichenmaier, M., and Sheinman, O.K.: The Sugawara construction and Casimir operators for Krichever-Novikov algebras., J. Math. Sci., New York **92**, no. 2, 3807–3834 (1998), q-alg/9512016.

[28] Schlichenmaier, M., and Sheinman, O.K.: Wess-Zumino-Witten-Novikov theory, Knizhnik-Zamolodchikov equations, and Krichever-Novikov algebras, I. Russian Math. Surv. (Uspeki Math. Naukii). **54**, 213–250 (1999), math.QA/9812083

[29] Tsuchiya, A., Ueno,K., and Yamada Y.: Conformal field theory on universal family of stable curves with gauge symmetries. Adv. Stud. Pure Math. **19**, 459–566 (1989)

[30] Tsujishita, T.: On the continuous cohomology of the Lie algebra of vector fields. Proc. Japan Acad. **53**, Sec. A, 134-138 (1977)

Alice Fialowski, *email*: `fialowsk@cs.elte.hu`
Martin Schlichenmaier, *email*: `schlichenmaier@math.uni-mannheim.de`

APPEARED IN: *Commun. in Contemporary Math.* **9** (2007), 41–79

Deformations of Four Dimensional Lie Algebras

Alice Fialowski
Institute of Mathematics
Eötvös Loránd University, Budapest

Michael Penkava
Department of Mathematics
University of Wisconsin, Eau Claire

Abstract We study the moduli space of four dimensional ordinary Lie algebras, and their versal deformations. Their classification is well known; our focus in this paper is on the deformations, which yield a picture of how the moduli space is assembled. Surprisingly, we get a nice geometric description of this moduli space essentially as an orbifold, with just a few exceptional points.

1. Introduction

Lie algebras of small dimension are still a central area of research, although their classification is basically known up to order 7 (for instance, see [12, 14, 15, 10, 13]). The reason for this is that they play a crucial role in physical applications (especially in dimension 4). Despite the classification of these algebras, the moduli space of Lie algebras in a given dimension is not well understood. We should mention [9], on the variety of n dimensional Lie algebra structures. Moreover, in the existing classifications there are often overlaps of families determined by parameters and the manner in which unique objects are singled out is somewhat artificial. Our solution of this problem is to consider the cohomology of the Lie algebras as well as their versal deformations, and use this information as a guide to their division into families. This is the additional information which provides us with a natural division of the moduli space of Lie algebras into families, as well giving us a geometric picture of the structure of the moduli space. We did a similar study for 3-dimensional Lie algebras in [7].

The goal of the present paper is to get an accurate picture of the moduli space of complex 4-dimensional Lie algebras. The key ingredient in our description will be the versal deformations of the elements in the moduli space; therefore cohomology will be a primary computational tool.

In this paper we will show that the moduli space of Lie algebras on $\mathbb{C}^4$ is essentially an orbifold given by the natural action of the symmetric group Σ_3 on the complex projective space $\mathbb{P}^2(\mathbb{C})$. In addition, there are two exceptional complex projective lines, one of which has an action of the symmetric group Σ_2. Finally, there are 6 exceptional points. The moduli space is glued together by the miniversal deformations, which determine the elements that one may deform to locally, so deformation theory determines the geometry of the space. The exceptional points play a role in refining the picture of how this space is glued together. By orbifold, we mean essentially a topological space factored out by the action of a group. In the case of $\mathbb{P}^n$, there is a natural action of Σ_{n+1} induced by the natural action of Σ_{n+1} on $\mathbb{C}^{n+1}$. An orbifold point is a point which is fixed by some element in the group. In the case of Σ_{n+1} acting on $\mathbb{P}^n$, points which have two or more coordinates with the same value are orbifold points, but there are some other ones, such as the point $(1 : -1) = (-1 : 1)$.

In the classical theory of deformations, a deformation is called a jump deformation if there is a 1-parameter family of deformations of a Lie algebra structure such that every nonzero value of the parameter determines the same deformed Lie algebra, which is not the original one (see [8]). There are also deformations which move along a family, meaning that the Lie algebra structure is different for each value of the parameter. There can be multiple parameter families as well.

In the picture we will assemble, both of these phenomena arise. Some of the structures belong to families and their deformations simply move along the family to which they belong. If there is a jump deformation from an element to a member of a family, then there will always be deformations from that element along the family as well, although they will typically not be jump deformations. In addition, there are sometimes jump deformations either to or from the exceptional points, so these exceptional points play an interesting role in the picture of the moduli space.

The structure of this paper is as follows. After some preliminary definitions and explanation of notation, we will explain our classification of four dimensional Lie algebras, giving a comparison between our description of the isomorphism classes of Lie algebras and the ones in [2] and [1]. Our division of the algebras into families is based primarily on cohomological considerations; elements with the same cohomological description are placed into the same family in our decomposition. The correlation between our decomposition and the one in [1] is very close. The main differences arise out of our intention to divide up our families as projective spaces, a point of view which only partially occurs in [1].

After giving a description of the elements of the moduli space, we then study in detail miniversal deformations of each element, and determine how the local deformations behave. The main tool used in this paper is a constructive approach to the computation of miniversal deformations, which was first given in [5, 6]. We do not provide complete details about the method of construction, but try to provide enough information that the reader might be able to reconstruct miniversal deformations from the data we provide. Our goal here is to use the constructions to give a picture of the moduli space, rather than to demonstrate the constructions themselves.

Finally, we will assemble all the information we have collected to give a pictorial representation of the moduli space.

2. Preliminaries

In classical Lie algebra theory, the cohomology of a Lie algebra is studied by considering a differential on the dual space of the exterior algebra of the underlying vector space, considered as a cochain complex. If V is the underlying vector space on which the Lie algebra is defined, then its exterior algebra $\bigwedge V$ has a natural $\mathbb{Z}_2$-graded coalgebra structure as well. In this language, a Lie algebra is is simply a quadratic odd codifferential on the exterior coalgebra of a vector space. An odd codifferential is simply an odd coderivation whose square is zero. The space L of coderivations has a natural $\mathbb{Z}$-grading $L = \bigoplus L_n$, where L_n is the subspace of coderivations determined by linear maps $\phi : \bigwedge^n V \to V$. A Lie algebra is a codifferential in L_2, in other words, a quadratic codifferential. (L_∞ algebras are just arbitrary odd codifferentials.)

The space of coderivations has a natural structure of a $\mathbb{Z}_2$-graded Lie algebra. The condition that a coderivation d is a codifferential can be expressed in the form $[d, d] = 0$. The coboundary operator $D : L \to L$ is given simply by the rule $D(\varphi) = [d, \varphi]$ for $\varphi \in L$; the fact that $D^2 = 0$ is a direct consequence of the fact that d is an odd codifferential. Moreover, $D(L_n) \subseteq L_{n+1}$, which means that the cohomology $H(d) = \ker D/\operatorname{Im} D$ has a natural decomposition as a $\mathbb{Z}$-graded space: $H(d) = \prod H^n(d)$, where

$$H^n(d) = \ker(D : L_n \to L_{n+1})/\operatorname{Im}(D : L_{n-1} \to L_n).$$

Recall that for an arbitrary vector space V of dimension n, the dimension of $\bigwedge^k V$ is just $\binom{n}{k}$. If $\{e_1, ..., e_n\}$ is a basis of V, $I = (i_1, \dots, i_k)$ is a multi-index with $i_1 < \cdots < i_k$, and we denote $e_I = e_{i_1} \cdots e_{i_k}$, then the e_I-s give a basis of $\bigwedge^k V$. Define $\varphi^I_j \in L_k$ by $\phi^I_j(e_J) = \delta^I_J e_j$, where δ^I_J is the Kronecker delta. The elements of L_k are all even if k is odd, and odd if k is even; to stress this difference, we will denote even elements as ϕ^I_j, but odd ones as ψ^I_j. Because we will be working with a four dimensional space, only L_0, L_1, L_2, L_3 and L_4 are nonzero, so 1 and 3 cochains are even, while 0, 2 and 4 cochains are odd. In general, the dimension of L_k is just $n\binom{n}{k}$, so for our case, $\dim L_0 = 4$, $\dim L_1 = 16$, $\dim L_2 = 24$, $\dim L_3 = 16$ and $\dim L_4 = 4$.

The Lie algebra structures are codifferentials in L_2. In order to represent a codifferential d as a matrix, we choose the following order for the increasing pairs $I = (i_1, i_2)$

of indices:

$$\{(1,2),(1,3),(2,3),(1,4),(2,4),(3,4)\},$$

and denote the ith element of this ordered set by $S(i)$. Using this order and the Einstein summation convention, we can express

$$d = a^i_j \varphi_i^{S(j)}.$$

Let $A = (a^i_j)$ be the matrix of coefficients of d. The first column represents $d(e_1e_2)$, the second $d(e_1e_3)$, etc. The Jacobi identity of the Lie algebra is given by the equation $[d,d] = 0$, which can be expressed in matrix form as $AB = 0$, where B is the matrix

$$B := \begin{bmatrix} a^1_6 & -a^2_6 & -a^2_5 - a^1_4 & -a^1_2 - a^2_3 \\ -a^1_5 & -a^3_6 - a^1_4 & -a^3_5 & a^1_1 - a^3_3 \\ -a^3_6 - a^2_5 & -a^2_4 & a^3_4 & a^3_2 + a^2_1 \\ a^1_3 & -a^4_6 + a^1_2 & -a^4_5 + a^1_1 & -a^4_3 \\ -a^4_6 + a^2_3 & a^2_2 & a^4_4 + a^2_1 & a^4_2 \\ a^4_5 + a^3_3 & a^4_4 + a^3_2 & a^3_1 & -a^4_1. \end{bmatrix}$$

Since AB is a 4×4 matrix, we obtain 16 quadratic relations among the coefficients that must be satisfied. In principle, it should be possible to use a computer algebra system to determine the solutions, but in our experience, this method has some drawbacks, unless one reduces the problem to some special cases, which we will do below.

In order to classify the solutions, we note that the dimension of the derived algebra is just the rank of A. We will show that the rank of A is never larger than 3. From this it follows that there is an ideal I of dimension 3 in the Lie algebra L, which gives an exact sequence of Lie algebras

$$0 \to I \to L \to \mathbb{K} \to 0,$$

where $\mathbb{K}$ is the abelian Lie algebra of dimension 1. But then, the structure of L is completely determined by the structure of I as a Lie algebra, and an outer derivation δ of I. In [7], the moduli space of three dimensional Lie algebras was studied, and we will use the classification given there, because we will use in our classification the structure of the cohomology of these Lie algebras, which is given in detail in that paper.

3. Dimension of the Derived Algebra

We separate the types of Lie algebras into two distinct cases.

(1) Every independent pair of vectors spans a two dimensional subalgebra.
(2) There are independent vectors x, y and z so that $d(xy) = z$.

The first case is interesting, in that, up to isomorphism, over any field $\mathbb{K}$, there is exactly one nonabelian Lie algebra in each dimension greater than one satisfying this property, and it is given as an extension of a one dimensional Lie algebra by an abelian ideal. To see this, suppose that L has dimension at least two, is nonabelian, and satisfies the property that every independent pair of vectors spans a two dimensional subalgebra.

Let x'_1 and y' be two independent elements whose bracket $[x'_1, y'] = ax'_1 + by'$ does not vanish. We may assume that $a \neq 0$. If $x_1 = x'_1 + b/ay'$ and $y = 1/ay'$, then $[x_1, y] = x_1$. Next, suppose that x'_2 is independent of x_1 and y. Let $[x'_2, y] = ax'_2 + by$. Then $x_1 + ax'_2 + by = [x_1 + x'_2, y] = p(x_1 + x'_2) + qy$ for some p and q, so $a = 1$. Let $x_2 = x'_2 + by$. Then $[x_2, y] = x_2$. Now, express $[x_1, x_2] = ax_1 + bx_2$. Then $ax_1 + bx_2 - x_2 = [x_1 + y, x_2] = p(x_1 + y) + qx_2$, which implies that $a = 0$. Similarly, $x_1 + bx_2 = [x_1, y + x_2] = px_1 + q(y + x_2)$, so $b = 0$ and thus $[x_1, x_2] = 0$. The process can be repeated indefinitely, so we obtain a basis $\{x_1, \dots, x_n, y\}$ satisfying $[x_i, y] = x_i$, $[x_i, x_j] = 0$.

Finally, let us show that the bracket of any two elements is linearly dependent on them. Let $u = a^i x_i + by$ and $v = c^i x_i + dy$, then $[u, v] = a^i dx_i - c^i bx_i = du - bv$. Clearly, the x_i-s span an abelian ideal in the algebra, so L is an extension of the one dimensional Lie algebra (spanned by y) by this ideal. It follows that there is an abelian ideal of dimension n; moreover, this ideal coincides with the derived algebra, so the rank of the matrix A is precisely n, one less than the dimension of the vector space. In fact, the matrix A has precisely the form $A = \left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]$, where I is the $n \times n$ identity matrix. This completes the description of the first case.

For the second case, suppose that there are linearly independent vectors such that $d(e_1e_2) = e_3$, so the matrix A of d satisfies $a_1^1 = a_1^2 = a_1^4 = 0$, $a_1^3 = 1$. One can easily check the possible solutions by considering subcases of this second case. For example, either e_1, e_2 and e_3 span a subalgebra, or we can assume that $d(e_1e_2) = e_4$. Since it is well known that the derived subalgebra of any 4 dimensional Lie algebra has dimension at most 3, we will not give a detailed analysis of this issue, and simply point out that the division into subcases can be carried out relatively easily. However, we note that even without breaking up the second case into subcases, we can solve the Jacobi identity using Maple, yielding around 40 solutions all of which have matrices of rank less than or equal to three. We note that the solutions are well defined over any field, so the fact that the derived algebra has dimension 3 is independent of the field $\mathbb{K}$ as well.

4. Extensions of $\mathbb{C}$ by a three dimensional ideal

From now on, in this paper, we shall assume that we are working over the base field $\mathbb{C}$. It is not difficult to classify the moduli space over $\mathbb{R}$ as well. Over fields of finite characteristic, and over other fields, even the classification of 3 dimensional Lie algebras is quite complicated.

Since the dimension of the derived algebra is never more than 3, every 4 dimensional Lie algebra is given as an extension of $\mathbb{C}$ by some three dimensional ideal. In [7], a complete classification of three dimensional algebras and their cohomology was given. We summarize the results about the cohomology in Table 3. Here we have realigned

Type	Codiff	H^1	H^2	H^3
$d_1 = \mathfrak{n}_3$	ψ_1^{23}	4	5	2
$d_2 = \mathfrak{r}_{3,1}(\mathbb{C})$	$\psi_1^{13} + \psi_2^{23}$	3	3	0
$d_2(1:1) = \mathfrak{r}_3(\mathbb{C})$	$\psi_1^{13} + \psi_1^{23} + \psi_2^{23}$	1	1	0
$d_2(\lambda:\mu) = \mathfrak{r}_{3,\mu/\lambda}(\mathbb{C})$	$\psi_1^{13}\lambda + \psi_1^{23} + \psi_2^{23}\mu$	1	1	0
$d_2(1:0) = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$\psi_1^{13} + \psi_1^{23}$	2	1	0
$d_2(1:-1) = \mathfrak{r}_{3,-1}(\mathbb{C})$	$\psi_1^{13} + \psi_1^{23} - \psi_2^{23}$	1	2	1
$d_3 = \mathfrak{sl}_2(\mathbb{C})$	$\psi_3^{12} + \psi_2^{13} + \psi_1^{23}$	0	0	0

Table 3. Cohomology of Three Dimensional Algebras

the family of codifferentials as presented in [7] in order to identify elements which have the same cohomological type as belonging to the same family. The changes are actually modest: the family $d_2(\lambda:\mu)$ coincides with $d(\mu/\lambda)$ of that paper except that the new element d_2 was given as $d(1)$ in the paper, and the element $d(1:1)$ corresponds to the element d_2 in the previous paper. In addition, we have introduced projective notation for the family $d_2(\lambda:\mu)$. It should be noted that $d_2(\lambda:\mu) = d_2(\mu:\lambda)$, so the family can be identified with $\mathbb{P}^1(\mathbb{C})/\Sigma_2$, which makes it an orbifold with orbifold points at $d_2(1:1)$ and $d_2(1:-1)$, where there is some atypical phenomena in the moduli space. At the point $d_2(1:1)$, there is a doppelganger d_2, whose neighborhoods coincide with those of the point $d_2(1:1)$, and which also deforms infinitesimally into $d_2(1:1)$. At the point $d_2(1:-1)$, there is a deformation in the d_3 direction as well as a deformation in the direction of the family. Otherwise, members of the family $d_2(\lambda:\mu)$ deform only in the direction of the family itself. The codifferential d_1 has deformations into every other type of codifferential except d_2, which accounts for why it has such a large dimension of H^2.

In order to determine all the codifferentials of degree 4, it is only necessary to study the equivalence classes of codifferentials given by extending $\mathbb{C}$ by a 3 dimensional algebra, via an outer derivation. For this reason, in Table 3, we have denoted by H^1 the dimension of the outer derivations, unlike our convention in [7]. In most cases, an extension of $\mathbb{C}$ by a 3 dimensional algebra is equivalent to either an extension by the Heisenberg algebra d_1, or an extension by the zero algebra, that is, a three dimensional central extension of $\mathbb{C}$. For each of the types of 3 dimensional algebras in our classification in Table 3, we will analyze the extensions of $\mathbb{C}$, by studying the outer derivations.

Suppose that A is a matrix representing a codifferential d and A' is the matrix representing a codifferential d'. The codifferentials d and d' determine isomorphic Lie algebras, and we call them equivalent codifferentials, if there is a linear automorphism $g : V \to V$ such that $d' = g^{-1} d\tilde{g}$, where $\tilde{g} : \bigwedge^2 V \to \bigwedge^2 V$ is the induced isomorphism. If we represent g by the 4×4 matrix $G = (g^i_j)$, where $g(e_j) = g^i_j e_i$, then $\tilde{g}$ is represented by the 6×6 matrix Q, in other words, $\tilde{g}(e_{S(j)}) = Q^i_j e_{S(i)}$, then the coefficients of Q are given by the formula

$$Q^i_j = g^m_k g^n_l - g^m_l g^n_k, \text{ where } S(i) = (k, l) \text{ and } S(j) = (m, n).$$

It follows that d is equivalent to d' precisely when there is an invertible matrix G and a corresponding matrix Q such that $A' = G^{-1}AQ$. It is usually easier to check by computer whether there is a matrix G and corresponding Q so that $GA' = AQ$, but then one must be careful to check that $\det(G) \neq 0$.

a). **The simple Lie algebra $d_3 = \mathfrak{sl}_2(\mathbb{C})$.** Since $\mathfrak{sl}_2(\mathbb{C})$ is simple, all derivations are inner. As a consequence, any extension of $\mathbb{C}$ by $\mathfrak{sl}_2(\mathbb{C})$ is just a direct sum $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$. This 4 dimensional algebra is given by the codifferential

$$d_3 = \psi^{12}_3 + \psi^{13}_2 + \psi^{23}_1, \tag{4.60}$$

which represents the simple algebra $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$ in the BS list [2].

b). **The solvable Lie algebra $d_2 = r_3(\mathbb{C})$.** This algebra is given by the codifferential

$$d_2 = \psi^{13}_1 + \psi^{23}_2.$$

$H^1(d_2) = \langle \varphi^2_2, \varphi^1_2, \varphi^2_1 \rangle$. Thus a generic outer derivation of d_2 is given by $\delta = \varphi^2_2 x + \varphi^1_2 y + \varphi^2_1 z$. An extension of $\mathbb{C}$ by δ is given by the rule $d(e_i e_4) = \delta(e_i)$. We compute

$$d(e_1 e_4) = e_2 y \quad d(e_2 e_4) = e_1 z + e_2 x \quad d(e_3 e_4) = 0,$$

so that the general formula for an extension d of $\mathbb{C}$ by d_2 is

$$d = \psi^{13}_1 + \psi^{23}_2 + \psi^{14}_2 y + \psi^{24}_1 z + \psi^{24}_2 x.$$

When $x^2 + 4yz \neq 0$, d is equivalent to the codifferential

$$d^\sharp_2 = \psi^{12}_1 + \psi^{34}_3, \tag{4.61}$$

which represents the Lie algebra $\mathfrak{r}_2 \oplus \mathfrak{r}_2$ in the BS list. When $x^2 + 4yz = 0$ and the three parameters are not all 0, then the matrix can be transformed into the matrix of the codifferential

$$d_1(1 : 0) = \psi^{12}_3 + \psi^{13}_3 + \psi^{23}_4, + \psi^{14}_4,$$

which represents the Lie algebra $\mathfrak{g}_8(0)$ in the BS list.

c). **The solvable algebra $d_2(\lambda : \mu)$.** This algebra is given by the codifferential

$$d_2(\lambda : \mu) = \psi^{13}_1 \lambda + \psi^{23}_1 + \psi^{23}_2.$$

If we consider the trivial extension of $\mathbb{C}$ by $d_2(\lambda : \mu)$, then $\{e_1, e_2, e_4\}$ span an abelian ideal, so this case reduces to an extension of $\mathbb{C}$ by an abelian ideal. To analyze nontrivial extensions, first note that

$$H^1(d_2(1:0)) = \langle \varphi^1_1 + \varphi^2_2, \varphi^3_2 \rangle$$
$$H^1(d_2(\lambda : \mu)) = \langle \varphi^1_1 + \varphi^2_2 \rangle \quad \text{otherwise}$$

If we extend our codifferential by the derivation $\delta = (\varphi_1^1 + \varphi_2^2)x + \varphi_2^3 y$, the extended codifferential is

$$d = d_2(\lambda : \mu) + (\varphi_1^{14} + \varphi_2^{24})x + \varphi_2^{34} y.$$

When $x \neq 0$ then if $\lambda = \mu$, the extended codifferential is equivalent to the codifferential $d_1(1:0)$, otherwise it is equivalent to the codifferential $d_2^\sharp$.
When $x = 0$ and $\mu \neq 0$, the codifferential is equivalent to the unextended codifferential which we will identify with the codifferential

$$d_3(\lambda : \mu : 0) = \psi_1^{14}\lambda + \psi_1^{24} + \psi_2^{24}\mu + \psi_2^{34},$$

which represents the Lie algebra $\mathfrak{r}_{3,\mu/\lambda}(\mathbb{C}) \oplus \mathbb{C}$ (unless $\lambda = \mu$, in which case it represents the Lie algebra $\mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}$). When $\mu = 0$ and $x = 0$, then if $y \neq 0$, the extended codifferential is equivalent to $d_3(1:0:0)$, which represents the Lie algebra $\mathfrak{g}_2(0,0)$, but when $y = 0$, the unextended codifferential is equivalent to the codifferential

$$d_3(0:1) = \psi_2^{34} + \psi_3^{34},$$

which represents the Lie algebra $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$.

d). **The Heisenberg Algebra $d_1 = n_3(\mathbb{C})$.** Let $d_1 = \psi_1^{23}$ be the three dimensional Heisenberg algebra. Then

$$H^1(d_1) = \langle \varphi_3^2, \varphi_2^3, \varphi_1^1 + \varphi_2^2, \varphi_1^1 + \varphi_3^3 \rangle$$

so $H^1(d_1)$ is four dimensional. If we consider a generic outer derivation

$$\delta = \varphi_3^2 a + \varphi_2^3 b + (\varphi_1^1 + \varphi_2^2)c + (\varphi_1^1 + \varphi_3^3)d,$$

the term $\psi_3^{24}a + \psi_2^{24}c + \psi_2^{34}b + \psi_3^{34}d + \psi_1^{14}(c+d)$ would be added to d_1 obtain the extended codifferential. If we set $a = a_5^3$, $b = a_6^2$, $c = a_5^2$ and $d = a_6^3$, then we get the extended codifferential

$$d = \psi_1^{23} + \psi_1^{14}(a_5^2 + a_6^3) + \psi_2^{24}a_5^2 + \psi_2^{34}a_6^2 + \psi_3^{24}a_5^3 + \psi_3^{34}a_6^3,$$

with matrix A given by $A = \begin{bmatrix} 0&0&1&a_5^2+a_6^3&0&0 \\ 0&0&0&0&a_5^2&a_6^2 \\ 0&0&0&0&a_5^3&a_6^3 \\ 0&0&0&0&0&0 \end{bmatrix}$. Let g be the linear transformation whose matrix is $G = \begin{bmatrix} 1&0&0&0 \\ 0&p&r&0 \\ 0&q&s&0 \\ 0&0&0&1 \end{bmatrix}$. Let $R = \left[\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right]$, and assume $\det(R) = 1$. Now the matrix Q is given in block form by $Q = \begin{bmatrix} R&0&0 \\ 0&I&0 \\ 0&0&R \end{bmatrix}$. The matrix of $d' = g^{-1}d\tilde{g}$ is $A' = \begin{bmatrix} 0&0&1&a_5^2+a_6^3&0&0 \\ 0&0&0&0&a_5'^2&a_6'^2 \\ 0&0&0&0&a_5'^3&a_6'^3 \\ 0&0&0&0&0&0 \end{bmatrix}$, where

$$\begin{bmatrix} a_5'^2 & a_6'^2 \\ a_5'^3 & a_6'^3 \end{bmatrix} = R^{-1} \begin{bmatrix} a_5^2 & a_6^2 \\ a_5^3 & a_6^3 \end{bmatrix} R,$$

which means that if $V = \begin{bmatrix} a_5^2 & a_6^2 \\ a_5^3 & a_6^3 \end{bmatrix}$, then similar submatrices give equivalent codifferentials. Note that the a_4^1 coefficient $a_5^2 + a_6^3$ is just the trace of the matrix V, which is invariant under similarity transformations. Therefore, looking at the submatrix V alone, we have the following cases

- $V = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \mu \end{smallmatrix}\right]$, corresponding to the codifferential

$$d_1(\lambda : \mu) = \psi_1^{23} + \psi_1^{14}(\lambda + \mu) + \psi_2^{24}\lambda + \psi_2^{34} + \psi_3^{34}\mu. \tag{4.62}$$

  This family of codifferentials should be thought of as a projective family, parameterizing $\mathbb{P}^1(\mathbb{C})$. There is an action of Σ_2 on this space which identifies $d_1(\lambda : \mu)$ with $d_1(\mu : \lambda)$. There are two orbifold points under this action: $d_1(1:1)$ and $d_1(1,-1)$. We can reasonably expect something unusual to happen at these orbifold points. In fact, $d_1(1:-1)$ represents the Lie algebra $\mathfrak{g}_7$ on the BS list while for all other values, *i.e.*, when $\lambda + \mu \neq 0$, $d_1(\lambda : \mu)$ represents the Lie algebra $\mathfrak{g}_8\left(\frac{\lambda\mu}{(\lambda+\mu)^2}\right)$.

The diagonal matrix $V = \operatorname{diag}(1,1)$. This is the only nonzero diagonalizable matrix which does not show up in the case above. Its associated codifferential is given by the formula

$$d_1^\sharp = \psi_1^{23} + 2\psi_1^{14} + \psi_2^{24} + \psi_3^{34}, \tag{4.63}$$

representing the Lie algebra $\mathfrak{g}_6$.

- $V = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then the extended codifferential is equivalent to

$$d_2^\star = \psi_1^{24} + \psi_2^{34}, \tag{4.64}$$

representing the Lie algebra $\mathfrak{n}_4(\mathbb{C})$.

- $V = 0$. This is the original, unextended codifferential, which is equivalent to the codifferential

$$d_1 = \psi_1^{24}, \tag{4.65}$$

representing the Lie algebra $\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$.

e). **Extensions of $\mathbb{C}$ by an abelian ideal.** Since $H^1(0) = L_1(\mathbb{C}^3)$, the whole 9 dimensional cochain space, an extension of $\mathbb{C}$ by $\mathbb{C}^3$ is given by a matrix of the form $A = \begin{bmatrix} 0 & 0 & 0 & a_4^1 & a_5^1 & a_6^1 \\ 0 & 0 & 0 & a_4^2 & a_5^2 & a_6^2 \\ 0 & 0 & 0 & a_4^3 & a_5^3 & a_6^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If we let $V = \begin{bmatrix} a_4^1 & a_5^1 & a_6^1 \\ a_4^2 & a_5^2 & a_6^2 \\ a_4^3 & a_5^3 & a_6^3 \end{bmatrix}$, then any matrix V' which is similar to V up to multiplication by a nonzero constant determines an equivalent codifferential. Since matrices which are constant multiples of each other determine the same codifferential, we can think of the nonequivalent codifferentials as being parameterized projectively. The decomposition of these matrices into distinct equivalence classes is as follows.

- The codifferential

$$d_3(\lambda:\mu:\nu) = \psi_1^{14}\lambda + \psi_1^{24} + \psi_2^{24}\mu + \psi_2^{34} + \psi_3^{34}\nu \tag{4.66}$$

for $(\lambda:\mu:\nu) \in \mathbb{P}^2(\mathbb{C})/\Sigma_3$, where the action of Σ_3 is given by permutation of the coordinates. These points determine an orbifold with orbifold points occurring along certain lines ($\mathbb{P}^1(\mathbb{C})$) where some of the parameters coincide. It might seem more natural to use diagonal matrices to represent this two parameter family; the choice here is based on cohomological considerations.

The codifferential

$$d_3(\lambda:\mu) = \psi_1^{14}\lambda + \psi_2^{24}\lambda + \psi_2^{34} + \psi_3^{34}\mu \tag{4.67}$$

for $(\lambda:\mu) \in \mathbb{P}^1(\mathbb{C})$. Here there is no action of the symmetric group.

- The Heisenberg algebra $d_1 = \psi_1^{24}$. The only eigenvalue of the matrix is zero, and it has two Jordan blocks. We will see that every point in $d_3(\lambda,\mu)$ is infinitesimally close to this point.
- The solvable algebra d_2^*. The matrix has one Jordan block, with eigenvalue zero.
- The identity matrix determines the codifferential

$$d_3^* = \psi_1^{14} + \psi_2^{25} + \psi_2^{34}, \tag{4.68}$$

which represents the Lie algebra $\mathfrak{g}_1(1)$.

- The zero algebra $d = 0$. Every point is infinitesimally close to this zero point.

We summarize these results and give the Lie bracket operations in standard terminology in the table below.

5. Comparison with the Burde-Steinhoff and Agaoka Lists

The comparison between the Burde-Steinhoff (BS) list and ours is slightly complicated. On the other hand, our decomposition is essentially the same as Agaoka's list, so we will just note the corresponding element, which is of the form $\mathbf{L}_i(\alpha)$ (see [1]).

Type	Brackets
$d_1(\lambda:\mu)$	$[e_2,e_3]=e_3,[e_1,e_4]=(\lambda+\mu)e_1,$ $[e_2,e_4]=\lambda e_2,[e_3,e_4]=e_2+\mu e_3$
$d_3(\lambda:\mu:\nu)$	$[e_1,e_4]=\lambda e_1,[e_2,e_4]=e_1+\mu e_2,[e_3,e_4]=e_2+\nu e_3$
$d_3(\lambda:\mu)$	$[e_1,e_4]=\lambda e_1,[e_2,e_4]=\lambda e_2,[e_3,e_4]=e_2+\mu e_3$
d_1	$[e_2,e_4]=e_1$
$d_1^\sharp$	$[e_2,e_3]=e_1,[e_1,e_4]=2e_1,[e_2,e_4]=e_2,[e_3,e_4]=e_3$
d_2^*	$[e_1,e_2]=e_1,[e_3,e_4]=e_2$
$d_2^\sharp$	$[e_1,e_2]=e_1,[e_3,e_4]=e_3$
d_3	$[e_1,e_2]=e_3,[e_1,e_3]=e_2,[e_2,e_3]=e_1$
d_3^*	$[e_1,e_4]=e_1,[e_2,e_4]=e_2,[e_3,e_4]=e_3$

TABLE 4. Table of Lie Bracket Operations

a). $d_1(\lambda:\mu)=\mathbf{L}_8(\mu/\lambda)=\psi_1^{23}+\psi_1^{14}(\lambda+\mu)+\psi_2^{24}\lambda+\psi_2^{34}+\psi_3^{34}\mu$.

(1) When $\lambda+\mu\neq 0$, then
$$d_1(\lambda:\mu)=\mathfrak{g}_8\left(\tfrac{\lambda\mu}{(\lambda+\mu)^2}\right).$$
(2) When $\lambda+\mu=0$, then we have the codifferential $d_1(1:-1)$ and
$$d_1(1:-1)=\mathfrak{g}_7.$$

b). $d_3(\lambda:\mu:\nu)=\mathbf{L}_7(\lambda/\nu,\mu/\nu)=\psi_1^{14}\lambda+\psi_1^{24}+\psi_2^{24}\mu+\psi_2^{34}+\psi_3^{34}\nu$.

(1) When the trace $\lambda+\mu+\nu$ of the matrix V is nonzero and none of the parameters are equal to zero, then
$$d_3(\lambda:\mu:\nu)=\mathfrak{g}_2\left(\tfrac{\lambda\mu\nu}{(\lambda+\mu+\nu)^3},\tfrac{\lambda\mu+\lambda\nu+\mu\nu}{(\lambda+\mu+\nu)^2}\right).$$
(2) When exactly one of the parameters vanishes and the other two are not equal, then
$$d_3(\lambda:\mu:0)=\mathfrak{r}_{3,\mu/\lambda}(\mathbb{C})\oplus\mathbb{C}.$$
(3) When one of the parameters vanishes and the other two are equal we have the special point
$$d_3(1:1:0)=\mathfrak{r}_3(\mathbb{C})\oplus\mathbb{C}.$$
(4) When two of the parameters vanish, then we have the special point
$$d_3(1:0:0)=\mathfrak{g}_2(0,0).$$
(5) When the trace of V is zero, none of the parameters is equal to zero, and the parameters are not the three distinct roots of unity, then we have
$$d_3(\lambda:\mu:-\lambda-\mu)=\mathfrak{g}_3\left(\tfrac{(\lambda^2+\lambda\mu+\mu^2)^3}{(\lambda\mu(\lambda+\mu))^2}\right).$$
(6) When λ, μ and ν are the three distinct cube roots of unity, then
$$d_3(1:-1/2+1/2i\sqrt{3}:-1/2-1/2i\sqrt{3})=\mathfrak{g}_4.$$

c). $d_3(\lambda:\mu)=\mathbf{L}_4(\mu/\lambda)=\psi_1^{14}\lambda+\psi_2^{24}\lambda+\psi_2^{34}+\psi_3^{34}\mu$.

(1) When neither of the parameters vanish or are equal, then we have
$$d_3(\lambda:\mu)=\mathfrak{g}_1(\mu/\lambda).$$
(2) When $\mu=0$, then we have the special point
$$d_3(1:0)=\mathfrak{r}_{3,1}(\mathbb{C})\oplus\mathbb{C}.$$
(3) When $\lambda=0$ then we have the special point
$$d_3(0:1)=\mathbf{L}_4(\infty)=\mathfrak{r}_2(\mathbb{C})\oplus\mathbb{C}^2.$$
(4) When $\lambda=\mu$ then we have the special point
$$d(0)=d_3(1:1)=\mathfrak{g}_5.$$

d). **The special cases.**

$$
\begin{aligned}
d_1 &= \mathbf{L}_1 = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C} = \psi_1^{24} \\
d_1^\sharp &= \mathbf{L}_5 = \mathfrak{g}_6 = \psi_1^{23} + 2\psi_1^{14} + \psi_2^{24} + \psi_3^{34} \\
d_2^* &= \mathbf{L}_2 = \mathfrak{n}_4(\mathbb{C}) = \psi_1^{24} + \psi_2^{34} \\
d_2^\sharp &= \mathbf{L}_9 = \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C}) = \psi_1^{12} + \psi_3^{34} \\
d_3 &= \mathbf{L}_6 = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C} = \psi_3^{12} + \psi_2^{13} + \psi_1^{23} \\
d_3^* &= \mathbf{L}_3 = \mathfrak{g}_1(1) = \psi_1^{14} + \psi_2^{24} + \psi_3^{34}.
\end{aligned}
$$

6. Deformations of the Lie Algebras

For the basic notion of deformations, we refer to [8, 11, 3, 4, 5]. In some previous papers, we considered deformations of L_∞ algebras [6, 7]. In this paper, we will only consider Lie algebra deformations of our Lie algebras, which are determined by cocycles coming from H^2. We will not explore L_∞ deformations of the Lie algebras we study in this paper, but it would not be difficult to construct them from the cohomology computations we provide here.

In Table 5, we give a classification of the codifferentials according to their cohomology. Note that for the most part, elements from the same family have the same cohomology. In fact, the decomposition of the codifferentials into families was strongly influenced by the desire to associate elements with the same pattern of cohomology in the same family. This is why our family $d_3(\lambda : \mu : \nu)$ was not chosen to be the diagonal matrices. Similar considerations influenced our selection of the family $d_3(\lambda : \mu)$.

Type	H^1	H^2	H^3	H^4
d_3	1	0	1	1
$d_2^\sharp$	0	0	0	0
$d_1(1 : -1)$	2	2	2	1
$d_1(1 : 0)$	1	2	1	0
$d_1(\lambda : \mu)$	1	1	0	0
$d_1^\sharp$	3	3	0	0
$d_3(1 : -1 : 0)$	3	5	5	2
$d_3(\lambda : \mu : \lambda + \mu)$	2	3	1	0
$d_3(\lambda : \mu : 0)$	3	3	1	0
$d_3(\lambda : \mu : -\lambda - \mu)$	2	2	1	1
$d_3(\lambda : \mu : \nu)$	2	2	0	0
$d_3(1 : 0)$	5	7	3	0
$d_3(0 : 1)$	6	6	2	0
$d_3(1 : 2)$	4	5	1	0
$d_3(1 : -2)$	4	4	1	1
$d_3(\lambda : \mu)$	4	4	0	0
d_1	8	13	10	3
d_2^*	4	6	5	2
d_3^*	8	8	0	0

Table 5. Table of the Cohomology

a). **The codifferential $d_3 = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$.** It is an easy calculation to show that

$$
\begin{aligned}
H^1 &= \langle \varphi_4^4 \rangle \\
H^2 &= 0 \\
H^3 &= \langle \varphi_4^{123} \rangle \\
H^4 &= \langle \varphi_4^{1234} \rangle
\end{aligned}
$$

Since H^2 vanishes, this algebra is rigid in terms of deformations in the Lie algebra sense.

b). **The codifferential $d_2^\sharp = \mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$.** Since the cohomology vanishes entirely, this algebra has no interesting deformations or extensions. This algebra is the only four dimensional Lie algebra which is truly rigid in the L_∞ algebra sense, although d_3 is also rigid in the Lie algebra sense.

c). **The codifferential $d_1(\lambda : \mu)$.** In the generic case we have

$$H^1 = \langle 2\varphi_1^1 + \varphi_2^2 + \varphi_3^3 \rangle$$
$$H^2 = \langle \psi_1^{14} + \psi_2^{24} \rangle$$

and all higher cohomology vanishes. Thus, generically, the infinitesimal deformation is given by

$$d^{\,\text{inf}} = d_1(\lambda + t, \mu). \tag{6.69}$$

Since $d^{\,\text{inf}}$ is actually a member of the family $d_1(\lambda : \mu)$, it is clear that $[d^{\,\text{inf}}, d^{\,\text{inf}}] = 0$, so the infinitesimal deformation is the miniversal deformation d^∞ and the base of the miniversal deformation is $\mathbb{C}[[t]]$. Moreover, it is transparent in this case that the deformations run in the direction of the family.

d). **The codifferential $d_1(1 : -1)$.** For this special case there are more cohomology classes than in the generic case. We have

$$H^1 = \langle 2\varphi_1^1 + \varphi_2^2 + \varphi_3^3, \varphi_1^4 \rangle$$
$$H^2 = \langle \psi_1 = \psi_1^{14} + \psi_2^{24}, \psi_2 = \psi_4^{23} \rangle$$
$$H^3 = \langle \varphi_4^{123}, \varphi_1^{123} + \varphi_4^{234} \rangle$$
$$H^4 = \langle \psi_4^{1234} \rangle.$$

Consider the universal infinitesimal deformation

$$d^{\,\text{inf}} = d_1(1 : -1) + \psi_1 t^1 + \psi_2 t^2.$$

Then we have $\frac{1}{2}[d^{\,\text{inf}}, d^{\,\text{inf}}] = -(\varphi_1^{123} + \varphi_4^{234}) t^1 t^2$, which is a nontrivial cocycle. It follows that the infinitesimal deformation is miniversal, and the base of the miniversal deformation is $\mathcal{A} = \mathbb{C}[[t^1, t^2, t^3]]/(t^1 t^2)$.
When $t^1 t^2 \neq 0$, the miniversal deformation $d^\infty = d^{\,\text{inf}}$ does not correspond to an actual deformation. The cohomology class of the cocycle $(\varphi_1^{123} + \varphi_4^{234})$ is called an obstruction to the extension of the infinitesimal deformation to higher order. In order to obtain an actual deformation out of the miniversal deformation, we need to restrict ourselves to the lines $t^1 = 0$ or $t^2 = 0$, along which the obstruction term vanishes. The cohomology H^2, which gives the tangent space to the moduli space, has dimension 2, but the deformations actually lie on two curves. Thus the dimension of the tangent space does not reveal the complete situation in terms of the deformations; one needs to construct the versal deformation to get the true picture.
A deformation along the line $t^2 = 0$ gives $d_1(1 + t_1, : -1 + t_1)$, the same pattern as we observed generically. On the other hand, a deformation along the line $t^1 = 0$ yields a coderivation which is equivalent to the codifferential d_3 . This is an example of a jump deformation, because $d_1(1 : -1) + \psi_2 t^2 \sim d_3$ for all values of t^2. In the classical language of Lie brackets, we get the following bracket table:

$$[e_1, e_3] = e_1 + t^2 e_4, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 - e_3.$$

Both orbifold points in the family $d_1(\lambda : \mu)$ have some unusual features. The point $d_1(1 : -1)$, because it has a jump deformation out of the family to d_3, and the point $d_1(1 : 1)$ because, as we will see shortly, there is a jump deformation to it from the element $d_1^\sharp$, which lies outside of the family. What is surprising is that the point $d_1(1 : 0)$, which is not an orbifold point, is also special.

e). **The codifferential** $d_1(1:0)$**.** The cohomology H^1 is the same as the generic case, while H^2 and H^3 are given by

$$H^2 = \langle \psi_1 = \psi_1^{14} + \psi_3^{34}, \psi_2 = \psi_2^{13} \rangle$$
$$H^3 = \langle \phi = \varphi_2^{134} \rangle.$$

The universal infinitesimal deformation $d^{\,\text{inf}} = d_1(1:0) + \psi_i t^i$ satisfies $\frac{1}{2}[d^{\,\text{inf}}, d^{\,\text{inf}}] = -2\phi t^1 t^2$, so it is miniversal and the base of the versal deformation is $\mathbb{C}[[t^1, t^2]]/(t^1 t^2)$. Along the line $t^2 = 0$, $d^\infty = d_1(1, t^1)$, so we deform along the family as in the generic case.
Along the line $t^1 = 0$, the deformation $d^{\,\text{inf}}$ is equivalent to $d_2^\sharp$ for all values of t^2. Thus $d_1(1:0)$ has a jump deformation to the element $d_2^\sharp$. The classical form of the Lie brackets for the case $t^1 = 0$ is

$$[e_1, e_3] = t^2 e_2, [e_2, e_3] = e_1, [e_1, e_4] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2.$$

f). **The codifferential** $d_1^\sharp$**.** The cohomology is given by

$$H^1 = \langle \varphi_1^1 + \varphi_2^2, \varphi_3^2, \varphi_2^3 \rangle$$
$$H^2 = \langle \psi_1 = \psi_3^{24}, \psi_2 = \psi_2^{34}, \psi_3 = \psi_1^{14} + \psi_3^{34} \rangle.$$

The universal infinitesimal deformation $d^{\,\text{inf}} = d_1^\sharp + \psi_i t^i$ is miniversal as $[d^{\,\text{inf}}, d^{\,\text{inf}}] = 0$, so the base of the miniversal deformation is just $\mathcal{A} = \mathbb{C}[[t^1, t^2, t^3]]$.
Now let us consider which codifferential $d^\infty = d^{\,\text{inf}}$ is equivalent to. Even though the deformation is defined for all values of the parameters, which element we deform to depends in a complicated manner on the parameters.
Except on the plane $t^1 = 0$, we have $d^\infty \sim d_1(\alpha : \beta)$ where

$$(\alpha, \beta) = 2 + t^3 \pm \sqrt{(t^3)^2 + 4t^1 t^2}.$$

On the plane $t^1 = 0$, we have $d^\infty \sim d_1(1 + t^3 : 1)$. In particular, if $t^3 = 0$, we have $d^\infty \sim d(1:1)$. In fact, along the entire surface given by $(t^3)^2 + 4t^1 t^2 = 0$, we have $d^\infty \sim d(1:1)$, so there is a two parameter family of jump deformations to $d_1(1:1)$. Thus $d_1^\sharp$ has a jump deformation to $d_1(1:1)$ and deforms along the family $d_1(\alpha:\beta)$ as if it were the element $d_1(1:1)$ in this family. This is a pattern which will always emerge: *If a codifferential has a jump deformation to another codifferential, then it will deform also to every codifferential to which the element it jumps to deforms.*
We give the classical form of the Lie brackets for d^∞:

$$[e_2, e_3] = e_1, [e_1, e_4] = (2 + t^3)e_1, [e_2, e_4] = e_2 + t^1 e_3, [e_3, e_4] = t^2 e_2 + (1 + t^3)e_3.$$

Let us consider the picture including only the codifferentials d_3, $d_1^\sharp$ and $d_1(\lambda:\mu)$. The picture is very similar to that of the moduli space of three dimensional Lie algebras. The family $d_1(\lambda:\mu)$ consists of a $\mathbb{P}^1$ with an action of the symmetric group Σ_2, with orbifold points at $(1:1)$ and $(1:-1)$. The point $d_1(1:-1)$ has a jump deformation to d_3, the four dimensional simple Lie algebra, while there is a jump deformation from the point $d_1^\sharp$ to $d_1(1:1)$. The point $d_1(1:0)$ is not an orbifold point, but is still special, with a jump deformation to the point $d_2^\sharp$. We did not see this phenomenon in the 3 dimensional picture, but there was something special about the point $d_2(1:0)$ in the family of codifferentials $d_2(\lambda:\mu)$ (see Table 3), because $\dim(H^1(d_2(1:0))) = 2$, instead of the generic value. Since H^1 influences the extensions of $\mathbb{C}$ by a Lie algebra, it is perhaps natural to expect that the 4 dimensional counterpart to $d_2(1:0)$ should not behave generically.

g). **The codifferential** $d_3(\lambda:\mu:\nu)$**.** Before examining the cohomology in the generic case, we want to make some general remarks about the family $d_3(\lambda:\mu:\nu)$, which we will call *the big family*, relating to the fact that the points correspond to $\mathbb{P}^2/\Sigma_3$, in contrast to $d_3(\lambda:\mu)$, which we will refer to as *the small family.* Let us refer to elements in the orbit of a point under the action of the symmetric group as conjugates. Most points in $\mathbb{P}^2$ have precisely 6 conjugates, and the stabilizer of the point is the trivial subgroup. The few exceptional cases are as follows.

(1) The points $(\lambda:\lambda:\mu)$, where $\lambda\neq\mu$ and their conjugates are stabilized by a subgroup of order 2, so they each have only 3 conjugates.
(2) The point $(1:-1:0)$ and its 3 conjugates are also stabilized by a subgroup of order 2.
(3) The point $(1:r:r^2)$, where r is a primitive cube root of unity, and its 2 conjugates, are stabilized by the alternating group A_3.
(4) The point $(1:1:1)$ is stabilized by the entire group Σ_3.

Next, consider the lines $(\mathbb{P}^1)$ in $\mathbb{P}^2$ and the induced action of Σ_3 on the set of lines. For most lines, the stabilizer of the line is just the trivial subgroup, but again, there are a few exceptions.

(1) The line $(\lambda:\lambda:\mu)$ and its 3 conjugates are stabilized by subgroups of order 2.
(2) The lines $(\lambda:\mu:c(\lambda+\mu))$ and their conjugates are also stabilized by subgroups of order 2.

It turns out that when $c=0$ or $c=\pm1$, the cohomology of the codifferentials corresponding to points on the line $(\lambda:\mu:c(\lambda+\mu))$ does not follow the generic pattern. The cohomology of the codifferentials corresponding to points on the line $(\lambda:\lambda:\mu)$ is the same as the generic case with the exception of the points $(1:1:0)$, $(1:1:2)$ and $(1:1:-2)$, which are the points of intersection of this line with the three other special lines. Note also that the lines $(\lambda:\mu:c(\lambda+\mu))$ all intersect in precisely the point $(1:-1:0)$, which makes this point very special.
To determine the cohomology of a codifferential of type $d_3(\lambda:\mu:\nu)$, read Table 5 in descending order, and whichever is the first pattern it matches, that gives its cohomology. However, we will present the description of the cohomology in ascending order, because it is more natural to begin with the generic pattern, and then proceed to the more exotic cases.

h). **The codifferential $d_3(\lambda:\mu:\nu)$: the generic case.** Generically, we have

$$\begin{aligned}H^1 =&\langle\varphi_1^1(\lambda-\mu)+\varphi_1^2+\varphi_2^2(\mu-\nu)+\varphi_2^3,\\&\varphi_1^1(-\lambda\mu+\lambda^2-\lambda\nu+\mu\nu)+\varphi_1^2(\lambda-\nu)+\varphi_1^3\rangle.\end{aligned}$$

For most generic values of $(\lambda:\mu:\nu)$ a natural basis to choose for H^2 would be $H^2=\langle\psi_2^{24},\psi_3^{34}\rangle$. Then $d^{\text{ inf}}=d_3(\lambda,\mu+t^1,\nu+t^2)$, so there is no difficulty in seeing what the deformations are equivalent to. However, for certain generic values of the parameters, the two cocycles above are not a basis of H^2, so we need to work with a more complex solution, which yields a basis of H^2 for all generic values. Let us take

$$H^2=\langle\psi_1=\psi_3^{24},\psi_2=\psi_3^{14}\rangle.$$

The universal infinitesimal deformation $d^{\text{ inf}}=d_3(\lambda:\mu:\nu)+\psi_i t^i$ is miniversal, with base $\mathcal{A}=\mathbb{C}[[t^1,t^2]]$. It is a bit more difficult to identify what the miniversal deformation $d^\infty=d^{\text{ inf}}$ is equivalent to when we take this more complicated basis of H^2. In fact, if we let x be a root of the polynomial

$$z^3+(-\nu+2\lambda-\mu)z^2+(\mu\nu-\lambda\nu-\lambda\mu+\lambda^2-t^1)z-t^2,$$

and y be a root of the polynomial

$$z^2+(-x-\nu+\mu)z+x^2+x(\lambda-\mu)-t^1,$$

then if g is given by the matrix

$$G=\begin{bmatrix}1&0&0&0\\x&1&0&0\\x(\lambda-\mu)+x^2&y&1&0\\0&0&0&1\end{bmatrix},$$

we have

$$g^*(d^\infty)=d_3(\lambda+x:\mu+y-x:\nu-y).$$

Thus for generic values of $(\lambda:\mu:\nu)$, all deformations of $d_3(\lambda:\mu:\nu)$ simply move along this same big family.

i). **The codifferential** $d_3(\lambda:\mu:-\lambda-\mu)$**.** In this case H^1 is the same as the generic case, and for most values of μ, one can use the cocycles ψ_1^{14} and ψ_2^{24} as the basis of H^2. Since the brackets of these two cocycles vanish, the resulting infinitesimal deformation

$$d^{\,\mathrm{inf}} = d_3(\lambda:\mu:-\lambda-\mu) + \psi_1^{14}t^1 + \psi_2^{24}t^2$$

is miniversal, and in fact coincides with $d_3(\lambda+t^1,\mu+t^2,-\mu-\lambda)$. In this case, it is obvious that the deformations of our codifferential just lie along the big family. The values for which these two elements do not form a basis of H^2 are $(1:-1:0)$, which we will cover separately, and $(1:1:-2)$, for which $\psi_1=\psi_3^{14}$ and $\psi_2=\psi_2^{24}$ give a basis of H^2. It is still true that the brackets of these cocycles vanish, and the deformations lie along the big family, although the expression of the member of the family corresponding to the element d^∞ is more complicated in this case, and will be omitted.
Thus the family $d_3(\lambda:\mu:-\lambda-\mu)$ is not special in deformation theory. This is a bit surprising, since H^3 does not vanish for elements of this subfamily, so it would not have been unreasonable to expect that there would be some obstructions to the extension of an infinitesimal deformation.

j). **The codifferential** $d_3(\lambda:\mu:0)=\mathfrak{r}_{3,\mu/\lambda}(\mathbb{C})\oplus\mathbb{C}$**.** . The dimensions of H^1 and H^2 increase to 3, and H^3 is 1-dimensional as well. The two cocycles ψ_1 and ψ_2 chosen as basis elements for H^2 in the generic case remain nontrivial and one can find an independent nontrivial cocycle $\psi_1^{13}+\psi_2^{23}$. However, this choice of a basis turns out to be inconvenient, and a slight modification of the basis will make the presentation simpler. We have

$$\begin{aligned}H^1 =&\langle\varphi_1^1\lambda(\lambda-\mu)+\varphi_1^2\lambda+\varphi_1^3,\varphi_1^1+\varphi_2^2+\varphi_3^3,\varphi_3^4\rangle\\ H^2 =&\langle\psi_1=\psi_3^{24}+\psi_3^{14}\lambda,\psi_2=\psi_3^{14},\psi_3=\psi_1^{13}+\psi_2^{23}\rangle.\end{aligned}$$

Let $d^{\,\mathrm{inf}}=d_3(\lambda:\mu:0)+\psi_it^i$. We compute

$$\begin{aligned}&[\psi_1,\psi_3]=-\varphi_1^{124}+\varphi_3^{234}+\varphi_3^{134}\lambda+\varphi_2^{124}\lambda=-D(\psi_2^{12}+\psi_3^{13})\\ &[\psi_2,\psi_3]=\varphi_3^{134}+\varphi_2^{124},\end{aligned}$$

so that

$$\tfrac{1}{2}[d^{\,\mathrm{inf}},d^{\,\mathrm{inf}}]=-D(\zeta_1)t^1t^3+(\varphi_3^{134}+\varphi_2^{124})t^2t^3,$$

where $\zeta_1=\psi_2^{12}+\psi_3^{13}$.
Note that in the case $t^3=0$, since ψ_1 and ψ_2 span the same subspace as the ones we used in the generic case, a deformation with $t^3=0$ is equivalent to one in the family. Thus there is a two parameter family of deformations of $d_3(\lambda:\mu:0)$ along the big family $d_3(\alpha:\beta:\eta)$.
On the other hand, when t^3 does not vanish, we will have to consider how $d^{\,\mathrm{inf}}$ extends to a higher order deformation. It turns out that when $\lambda=\mu$, the codifferential $\varphi_3^{134}+\varphi_2^{124}$ is a coboundary, but otherwise, it can be taken as a basis of H^3, and so is an obstruction to the extension of $d^{\,\mathrm{inf}}$ to a higher order deformation. We will first consider this obstructed case.

j).1. $\mu\neq\lambda$. In this case we have

$$H^3=\langle\phi=\varphi_3^{134}+\varphi_2^{124}\rangle.$$

We extend $d^{\,\mathrm{inf}}$ to the second order deformation

$$d^2=d^{\,\mathrm{inf}}+\zeta_1t^1t^3.$$

Since $\frac{1}{2}[d^2,d^2]=\phi t^2t^3$, the second order deformation is miniversal and the base of the miniversal deformation is $\mathcal{A}=\mathbb{C}[[t^1,t^2,t^3]]/(t^2t^3)$.
Thus any true deformation is given by taking $d^\infty=d^2$ with either $t^2=0$ or $t^3=0$. Since the case $t^3=0$ has already been examined, we consider the case $t^2=0$. In this

case, the matrix A of the deformation d^∞ is given by

$$A=\begin{bmatrix}0&t^2&0&\lambda&1&0\\ t^1t^3&0&t^3&0&\mu&1\\ 0&t^1t^3&0&\lambda t^1&t^1&0\\ 0&0&0&0&0&0\end{bmatrix}.$$

When $t^1=-\frac{(\lambda-\mu)^2}{4}$, then $d^\infty\sim d_1(1:0)$. Since $\lambda\neq\mu$, note that this deformation is not a jump deformation of $d_3(\lambda:\mu:0)$ but occurs some "distance" away from the original codifferential.
When $t^1\neq-\frac{(\lambda-\mu)^2}{4}$, then $d^\infty=d_2^\sharp$. This is a jump deformation, since it is independent of the value of t^1, as long as it is small.
Thus we obtain that the deformations of $d_3(\lambda:\mu:0)$, for $\lambda\neq\mu$, live along two planes in the (t^1,t^2,t^3) space. One is the plane $t^3=0$ determining deformations along the family $d_3(\lambda:\mu:\nu)$, while those which lie in the plane $t^2=0$ are equivalent to $d_2^\sharp$, except along the line $t^1=-\frac{(\lambda-\mu)^2}{4}$, which is not important to us, because this line does not include the origin. We say that a family of deformations is *not local* if the origin in the t-parameter space is not part of the family. Thus the deformations along the line $t^1=-\frac{(\lambda-\mu)^2}{4}$ are not local, in this sense. Only local deformations play a role in determining how the moduli space is glued together.

j).2. $\mu=\lambda$. This is the codifferential $d_3(1:1:0)=\mathfrak{r}_3(\mathbb{C})\oplus\mathbb{C}$. We have

$$[\psi_2,\psi_3]=\varphi_3^{134}+\varphi_2^{124}=-D(\zeta_2),$$

where $\zeta_2=-\psi_3^{23}+\psi_3^{13}+\psi_4^{24}+\psi_2^{12}$. So the second order deformation is given by

$$d^2=d^{\text{ inf}}+\zeta_1t^1t^3+\zeta_2t^2t^3.$$

Since $\varphi=\varphi_3^{124}$ is a nontrivial 3-cocycle, we can take

$$H^3=\langle\phi=\varphi_3^{124}\rangle.$$

Now

$$\tfrac{1}{2}[d^2,d^2]=-2\phi t^2t^3(t^1+t^3)-D(\zeta_3)t^2(t^3)^2+(\varphi_4^{124}-\varphi_3^{123})t^2(t^3)^2(t^1+t^2),$$

where $\zeta_3=\psi_4^{23}$. We did not obtain any second order relations, but because of the term involving φ in the bracket above, there is a third order relation $t^2t^3(t^1+t^2)$. The last term in the bracket is of higher order, so can be ignored in computing the third order deformation. We can take

$$d^3=d(1:1:0)+\psi_it^i+\zeta_1t^1t^3+\zeta_2t^2t^3+\zeta_3t^2(t^3)^2.$$

One computes that

$$\begin{aligned}\tfrac{1}{2}[d^3,d^3]=&-2\phi t^2t^3(t^1+t^3)+2(\varphi_4^{124}-\varphi_3^{123})t^2(t^3)^2(t^1+t^2)\\&+2\varphi_4^{123}t^2(t^3)^3(t^1+t^2).\end{aligned}$$

But this term is equal to zero, using the third order relation. Thus the base of a versal deformation is $\mathcal{A}=\mathbb{C}[[t^1,t^2,t^3]]/(t^2t^3(t^1+t^2))$, $d^\infty=d^3$, and the formal deformation corresponds to an actual deformation along the three planes $t^3=0$, $t^2=0$ and $t^2=-t^1$.
The plane $t^3=0$ corresponds to the generic case, which gives a 2 parameter space of deformations along the family $d_3(\lambda:\mu:\nu)$.
Now consider the plane spanned by $t^2=0$. If neither t^1 nor t^3 vanish, we then have $d^\infty\sim d_2^\sharp$, so there is a jump deformation from $d_3(1:1:0)$ to $d_2^\sharp$, just as for the other points on the line $d_3(\lambda:\mu:0)$.
When $t^2=t^3=0$, we are on the plane $t^3=0$, which we discussed already. When $t^2=t^1=0$, then we get a jump deformation to the point $d_1(1:0)$. This is not like the generic case.
Finally, let us consider the case when $t^2=-t^1$ and $t^3\neq0$. Let us express $t^1=\frac{\alpha\beta}{(\alpha+\beta)^2}$, and let $x=\frac{\alpha+\beta}{t^3}$. Then $g^*(d^3)=d_1(\alpha,\beta)$ if g is given by the matrix $G=$

$\begin{bmatrix} -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & \frac{\beta}{t^3} & -\frac{1}{t^3} & x \\ 0 & -\beta & 1 & 0 \end{bmatrix}$. Note that $(\alpha : \beta)$ is independent of t^3 as long as t^3 is nonzero. On the plane $t^2 = -t^1$, we see that for $t^1 = 0$, the deformation jumps to $d_1(1:0)$, but when $t^1 \neq 0$, we deform along the family $d_1(\alpha : \beta)$. It is as if $d_3(1:1:0)$ sits just above the point $d_1(1:0)$ and deforms as if it were that element. This is the usual pattern we have already discussed when there is a jump deformation to a point.

k). **The codifferential $d_3(\lambda : \mu : \lambda + \mu)$.** We exclude the codifferential $d_3(1:0:1)$ from our consideration here, because it coincides with the codifferential $d_3(1:1:0)$, which we treated previously. H^1 is the same as the generic case, and we have

$$\begin{aligned} H^2 = \langle \psi_1 =& 2\psi_1^{12}\lambda\mu(\lambda+\mu)^2 + \psi_2^{12}\lambda\mu^2(\lambda+\mu)(\lambda+2\mu) \\ &+ \psi_3^{12}\lambda^2\mu^2(\lambda+\mu)^2 - \psi_1^{13}\mu(\mu^2+2\lambda\mu+2\lambda^2) \\ &- \psi_2^{13}\mu^2(\lambda+\mu) + \psi_1^{23}(\mu^2+\lambda\mu+\mu^2) \\ &+ (\psi_1^{14}+\psi_2^{23})\lambda^2\mu^2(\lambda+\mu) \\ , \psi_2 =& \psi_1^{14} \\ , \psi_3 =& \psi_2^{14}\rangle. \end{aligned}$$

The brackets of ψ_1 with itself and ψ_3 are coboundaries, its bracket with ψ_2 is a nontrivial cocycle, and the rest of the brackets vanish. From this, one sees immediately that the second order relation is $t^1t^2 = 0$, but it is not so obvious what higher order terms might be necessary to add in order to obtain the relation on the base of the miniversal deformation. Since the space of 3-cocycles is 12 dimensional, we know that a miniversal deformation can be expressed in the form

$$d^\infty = d_3(\lambda, \mu, \lambda+\mu) + \psi_i t^i + \zeta_i x^i,$$

where $\zeta_1, \dots, \zeta_{11}$ is a pre-basis of the space of 3-coboundaries. In fact, we can give this pre-basis as

$$\{\zeta_i, i = 1, \dots 11\} = \{\psi_1^{12}, \psi_2^{12}, \psi_3^{12}, \psi_4^{12}, \psi_1^{13}, \psi_2^{13}, \psi_3^{13}, \psi_4^{13}, \psi_1^{23}, \psi_2^{23}, \psi_4^{23}\}.$$

Note that the first 10 of these vectors are just the first 10 elementary 2-cochains. Also

$$H^3 = \langle \phi = \varphi_3^{124} \rangle,$$

and we can complete the linearly independent set given by the $D(\zeta_i)$ and ϕ to a basis $\{D(\zeta_1), \dots, D(\zeta_(11)), \phi, \tau_1, \dots \tau_4\}$ of L_3. Then we must have

$$[d^\infty, d^\infty] = D(\zeta_1)s^1 + \dots + D(\zeta_{11})s^{11} + \phi s^{12} + \tau_1 s^{13} + \cdots \tau_4 s^{16},$$

for some coefficients $s^1, \dots s^{16}$, where these coefficients are expressed as polynomials in the variables t^i and x^i. Now all of these coefficients must be equal to zero, once we take into account the relation on the base of the miniversal deformation, which is the coefficient s^{12}. The expression one obtains for s^{12} by direct computation from the form of d^∞ will have the variables x^i in it, but it should depend only on the variables t^i. The trick is to solve the first 11 equations for x^i as functions of the variables t^i, and then substitute these into the formula for s^{12} to obtain the relation on the base. The relation on the base of the miniversal deformation is simply $t^1t^2 = 0$, which is exactly the second order relation. If you solve for the coefficients of $s^{13}, \dots, s^{16}$, then they turn out to be multiples of s^{12}, so they are equal to zero using the relation on the base.

Let us study the deformations of $d(\lambda : \mu : \lambda + \mu)$. Since the relation on the base of the miniversal deformation is $t^1t^2 = 0$, in any true deformation, we must have either $t^1 = 0$ or $t^2 = 0$.

When $t^1=0$, then $d^\infty = d_3(\lambda:\mu:\lambda+\mu)+\psi_2t^2+\psi_3t^3$, so that for any values of t^2 and t^3 we have a deformation along the big family. In fact, $d^\infty\sim d_3(\alpha:\beta:\eta)$ where

$$\begin{aligned}\alpha =&\tfrac{1}{2}(\lambda+\mu+t^2+\sqrt{(t^2+\lambda-\mu)^2+4t^3})\\ \beta =&\tfrac{1}{2}(\lambda+\mu+t^2-\sqrt{(t^2+\lambda-\mu)^2+4t^3})\\ \eta =&\lambda+\mu.\end{aligned}$$

The interesting case is when $t^2=0$. The matrix of d^∞ is quite complicated, so we won't reproduce it here, but it should be noted that some terms have $t^3-\lambda\mu$ in the denominator, so that $t^3=\lambda\mu$ may not correspond to an actual deformation. When $t^1\neq 0$, then $d^\infty\sim \mathfrak{g}_8\left(\frac{\lambda\mu-t^3}{(\lambda+\mu)^2}\right)$. In particular, if we set $t^3=0$, we see that there is a jump deformation to $d_1(\lambda:\mu)$, and that we also deform along the family $d_1(\lambda:\mu)$ when $t^3\neq 0$.

l). **The codifferential** $d_3(1:-1:0)$**.** For this codifferential, from the fact that H^2 and H^3 both have dimension 5, we expect to see some interesting phenomena, both because the tangent space to the space of deformations has dimension 5, and since H^3 has high dimension, the dimension of the variety of deformations would likely be lower than 5. We can give bases for the cohomology as follows:

$$\begin{aligned}H^1 &= \langle 2\varphi^1_1+\varphi^2_2+\varphi^3_1, \varphi^1_1+\varphi^2_2+\varphi^3_3, \varphi^4_3\rangle\\ H^2 &= \langle \psi_1=\psi^{24}_{,}\psi_2=\psi^{14}_3, \psi_3=\psi^{12}_3-\psi^{13}_3-\psi^{23}_3+\psi^{14}_4+\psi^{24}_4\\ &\quad ,\psi_4=\psi^{23}_1-2\psi^{23}_2, \psi_5=\psi^{12}_4-\psi^{13}_4-\psi^{23}_4\rangle\\ H^3 &= \langle \phi_1=\varphi^{124}_1, \phi_2=\varphi^{124}_3, \phi_3=\varphi^{123}_2+\varphi^{123}_3-\varphi^{234}_4\\ &\quad ,\phi_4=\varphi^{123}_4, \phi_5=\varphi^{123}_2+\varphi^{124}_4-\varphi^{234}_4\rangle.\end{aligned}$$

A pre-basis of the 3-coboundaries is

$$\{\zeta_1,\dots,\zeta_9\}=\{\psi^{12}_1,\psi^{12}_2,\psi^{12}_3,\psi^{12}_4,\psi^{13}_1,\psi^{13}_2,\psi^{13}_3,\psi^{13}_4,\psi^{14}_4\}.$$

A miniversal deformation is given by

$$d^\infty = d_3(1:-1:0)+\psi_it^i+\zeta_ix^i,$$

where the x^i are expressible as power series in the variables t^i. Since not all of the brackets of the ψ_i vanish, we do not expect that the coefficients x^i are all equal to zero, in general.

We can express

$$[d^\infty,d^\infty]=D(\zeta_i)s^i+\phi_is^{9+i}+\tau_is^{14+i},$$

where $D(\zeta_i)$, ϕ_i and τ_i form a basis of L_3. Solving $s^1=\cdots=s^9=0$ for $x^1,\dots,x^9$ in terms of $t^1,\dots,t^9$, and substituting these values of the x^i into the formulas for $s^{10},\dots,s^{14}$, we obtain 5 relations on the base of the versal deformation, the simplest of which is

$$\frac{(t^3(t^1)^2+4t^1t^2t^4+4t^2t^4)}{t^1-2}=0.$$

Some of these relations have t^1-1 or t^1-2 as a factor of the denominator, which means that there may not be a solution when t^1 takes on these values. There should be an actual, rather than just a formal power series solution for all values of t^i which make all 5 of the relations vanish. When we solved for the zeros of the relations, we obtained the following 5 solutions:

1) $t^1=t^2=t^4=0$
2) $t^3=t^4=t^5=0$
3) $t^2=t^3=t^5=0$
4) $t^3=t^5=0,\qquad t^1=-1$
5) $t^2=\frac{-t^1(t^1-2)^2}{8}, t^3=\frac{t^4(t^1-2)^2(t^1+1)}{2t^1}, t^5=\frac{(t^4)^2(t^1-2)^2(t^1+1)}{(t^1)^2}.$

Note that each of these solutions is only a 2-dimensional subvariety of the 5-dimensional tangent space.
For the first solution, the matrix of the corresponding d^∞ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ t^3 & -t^3 & -t^3 & 0 & 0 & 0 \\ t^5 & -t^5 & -t^5 & t^3 & t^3 & 0 \end{bmatrix}.$$

Along the curve $t^5 = (t^3)^2$, $d^\infty \sim d_1(1:-1)$. For all other points on the (t^3, t^5)-plane, $d^\infty \sim d_3$. This fits with our prior observation that there is a jump deformation from $d_1(1:-1)$ to d_3. Thus we have jump deformations from $d_3(1:-1:0)$ to both $d_1(1:-1)$ and d_3.
For the second solution, the matrix of d^∞ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1+t^1 & 1 \\ 0 & 0 & 0 & t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Deformations corresponding to this solution give a two parameter family of deformations along the big family $d_3(\lambda:\mu:\nu)$.
For the third solution, the matrix of d^∞ is

$$A = \begin{bmatrix} 0 & -t^4t^1 & t^4 & 1 & 1 & 0 \\ 0 & 0 & -2t^4 & 0 & -1+t^1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

When $t^4 = 0$, this is just a special case of the previous solution, and in fact, in this case $d^\infty = d_3(1:-1+t^1:0)$. Supposing that $t^4 \neq 0$, then when $t^1 \neq -1$, then $d^\infty \sim d_2^\sharp$ which is a jump deformation.
The fourth solution has $t^1 = -1$, which means it is not local, so does not contribute to our picture of the moduli space. Although the solution is interesting, we will omit it here.
For the last solution, which is the most complicated of them all, the first three columns of the matrix of d^∞ are

$$\begin{bmatrix} 0 & t^1t^4 & t^4 \\ \frac{t^4(t^1-2)^2(t^1+1)}{4} & 0 & -2t^4 \\ \frac{t^4(t^1-2)^4(t^1+1)}{8t^1} & \frac{-t^4(t^1-2)^2(-4+3(t^1)^2)}{8t^1} & \frac{-t^4(t^1-2)^2(t^1+1)}{2t^1} \\ \frac{(t^4)^2(t^1+2)(t^1-2)^4(t^1+1)}{8(t^1)^2} & \frac{-(t^4)^2((t^1)^3-2(t^1)^2+4t^1+8)(t^1-2)^2}{8(t^1)^2} & \frac{-(t^4)^2(t^1+1)(t^1-2)^2}{(t^1)^2} \end{bmatrix}.$$

Note that t^1 appears in the denominator, so cannot vanish for this solution. If $t^4 = 0$, $t^1 = -1$ or $t^1 = 2$, then the fifth solution coincides with one of the previous four, so we will not consider these cases here. The matrix of A is so complicated that in order to determine which standard form the codifferential is equivalent to three, we first had to transform A into a matrix of an equivalent codifferential which had a simpler matrix. We found that $d^\infty \sim d_1(\alpha:\beta)$ where

$$\alpha = \frac{t^1+\sqrt{5(t^1)^2-16t^1+16}}{2}, \quad \beta = \frac{t^1-\sqrt{5(t^1)^2-16t^1+16}}{2}.$$

Note that if we were to set $t^1 = 0$ in the above, we would obtain the codifferential $d_1(1:-1)$, to which we already obtained a jump deformation in the first solution above.
The picture of the local deformations of $d_3(1:-1:0)$ is as follows. First, we can deform along the big family. Secondly, we can deform to $d_2^\sharp$, like any other member of the family $d_3(\lambda:\mu:0)$. Thirdly, like any other member of the family $d_3(\lambda:\mu:\lambda+\mu)$, we have a jump deformation to an element in the family $d_1(\lambda:\mu)$. Because the element we deform to is $d_1(1:-1)$, which has an extra deformation to the element d_3, we can also deform to this element, as well as deforming along the family $d_1(\lambda:\mu)$.

m). **The codifferential** $d_3(\lambda:\mu)$**.** This family does not have an action of the symmetric group, which is important to keep in mind. Generically, H^1 and H^2 are 4 dimensional. The generic basis of H^2 below consists of elements which are linearly independent nontrivial cocycles for generic values of λ and μ except in the special case $\lambda=\mu$, which we will treat separately. Of course, for those values of λ and μ for which $\dim H^2>4$, they do not span H^2. Generically, we have

$$H^1=\langle\psi_2^1,\psi_2^2(\lambda-\mu)+\psi_2^3,\psi_2^2+\psi_3^3,\psi_1^2(\lambda-\mu)+\psi_1^3\rangle$$
$$H^2=\langle\psi_1=\psi_3^{34},\psi_2=\psi_2^{14},\psi_3=\psi_1^{24},\psi_4=\psi_2^{24}\rangle.$$

All of the brackets of these nontrivial cocycles vanish, so the miniversal deformation is just the first order deformation $d^\infty=d_3(\lambda:\mu)+\psi_i t^i$, and there are no relations on the base. The matrix of d^∞ is given by

$$A=\begin{bmatrix}0&0&0&\lambda&t^3&0\\0&0&0&t^2&\lambda+t^4&1\\0&0&0&0&0&\mu+t^1\\0&0&0&0&0&0\end{bmatrix}.$$

If $t^3\neq 0$, then $d^\infty\sim d_3(\alpha,\beta,\eta)$, where

$$\alpha=\lambda+\frac{t^4+\sqrt{(t^4)^2+4t^2t^3}}{2},\quad \beta=\lambda+\frac{t^4-\sqrt{(t^4)^2+4t^2t^3}}{2},\eta=\mu+t^1.$$

If $t^1=t^2=t^4=0$, then $d^\infty\sim d_3(\lambda:\lambda:\mu)$, so there is a jump deformation from $d_3(\lambda:\mu)$ to $d_3(\lambda:\lambda:\mu)$. Thus we see that the codifferential $d_3(\lambda:\mu)$ sits over the codifferential $d_3(\lambda:\lambda:\mu)$ and deforms along the big family as if it were that codifferential. All of the deformations of $d_3(\lambda:\mu)$ which do not lie along the hyperplane $t^3=0$ lie along the big family.
Now, consider the hyperplane $t^3=0$. The eigenvalues of the submatrix

$$B=\begin{bmatrix}\lambda&0&0\\t^2&\lambda+t^4&1\\0&\mu+t^1&\end{bmatrix}$$

are λ, $\lambda+t^4$ and $\mu+t^1$. If these eigenvalues are all distinct, then $d^\infty\sim d_3(\lambda:\lambda+t^4:\mu+t^1)$. Otherwise, one of the conditions $t^4=0$, $t^1=\lambda-\mu$, or $t^1-t^4=\lambda-\mu$ holds. Of these conditions, only the first one is local, so we will not consider the other two. Consider the plane $t^3=t^4=0$. Unless $t^2=0$ or $t^1=\lambda-\mu$, d^∞ is still equivalent to $d_3(\lambda:\lambda+t^4:\mu+t^1)$. Again, the second condition is not local, so we will ignore it. On the line $t^2=0$, we have $d^\infty\sim d_3(\lambda:\mu+t^1)$, so we get a deformation along the $d_3(\lambda:\mu)$ family.
To summarize the generic deformation behavior of an element of the family $d_3(\lambda:\mu)$, we have the following picture. First, we can always deform along the family to which an element belongs, so there is a deformation along the family $d_3(\lambda:\mu)$. Secondly, there is a jump deformation to the element $d_3(\lambda:\lambda:\mu)$ in the big family. Whenever there is a jump deformation, then we can deform in any manner in which the element we jump to deforms, and thus there is a deformation along the big family as well. Note that the line $(\lambda:\lambda:\mu)$, which is one of the lines in $\mathbb{P}^2$ with nontrivial stabilizer, is the target of our jump deformations, so the elements $d_3(\lambda:\lambda:\mu)$ are special not in the sense that they have more deformations, but that there are extra deformations to them.

n). **The codifferential** $d_3(1:1)$**.** Even though the dimension of H^2 for this element is the same as the generic case of $d_3(\lambda:\mu)$, we have to use a different basis for H^2 than in the generic case.

$$H^2=\langle\psi_1=\psi_3^{14},\psi_2=\psi_1^{14},\psi_3=\psi_1^{24},\psi_4=\psi_3^{24}\rangle.$$

As in the generic case, the brackets of these cocycles all vanish, so the universal infinitesimal deformation is the miniversal deformation d^∞, with matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1+t^2 & t^3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & t^1 & t^4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

When $t^1 \neq 0$, we obtain a complicated deformation along the family $d_3(\alpha:\beta:\eta)$. To understand the solution a bit better, when we solve for a matrix transforming A into one representing a codifferential of the form $d_3(\alpha:\beta:\eta)$, we obtain a solution which satisfies

$$\begin{aligned} t^3 =& \frac{1/3(\alpha^3+\beta^3+\eta^3)(t^2+3)^3+(\alpha+\beta+\eta)(t^2+3)p(\alpha,\beta,\eta,t^2)+8(\alpha+\beta+\eta)^3}{t^1(\alpha+\beta+\eta)^3} \\ t^4 =& \frac{1/3(\alpha^2+\beta^2+\eta^2)(t^2+3)^2-1/2(\alpha+\beta+\eta)((t^2)^2+2t^2+3))}{(\alpha+\beta+\eta)^2}, \end{aligned}$$

where p is a polynomial which is homogeneous, quadratic and symmetric in α, β and η and quadratic in t^2. Consequently, when $t^2 \neq -3$, for any values of t^2, t^4 and t^3, we obtain exactly one solution up to the action of the symmetric group, and thus one member of the family $d_3(\alpha:\beta:\eta)$ is determined. This follows since the line $\alpha+\beta+\eta=0$ intersects the quadric surface determined by the equation for t^4 above in exactly the orbifold points $(1:\frac{-1+\sqrt{3}}{2}:\frac{-1-\sqrt{3}}{2})$ and $(1:\frac{-1-\sqrt{3}}{2}:\frac{-1:\sqrt{3}}{2})$, which do not lie on the cubic surface determined by the equation for t^3.
When $t^1 \neq 0$ and $t^2 = t^3 = t^4 = 0$, then $d^\infty \sim d_3(1:1:1)$, so there is a jump deformation to this element, as we expect from the generic case.
When $t^1 = 0$, then the eigenvalues of the submatrix $B = \left[\begin{smallmatrix} 1+t^2 & t^3 & 0 \\ 0 & 1 & 1 \\ t^1 & t^4 & 1 \end{smallmatrix}\right]$ are $1+t^2$ and $1 \pm \sqrt{t^4}$, so they are distinct unless $t^4 = 0$ or $t^4 = (t^2)^2$. Thus, except in these two cases we have $d^\infty \sim d_3(1+t^2, 1+\sqrt{t^4}, 1-\sqrt{t^4})$. On the plane $t^1 = t^4 = 0$ we have $d^\infty \sim d_3(1+t^2, 1, 1)$.
On the surface $t^1 = 0$, $t^4 = (t^2)^2$ except on the curve $t^3 = 0$ we have $d^\infty \sim d_3(1+t^2, 1+t^2, 1-t^2)$. Finally, on the curve $t^3 = 0$ on this surface we have $d^\infty \sim d_3(1+t^2, 1-t^2)$, so we obtain a deformation along the family $d_3(\lambda:\mu)$ on this curve.
Thus, just like any other generic value, there is one curve along which there is a jump deformation to the corresponding point $d_3(1:1:1)$ on the large family, another curve along which we deform along the $d_3(\lambda:\mu)$ family, and otherwise, all deformations are along the big family. In a way, it is surprising that the one point in $\mathbb{P}^2$ which is fixed by every permutation does not have any special properties in terms of deformation theory, but as we have seen, there just isn't anything particularly special about the deformations of this codifferential.

o). **The codifferential** $d_3(1:-2)$**.** We have

$$\begin{aligned} H^2 =& \langle \psi_1 = \psi_2^{14}, \psi_2 = \psi_3^{24}, \psi_3 = \psi_3^{34}, \psi_4 = \psi_1^{24} \rangle \\ H^3 =& \langle \varphi_4^{123} \rangle. \end{aligned}$$

Even though $H^3 \neq 0$, it turns out that the brackets of all the ψ's with each other vanish, so the miniversal deformation d^∞ coincides with the infinitesimal deformation, and its matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & t^4 & 0 \\ 0 & 0 & 0 & t^1 & 1 & 1 \\ 0 & 0 & 0 & 0 & t^2 & -2+t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because this matrix has no terms on the left hand side, it is natural to guess that the deformations are either along the family $d_3(\alpha:\beta:\eta)$ or the family $d_3(\lambda:\mu)$, with possibly a few exceptional codifferentials.

When $t^1 \neq 0$ we have a solution of the form

$$\begin{aligned} t^3 =& \frac{\alpha+\beta+\eta}{q} \\ t^2 =& -\frac{(\alpha+\eta-2q)(\alpha+\beta-2q)(\beta+\eta-2q)}{q^2(\alpha+\beta+\eta-3q)} \\ t^4 =& \frac{-(\alpha-q)(\beta-q)(\eta-q)}{t^1q^2(\alpha+\beta+\eta-3q)}, \end{aligned}$$

where q is a nonzero free parameter. These equations are symmetric in α, β and η. If $t^3 \neq 0$, then $\alpha+\beta+\eta \neq 0$, and we can solve the first equation for q and get

$$\begin{aligned} t^2 =& \frac{-((\alpha+\beta)t^3-2(\alpha+\beta+\eta))((\beta+\eta)t^3-2(\alpha+\beta+\eta))((\alpha+\eta)t^3-2(\alpha+\beta+\eta)}{(\alpha+\beta+\eta)^3(t^3-3)} \\ t^4 =& \frac{-(\alpha t^3-(\alpha+\beta+\eta))(\beta t^3-(\alpha+\beta+\eta))(\eta t^3-(\alpha+\beta+\eta))}{t^1(t^3-3)(\alpha+\beta+\eta)^3}. \end{aligned}$$

We can express these equations in the form

$$\begin{aligned} t^2 =& \frac{\alpha\beta\eta(t^3)^3+(\alpha+\beta+\eta)(t^3-2)p(\alpha,\beta,\eta,t^3)}{(\alpha+\beta+\eta)^3(t^3-3)} \\ t^4 =& \frac{-\alpha\beta\eta(t^3)^3+(\alpha+\beta+\eta)r(\alpha,\beta,\eta,t^3)}{t^1(t^3-3)(\alpha+\beta+\eta)^3}, \end{aligned}$$

where p and q are homogeneous, quadratic and symmetric in α, β and η. The surfaces represented by these two equations are both cubic, so there are 9 points of intersection. Since every cubic which is given by a symmetric, homogeneous polynomial either contains the line $\alpha+\beta+\eta=0$ or intersects this line in precisely the points $(1:-1:0)$, $(1:0:-1)$ and $(0:1:-1)$, there are six points in the intersection of these two cubics not on this line, which uniquely determine the codifferential $d_3(\alpha:\beta:\eta)$ to which d^∞ is equivalent. The matrix representing the transformation can be chosen with nonzero determinant, as long as $t^3 \neq 3$. The condition $t^3 \neq 0$ can also be overcome, because if we substitute $t^3 = 0$ in the above, then the problem still has a solution.Thus, whenever $t^1 \neq 0$ and $t^3 \neq 3$, the deformation is equivalent to a member of the family $d_3(\alpha:\beta:\eta)$.
When $t^1 = 0$, then as long as $t^2 \neq 0$ and $t^3 \neq 1$, $d^\infty \sim d_3(\alpha:\beta:1)$, where

$$\alpha = \frac{t^3-1+\sqrt{(t^3-3)^2+4t^2}}{2} \quad \text{and} \quad \beta = \frac{t^3-1-\sqrt{(t^3-3)^2+4t^2}}{2}.$$

When $t^1 = t^2 = 0$ and $t^4 \neq 0$ we have $d^\infty \sim d_3(1:1:t^3-2)$. As a consequence, if we set $t^3 = 0$, we have a jump deformation from $d_3(1:-2)$ to $d_3(1:1:-2)$. On the other hand, when $t^1 = t^2 = t^4 = 0$, then $d^\infty \sim d_3(1:t^3-2)$. When $t^1 = 0$ and $t^3 = 1$, then we also have a deformation along the big family. The upshot of all this analysis is that $d_3(1:-2)$ is really not special in terms of deformation theory. It deforms along its own family, jumps to $d_3(1:1:-2)$, and deforms along that family.

p). **The codifferential** $d_3(1:2)$**.** We have

$$\begin{aligned} H^2 &= \langle \psi_1 = \psi^{12}_2 - \psi^{12}_3 + \psi^{13}_2 + \psi^{13}_3, \psi_2 = \psi^{14}_1 + \psi^{34}_3, \\ &\qquad \psi_3 = \psi^{14}_2, \psi_4 = \psi^{34}_3, \psi_5 = \psi^{24}_1 \rangle \\ H^3 &= \langle \varphi^{124}_3 + \varphi^{134}_2 \rangle. \end{aligned}$$

This time, we do have some nonzero brackets, but only those brackets of ψ_1 with ψ_2, ψ_4 and ψ_5, with the first one being a nontrivial cocycle, so that the second order relation is $t^1t^2 = 0$. After some work, one obtains that there is one relation on the base of the versal deformation,

$$t^1t^2(-1-t^4+t^3t^5) = 0,$$

so that there are three distinct solutions for a true deformation, given by the three factors of the miniversal deformation. Notice that the third factor does not give rise to a local deformation.

Let us study the first solution, when $t^1 = 0$. This case is simplest. The matrix corresponding to d^∞ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1+t^4 & t^5 & 0 \\ 0 & 0 & 0 & t^3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2+t^2+t^4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

When $t^5 \neq 0$, then d^∞ is equivalent to $d_3(\alpha : \beta : 2 + t^2 + t^4)$, where $\{\alpha, \beta\} = \frac{2+t^4 \pm \sqrt{(t^4)^2 + 4t^3t^5}}{2}$. If $t^2 = t^3 = t^4 = 0$, then the deformation is equivalent to $d_3(1 : 1 : 2)$ for all $t^5 \neq 0$, giving the expected jump deformation.

What happens if $t^5 = 0$? As long as $t^2 \neq -1$ and $t^2t^3 + t^4t^3 + t^3 + t^4 \neq 0$, then the deformation is still along the big family. If $t^2 \neq -1$, but $t^2t^3 + t^4t^3 + t^3 + t^4 = 0$, then as long as $t^3 \neq 0$ and $t^4 \neq 0$, the deformation is in the big family. If $t^4 = 0$, then $t^3 = 0$ or $t^2 = -1$, and in both cases we deform along the family $d_3(\alpha : \beta)$. Thus, the first solution to the relations on the base does not have any surprises.

The second solution to the relations on the base is $t^2 = 0$. We may as well assume that $t^1 \neq 0$ and that $-1 - t^4 + t^3t^5 \neq 0$ for this case. Then the matrix of d^∞ is

$$A = \begin{bmatrix} -t^1t^5 & \frac{-t^1t^5}{-1-t^4+t^3t^5} & \frac{t^1(t^5)^2}{-1-t^4+t^3t^5} & 1+t^4 & t^5 & 0 \\ -t^1 & \frac{-t^1}{-1-t^4+t^3t^5} & \frac{t^1t^5}{-1-t^4+t^3t^5} & t^3 & 1 & 1 \\ t^1(-1-t^4+t^3t^5) & t^1 & -t^1t^5 & 0 & 0 & 2+t^4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The submatrix consisting of the first three columns of A has rank 1, so we can transform this matrix into a simpler matrix.

Recall that we assume that $t^1 \neq 0$. When $t^5 \neq 0$, then it turns out that $d^\infty \sim d_1(\alpha : \beta)$, where $\frac{\alpha\beta}{(\alpha+\beta)^2} = \frac{1-t^4-t^3t^5}{(2+t^4)^2}$. Also, if $t^5 = 0$ and $t^3 \neq 0$, then the deformation is equivalent to $d_1(1 + t^4 : 1)$. In particular, if $t^4 = 0$, we see that there is a jump deformation from our codifferential to the codifferential $d_1(1 : 1)$. On the other hand, if $t^3 = 0$ and $t^4 \neq 0$, we also deform to $d_1(1 + t^4 : 1)$. When $t^3 = t^4 = t^5 = 0$, there is a jump deformation of d^∞ to $d_1^\sharp$.

The picture for this element is more intriguing than for $d_3(1 : -2)$. In addition to the usual deformations along the family $d_3(\alpha, \beta)$, jump deformation to $d_3(1 : 1 : 2)$, and deformations along the big family, we see that $d_3(1 : 2)$ has a jump deformation to the codifferential $d_1^\sharp$. Because $d_1^\sharp$ itself has a jump deformation to $d_1(1 : 1)$, we get a jump deformation to this element as well, and deformations along the family $d_1(\lambda : \mu)$. Thus we pick up far more deformations than we would expect considering that the dimension of H^2 is only one more than in the generic case. Again, the explanation for this "impossibility" has to do with the fact that the three dimensional tangent space to this element of the moduli space does not accurately reflect the nature of the deformations, which are all tangent to one of three planes in this space. The true picture is captured by the versal deformation.

q). **The codifferential** $d_3(0 : 1) = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$**.** The cohomology is given by

$$\begin{aligned} H^1 &= \langle \psi_2^1, \psi_1^2 - \psi_1^3, \psi_2^2 - \psi_2^3, \psi_1^4, \psi_2^4, \psi_1^1 \rangle \\ H^2 &= \langle \psi_1 = -\psi_2^{12} + \psi_2^{13}, \psi_2 = \psi_1^{14}, \psi_3 = \psi_2^{14}, \\ &\qquad \psi_4 = \psi_1^{24}, \psi_5 = \psi_2^{24}, \psi_6 = \psi_1^{12} - \psi_1^{13} \rangle \\ H^3 &= \langle \varphi_1^{124}, \varphi_3^{124} \rangle. \end{aligned}$$

Not all of the brackets of the nontrivial 2-cocycles vanish, so we obtain some relations on the base of the versal deformation. The second order relations are $t^1t^2 + t^3t^6 = 0$ and $t^1t^4 + t^5t^6 = 0$. The relations on the base are obtained by adding higher order terms to these second order relations. We will omit them for brevity, but instead will describe the solutions which may give actual deformations. There are 8 solutions, 4 of which not local. The local solutions are

$$\begin{aligned} &1)\quad t^1 = t^6 = 0\\ &2)\quad t^1 = t^5 = t^3 = 0\\ &3)\quad t^2 = t^3 = 0, t^4 = \tfrac{-t^5t^6}{t^1}\\ &4)\quad t^6 = \tfrac{t^1t^2}{t^3}, t^4 = \tfrac{t^2t^5}{t^3}. \end{aligned}$$

The first solution corresponds to the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & t^2 & t^4 & 0\\ 0 & 0 & 0 & t^3 & t^5 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The codifferentials d^∞ associated to this matrix are easy to analyze. They usually lie in the big family, except for some special cases when they are in the small family. There is a jump deformation to $d_3(0:0:1)$.
The second solution corresponds to

$$A = \begin{bmatrix} t^6 & -t^6 & \frac{t^4t^6}{t^2-1} & t^2 & t^4 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $t^6 \neq 0$, then either $t^2 \neq 1$ or $t^4 = 0$, and the deformation is equivalent to $d_2^\sharp$. As a consequence, there is a jump deformation to $d_2^\sharp$. When $t^6 = 0$, then if $t^4 \neq 0$ or $t^2 \neq 1$, then the deformation is equivalent to $d_3(1:t_2:0)$.
In the third solution, let us first assume t^6 does not vanish. Then the solution has matrix

$$A = \begin{bmatrix} -(t^5-1)t^6 & (t^5-1)t^6 & \frac{t^5t^6}{t^1} & t^5 & \frac{t^5}{t^1} & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

When $t^5 \neq 1$, then the deformation is equivalent to $d_2^\sharp$, so there is a jump deformation to $d_2^\sharp$.
Now let us assume that $t^6 = 0$ in the third solution. We can assume $t^1 \neq 1$, since that corresponds to the first solution. The matrix simplifies to

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ t^1(t^5-1) & t^1 & 0 & 0 & t^5 & 1\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $t^5 \neq 1$, then the deformation is equivalent to $d_2^\sharp$ again.
Finally, let us consider the fourth solution, whose matrix is equivalent to

$$A = \begin{bmatrix} \frac{t^1(t^2+t^5-1)}{t^2-1} & \frac{t^1t^2(t^2+t^5-1)}{t^3(t^2-1)} & t^1 & t^2+t^5 & t^3 & 1\\ 0 & 0 & 0 & 0 & 0 & \frac{-t^2}{t^3}\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two special cases that need to be considered, when $t^2 = 0$, in which case, the restriction $t^3 \neq 0$ does not apply, and the case when $t^5 = 0$. (The case $t^5 = 1 - t^2$ is not local.) In these cases, the restriction $t^2 \neq 1$ does not apply. Let us first address these special cases.
When $t^2 = 0$, if $t^1 \neq 0$ and $t^5 \neq 1$ then we get $d_2^\sharp$. On the other hand, if $t^2 = 0$ and $t^1 = 0$, then if $t^3 = t^5 = 0$, we get $d_3(0:1)$.
From now on, we deal with the general case, assuming that $t^5 \neq 0$, $t^2 \neq 0$, $t^5 \neq 1-t^2$. Then $t^2 \neq 1$ and $t^3 \neq 0$.

When $t^1 \neq 0$, the deformation is equivalent to $d_2^\sharp$; otherwise it is equivalent to $d_3(1 : t^2 + t^5 : 0)$.
The subcases are a bit tricky, but the same codifferentials keep showing up, so the final analysis of the deformations of this codifferential is not difficult. We either obtain a jump deformation to $d_3(1 : 0 : 0)$ or to $d_2^\sharp$, or we obtain a deformation along the big or small families.

r). **The codifferential** $d_3(1 : 0) = \mathfrak{r}_{3,1}(\mathbb{C}) \oplus \mathbb{C}$**.** The cohomology is given by

$$\begin{aligned} H^1 =&\langle \varphi_1^1, \varphi_2^1, \varphi_2^2 + \varphi_3^3, \varphi_3^4, \varphi_1^2 + \varphi_3^3 \rangle \\ H^2 =&\langle -\psi_1^{12} + \psi_2^{24} + \psi_4^{34}, \psi_2^{12} + \psi_4^{14}, \psi_1^{23}, \psi_3^{34}, \psi_1^{24}, \psi_2^{24}, \psi_3^{14} \rangle \\ H^3 =&\langle \varphi_1^{124}, \varphi_3^{124}, \varphi_1^{234} \rangle. \end{aligned}$$

Some of the brackets of the nontrivial 2-cocycles do not vanish, and we have the second order relations

$$t^1t^4 + 2t^3t^7 + 2t^2t^5 = 0, \quad t^2t^4 - t^2t^6 + t^1t^7 = 0, \quad t^1t^5 + t^3t^4 + t^3t^6 = 0.$$

We omit the long expressions for the seven relations on the base of the miniversal deformation, but remark that $1 + t^4 + t^6$ appears in the denominator of two of them, so there may be an obstruction to the extension of an infinitesimal deformation to a formal one.
The solution to the relations is quite complex; however, if we confine ourselves to solutions which are local, then we can reduce the problem to 9 relatively simple cases.

1) $t^1 = (t^4 + t^6)\sqrt{\frac{t^2t^3}{t^4(1+t^6)}}, t^7 = \frac{t^3(t^4+t^6)}{t^1}, t^5 = \frac{-t^3(t^4+t^6)}{t^1}$
2) $t^3 = \frac{(t^1)^2((t^6-t^4+2)^2+2(t^4-t^6))}{8t^2(t^4+t^6+2)}, t^7 = \frac{(t^4-t^6)(t^4-t^6-2)t^1}{8t^3}, t^5 = -\frac{t^3(t^4+t^6)}{t^1}$
3) $t^1 = t^2 = t^3 = 0$
4) $t^1 = t^2 = t^7 = 0, \quad t^6 = -t^4$
5) $t^1 = 0, t^4 = -t^6, t^7 = \frac{t^6(1+t^6)}{t^5}, t^2 = \frac{-t^3t^6(1+t^6)}{(t^5)^2)}$
6) $t^1 = t^3 = t^5 = 0$
7) $t^2 = t^4 = t^7 = 0, \quad t^5 = \frac{-t^3t^6}{t^1}$
8) $t^4 = t^5 = t^6 = t^7 = 0$
9) $t^3 = t^4 = t^5 = 0, \quad t^7 = \frac{t^2t^6}{t^1}$.

In the first solution, if $t^6 = 0$, or $4t^4 \neq (t^6 - t^4)^2$ then the deformation is equivalent to $d_2^\sharp$; otherwise, we get $d_1(1 : 0)$.
In the second solution the differential is equivalent to $d_1(\alpha : \beta)$, where

$$(\alpha, \beta) = t^4 + t^6 + 2 \pm \sqrt{5(t^4)^2 - 12t^4 - 6t^4t^6 + 4t^6 + 5(t^6)^2 + 4}.$$

(It may be more revealing to recognize this element as $\mathfrak{g}_8\left(\frac{4t^4-(t^6-t^4)^2}{(t^4+t^6+2)^2}\right)$). Note that since t^1 is any nonzero number, this means that there is a jump deformation from $d_3(1 : 0)$ to $d_1(1 : 0)$.
In the third solution, all deformations are either along the big family or the family $d_3(\alpha : \beta)$. If $t^4 = t^6 = t^7 = 0$ and $t^5 \neq 0$, then the deformation is equivalent to $d_3(1 : 1 : 0)$, so there is a jump deformation from $d_3(1 : 0)$ to this element.
In the fourth solution, we get $d_1(1 + t^6 : -t^6)$, so that if $t^6 = 0$, we see that there is a jump deformation to $d_1(1 : 0)$.
In the fifth solution, when $t^6 = 0$ then if $t^3 = 0$, we get $d_3(1 : 1 : 0)$; otherwise we obtain $d_1(1 : 0)$. If $t^3 = 0$ and $t^6 \neq 0$, then we get $d_3(1+\sqrt{t^6(t^6 + 1)} : 1-\sqrt{t^6(t^6 + 1)})$ (assuming $t^6 \neq -1$). When neither t^6 nor t^3 vanish, we get a jump deformation to $d_2^\sharp$.
In the sixth solution, if $t^2 = 0$, this reduces to a previous case. If $t^6 = 0$, then we get $d_2^\sharp$, a jump deformation. Otherwise, when $t^7 \neq 0$ or $t^6 \neq 1$, we get $d_1(1 : t^6)$.

In the seventh solution, we always get $d_2^\sharp$.
In the eight solution, if $t^1 = t^2 = 0$, this is a previous case. If $t^2 = 0$ but $t^1 \neq 0$, or $t^1 = 0$ and $t^2 \neq 0$, we get $d_2^\sharp$ unless $t^3 = 0$, in which case we get $d_1(1:0)$. When neither t^1 nor t^2 vanish, then we get $d_2^\sharp$; unless $(t^3)^2 = 4t^1t^2$, when we get $d_1(1:0)$.
In the ninth solution, we always get $d_2^\sharp$.
To summarize, we note that $d_3(1:0)$ jumps to $d_3(1:0:0)$ and $d_1(1:0)$ and it deforms along the big and small families as usual. It also jumps to $d_2^\sharp$.

s). **The codifferential $d_3^\star$.** The cohomology is given by

$$H^1 = \langle \varphi_2^1, \varphi_3^1, \varphi_1^2, \varphi_2^2, \varphi_3^2, \varphi_1^3, \varphi_2^3, \varphi_3^3 \rangle$$
$$H^2 = \langle \psi_2^{14}, \psi_3^{14}, \psi_3^{34}, \psi_3^{24}, \psi_1^{14}, \psi_2^{34}, \psi_1^{34}, \psi_1^{24} \rangle.$$

The brackets of all 2-cocycles with each other vanish, so the infinitesimal deformation is miniversal. The matrix of d^∞ is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1+t^5 & t^8 & t^7 \\ 0 & 0 & 0 & t^1 & 1 & t^6 \\ 0 & 0 & 0 & t^2 & t^4 & 1+t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The deformations are easy to analyze, because they are given by the equivalence classes of similar matrices of the 3×3 submatrix given by the parameters. It is easy to see that there are jump deformations to $d_3(1:1:1)$ and $d_3(1:1)$, as well as deformations along the families these two codifferentials belong to. There are no other possibilities for local deformations.

t). **The codifferential $d_2^\star = \mathfrak{n}_4(\mathbb{C})$.** The cohomology is given by

$$H^1 = \langle \varphi_3^4, 2\varphi_1^1 + \varphi_2^2 + \varphi_4^4, \varphi_1^3, \varphi_1^1 + \varphi_2^2 + \varphi_3^3 \rangle$$
$$H^2 = \langle \psi_2^{24}, \psi_1^{13} + \psi_2^{23}, \psi_3^{24}, \psi_4^{23}, \psi_3^{14}, \psi_1^{12} + \psi_2^{13} + \psi_3^{23} \rangle$$
$$H^3 = \langle \varphi_2^{124}, \varphi_3^{124}, \varphi_1^{123}, \varphi_4^{123}, \varphi_4^{124} - \varphi_3^{123} \rangle.$$

With such a large H^3, it would be too much to imagine that the brackets of the cocycles vanish; in fact, there are 5 relations on the base of the miniversal deformation. Since they are fairly simple, we will give them:

$$4t^2t^5 + (t^1)^2t^6 = 0$$
$$2t^5t^6 - t^1t^2t^5 + t^1t^3t^6 = 0$$
$$t^1t^4 + t^2t^6 = 0$$
$$2t^2t^3t^4 - t^1t^4t^6 - t^1(t^2)^2t^6 = 0$$
$$2t^4t^5 + 2t^1t^3t^4 - (t^1)^2t^2t^6 + t^1(t^6)^2 = 0.$$

Note that the fourth relation has no second order term, and the fact that the relations have no denominators means that the miniversal deformation is constructed in a finite number of steps; in fact, since the highest degree term in a relation is of degree 4, the fourth order deformation is miniversal. The solution to the relations can be decomposed into 3 four dimensional subspaces and one more complex four dimensional piece as follows.

1) $t^4 = t^5 = t^6 = 0$
2) $t^2 = t^4 = t^6 = 0$
3) $t^1 = t^2 = t^5 = 0$
4) $t^6 = \frac{-t^1t^4}{2t^2}, \quad t^3 = \frac{-(t^1)^2(t^4+(t^2)^2)}{4(t^2)^2}, \quad t^5 = \frac{t^4(t^1)^3}{8(t^2)^2}.$

For the first solution, the matrix of d^∞ is

$$A = \begin{bmatrix} 0 & t^2 & 0 & 0 & 1 & 0 \\ t^2t^3 & 0 & t^2 & 0 & t^1 & 1 \\ 0 & t^2t^3 & 0 & 0 & t^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

When $t^2 = 0$, $d^\infty \sim d_3\left(0 : \frac{t^1+\sqrt{(t^1)^2+4t^3}}{2} : \frac{t^1-\sqrt{(t^1)^2+4t^3}}{2}\right)$. Assume $t^2 \neq 0$. Then, when $t^3 = 0$, if $t^1 \neq 0$, we have $d^\infty \sim d_2^\sharp$, and when $t^1 = 0$, we get $d_1(1:0)$. When $t^3 \neq 0$, if $(t^1)^2 + 4t^3 = 0$, then $d^\infty \sim d_1(1:0)$; otherwise it is equivalent to $d_2^\sharp$. Thus we get jump deformations to $d_2^\sharp$ and $d_1(1:0)$.
For the second solution, the matrix is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & t^1 & 1 \\ 0 & 0 & 0 & t^5 & t^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case $d^\infty \sim d_3(\alpha : \beta : \eta)$ where

$$\alpha + \beta + \eta = t^1 q, \quad \alpha\beta\eta = t^5 q^3, \quad \alpha\beta + \alpha\eta + \beta\eta = t^3 q^2,$$

where q is an arbitrary nonzero parameter. As a consequence, there is a jump deformation from d_2^* to every member of the family $d_3(\alpha : \beta : \eta)$.
For the third solution, the matrix is given by

$$A = \begin{bmatrix} t^6 & 0 & 0 & 0 & 1 & 0 \\ 0 & t^6 & 0 & 0 & 0 & 1 \\ t^6t^3 & 0 & t^6 & 0 & t^3 & 0 \\ 0 & 0 & t^4 & 0 & 0 & 0 \end{bmatrix}.$$

When $t^6 = 0$, if $t^3 = 0$, then we get $d_1(1:-1)$, while if $t^4 = 0$ we get $d_3(1:-1:0)$, both jump deformations. When $t^6 = 0$ and neither t^3 nor t^4 vanishes, then the deformation is equivalent to d_3, another jump deformation.
Assume $t^6 \neq 0$. If $(t^6)^2 \neq -t^3t^4$ then we get d_3; otherwise we get $d_1(1:-1)$, both jump deformations.
For the fourth solution, the matrix is quite complicated, so we omit it. When $t^1 = 0$, then if $t^2 = 0$, we get $d_1(1:-1)$, and if $t^4 = 0$, then we get $d_1(1:0)$, both jump deformations; otherwise, we get a deformation along the family $d_1(\alpha:\beta)$.
When $t^1 \neq 0$ and $t^4 = 0$, then if $t^2 = 0$, we get the jump deformation $d_3(1:1:0)$; otherwise we get a jump deformation to $d_1(1:0)$.
When $t^1 \neq 0$ and $t^4 \neq 0$ and $t^4 = -(t^2)^4$, then we get a jump deformation to the element $d_1(1+\sqrt{5}, 1-\sqrt{5})$, which is just $\mathfrak{g}_8(-1)$ on the Burde-Steinhoff list.
When none of the three conditions above hold, then the deformation is equivalent to $d_1(t^2 + \sqrt{(t^2)^2 - 4t^4}, t^2 - \sqrt{(t^2)^2 - 4t^4}) = \mathfrak{g}_8\left(\frac{t^4}{(t^2)^2}\right)$. Since t^1 is an arbitrary nonzero number, these deformations are also jump deformations. Thus there is a jump deformation to any element of the family $d_1(\alpha:\beta)$.
To summarize, the deformations of d_2^* are as follows. There are jump deformations to every member of the big family and everything they deform to, which means we get jump deformations to the elements $d_2^\sharp$, d_3 and every element in the family $d_1(\lambda:\mu)$.

u). **The codifferential $d_1 = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$.** The cohomology is given by

$$\begin{aligned} H^1 &= \langle \varphi_1^1 + \varphi_2^2, \varphi_1^1 + \varphi_4^4, \varphi_4^2, \varphi_1^3, \varphi_3^3, \varphi_2^4, \varphi_3^4 \rangle \\ H^2 &= \langle -\psi_3^{23}, \psi_4^{23}, \psi_1^{14}, \psi_2^{14}, \psi_3^{14}, \psi_3^{12}, \psi_3^{34}, \\ &\quad \psi_4^{24}, \psi_2^{34}\psi_4^{12}, \psi_1^{13} + \psi_2^{23}, \psi_4^{14} - \psi_2^{12}, \psi_4^{34} - \psi_1^{13} \rangle \\ H^3 &= \langle \varphi_2^{234}, \varphi_4^{234}, \varphi_4^{124}, \psi_2^{134}, \psi_3^{134}, \psi_2^{123} + \psi_4^{134}, \psi_3^{123}, \psi_4^{123}, \psi_2^{124}, \psi_3^{124} \rangle. \end{aligned}$$

There are 10 relations, none of which involve terms of higher order than 3, so in fact, the second order deformation is already miniversal. We will not give the relations

here explicitly. Because the miniversal deformation is obtained in a finite number of steps, the relations are polynomial, not rational, in the parameters.
If all the parameters but t^{13} and t^{11} vanish, and $t^{13} = -t^{11}$ then the relations are satisfied, and we have a jump deformation to $d_1^\sharp$.
If all the parameters but t^{12}, t^5 and t^6 vanish, then if $t^{12} \neq 0$, the deformation is equivalent to d_3, so there is a jump deformation to d_3.
If we assume that

$$\begin{aligned} t^8 =& t^3 = 0, \quad t^1 = \tfrac{-t^6t^7}{t^5}, \quad t^{12} = \tfrac{-t^6t^9}{t^5}, \\ t^2 =& \tfrac{(t^6)^2t^9}{(t^5)^2}, \quad t^4 = \tfrac{-t^{12}t^5}{t^6}, \quad t^{11} = \tfrac{-t^6t^9}{t^5}, \quad t^{10} = \tfrac{t^6t^{12}}{t^5}, \end{aligned}$$

then if q is a free parameter and

$$\begin{aligned} &\alpha+\beta+\eta = \tfrac{t^7q}{t^5} \\ &(\alpha+\beta)(\alpha+\eta)(\beta+\eta) = \tfrac{-t^9q^3}{(t^5)^2} \\ &(\alpha\beta+\alpha\eta+\beta\eta) = \tfrac{t^{12}q^2}{t^5t^6}, \end{aligned}$$

we obtain a solution to the relations for which $d^\infty \sim d_3(\alpha:\beta:\eta)$. Whenever t^5 and t^6 don't vanish, there is a solution for every $(\alpha:\beta:\eta)$, which means that there is a jump deformation from d_1 to every element in the big family.
If all the parameters vanish except t^3, t^4 and t^7, then we obtain a solution for which $d^\infty \sim d_3(\alpha:\beta)$, where

$$\alpha = t^7q, \quad \alpha+\beta = t^3q, \quad \alpha\beta = -t^4q^2,$$

where again, q is a nonzero free parameter, so we also have jump deformations to every member of this family.
Similarly, if all the parameters but t^5, t^8, t^{10} and t^1 vanish, and $t^8 = t^1$, the relations are all satisfied. If q is a nonzero parameter, then independently of the value of t^5 we have $d^\infty \sim d_1(\alpha:\beta)$, and

$$\begin{aligned} \alpha+\beta =& -t^8q \\ \alpha\beta =& t^{10}q^2, \end{aligned}$$

so there is a jump deformation from d_1 to every member of the family $d_1(\alpha:\beta)$.
If t^5 and t^8 do not vanish, $t^1 = t^{12} = t^8$ and all the other parameters vanish, then the relations are satisfied and $d^\infty \sim d_2^\sharp$, which gives another jump deformation.
If all the parameters except t^5 vanish, the relations are satisfied, and we get a jump deformation to d_2^*.
Thus finally, we observe that d_1 has jump deformations to every codifferential except d_3^*. This pattern is completely analogous to the three dimensional Lie algebra case, where the corresponding element d_1 has jump deformations to every element except d_2, which is exactly the analog of the element d_3^*, one dimension lower.

7. Description of the Moduli Space

In Figure (2), we give a pictorial representation of the moduli space. The big family $d_3(\lambda:\mu:\nu)$ is represented as a plane, although in reality it is $\mathbb{P}^2/\Sigma_3$. The families $d_1(\lambda:\mu)$, $d_3(\lambda:\mu)$ and the three subfamilies $d_3(\lambda:\mu:0)$, $d_3(\lambda:\lambda:\mu)$ and $d_3(\lambda:\mu:\lambda+\mu)$ are represented by circles, mainly to reflect that the three subfamilies of the big family intersect in more than one point, because they each represent not a single $\mathbb{P}^1$, but several copies of $\mathbb{P}^1$ which are identified under the action of the symmetric group.
In the picture, jump deformations from special points are represented by curly arrows. The jump deformations from the small family $d_3(\lambda:\mu)$ to $d_3(\lambda:\lambda:\mu)$ and the jump deformations from $d_3(\lambda:\mu:\lambda+\mu)$ to $d_1(\lambda:\mu)$ are represented by cylinders. The jump deformations from the family $d_3(\lambda:\mu:0)$ to $d_2^\sharp$ and those from d_1 to the small family are represented by cones. Finally, the jump deformations from d_2^* to the

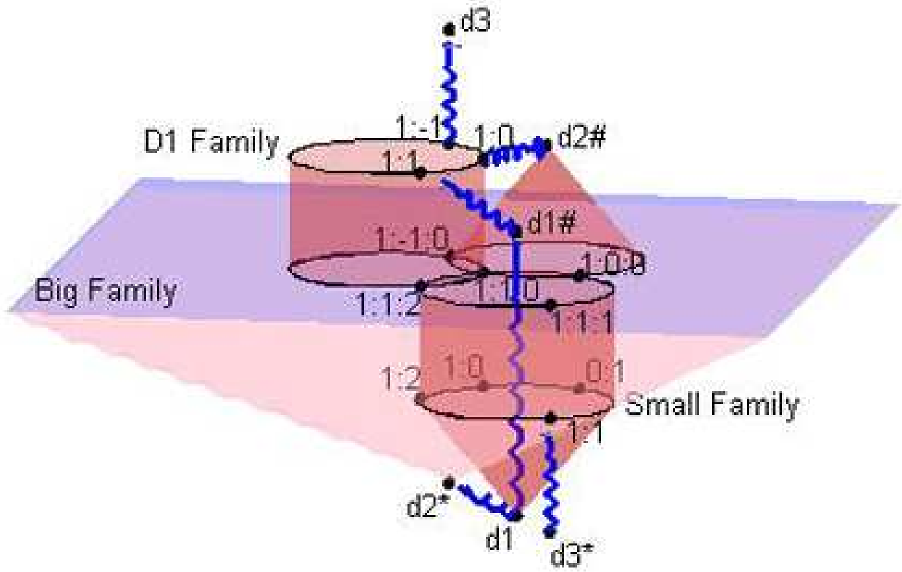


Figure 2. The Moduli Space of 4 dimensional Lie Algebras

big family are represented by an inverted pyramid shape. All jump deformations are either in an upwards or a horizontal direction.
The picture tries to capture the order of precedence of the deformations. For example, in the picture, you can trace a path of jump deformations from d_1 to $d_3(1:0)$ to $d_3(1:1:0)$ to $d_1(1:0)$ to $d_2^\sharp$.

8. Classifying a Particular Lie Algebra

In [1], it was shown that a four dimensional Lie algebra can be classified by computing certain invariants of the Lie algebra. Instead, our approach to classifying a Lie algebra, which we will outline here, used linear algebra.
Suppose that a codifferential d representing a Lie algebra structure has matrix A. Since the rank of the matrix is at most 3, it is easy to compute a new basis for which the matrix has the form $A = \left[\begin{smallmatrix} A' & \delta \\ 0 & 0 \end{smallmatrix}\right]$, where A' is a 3×3 matrix representing a 3 dimensional Lie algebra, and δ is a 3×3 matrix representing a derivation of this Lie algebra.
Next, consider the submatrix A'. If it has rank 3, then the codifferential is equivalent to d_3. Otherwise, we find a new basis in which the submatrix A' has been reduced to one with exactly as many rows as its rank. In fact, by using the classification methods for three dimensional Lie algebras, one can reduce the matrix A' to one of the standard forms.
At this point, the matrix δ representing the derivation on the three dimensional Lie algebra may not represent an outer derivation. However, by replacing the vector e_4 with a vector of the form $e_4' = ae_1 + be_2 + ce_3 + e_4$, one can replace the δ with one representing an outer derivation.
Once this has been accomplished, the classification scheme presented in this paper for determining the point in the moduli space corresponding to an extension of a three dimensional Lie algebra by an outer derivation can be applied. The precise identification scheme depends on which point in the moduli space of three dimensional Lie algebras occurs.
When computing versal deformations of the four dimensional Lie algebras, in most cases, we could identify the appropriate element by following a more simple scheme of solving for a matrix G such that the matrix $GA' = AQ$, where Q is the matrix representing the linear transformation g corresponding to G extended to $\bigwedge^2 V \to$

$\bigwedge^2 V$ and A' is a matrix representing one of the nine types of elements in the moduli space.

However, because our matrices involved many parameters, it was sometimes too difficult for the computer to solve for the values of the parameters for which the A' and A matrices are equivalent. In those cases, we followed the more complicated scheme outlined above. In practice, we found that it was only necessary to follow the steps partially, because after transforming the matrix to eliminate some of the rows, we then were able to apply the simple scheme, and obtain a solution.

9. Conclusions

The computation of the equivalence classes of non-isomorphic Lie algebra structures in a vector space V determines the elements of the moduli space of Lie algebra structures on V, but is only the first step in the classification of these structures. When classifying the algebras, there are different ways of dividing up the structures according to families; therefore, it is desirable to have a rationale for the division. In this paper, we have shown that there is a natural way to divide up the moduli space into families, using cohomology as a guide to the division, and versal deformations as a tool to refine the analysis.

The four dimensional Lie algebras can be decomposed into families, each of which is naturally an orbifold. If one takes into account the information about jump deformations, the division we have given is uniquely determined. The elements of the family which contain a Lie algebra structure d are precisely those Lie algebras which can be obtained as smooth deformations of d, but which are not smooth deformations of any Lie algebra structure d' which is a jump deformation of d. This rule allows us to distinguish between the algebra d_3^* and $d_3(1:1)$, for example. Even though d_3^* has smooth deformations to the family $d_3(\lambda:\mu)$, it also has a jump deformation to $d_3(1:1)$, which has smooth deformations to the same family. Thus $d_3(1:1)$, which has no jump deformations to any element having smooth deformations to the family, is the element which belongs to the family.

According to this system, there is one two-parameter family, two one-parameter families, and six singleton elements, giving rise to a two-dimensional orbifold, two one-dimensional orbifolds, and six one-dimensional orbifolds. The jump deformations provide maps between the families which either are smooth maps of orbifolds (or suborbifolds as in the case of the map $d_3(\lambda:\mu:\lambda+\mu)\to d_1(\lambda:\mu)$), or, in the case of some of the singletons, identify the element with a whole family.

The cohomology of a Lie algebra determines the tangent space to the Lie algebra, but the tangent space does not contain enough information to give a good local description of the moduli space. The relations on the base of the versal deformation determine the manner in which the moduli space contacts the tangent space. In one example here, the tangent space was two dimensional, but deformations were only along two curves. In another case, the tangent space was three dimensional, but the deformations were confined to three planes. It is clear that the cohomology is not sufficient to get an accurate picture of the moduli space. Versal deformations provide important detail that characterizes the moduli space completely.

10. Acknowledgments

The authors would like to thank E. Vinberg for helpful discussions and the Max-Planck-Institut für Mathematik, Bonn for hosting both authors while they were finishing this paper.

References

1. Y. Agaoka, *An algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras*, Linear Algebra and its Applications **345** (2002), 85–118.
2. D. Burde and C. Steinhoff, *Classification of orbit closures of 4-dimensional complex Lie algebras*, Journal of Algebra **214** (1999), 729–739.

3. A. Fialowski, *Deformations of Lie algebras*, Mathematics of the USSR-Sbornik **55** (1986), no. 2, 467–473.
4. A. Fialowski, *An example of formal deformations of Lie algebras*, Proc. NATO Conf. on Deformation Theory of Algebras and Appl., Kluwer 1988, 375–401.
5. A. Fialowski and D. Fuchs, *Construction of miniversal deformations of Lie algebras*, Journal of Functional Analysis (1999), no. 161(1), 76–110.
6. A. Fialowski and M. Penkava, *Deformation theory of infinity algebras*, Journal of Algebra **255** (2002), no. 1, 59–88, math.RT/0101097.
7. ———, *Versal deformations of three dimensional Lie algebras as L_∞ algebras*, Communications in Contemporary Mathematics **7** (2005), no. 2, 145–165, math.RT/0303346.
8. M. Gerstenhaber, *On the deformations of rings and algebras I–IV*, Annals of Mathematics **79** (1964), 59–103; II, Annals of Mathematics **84** (1966), 1–19; III, Annals of Mathematics **88** (1968), 1–34; IV, Annals of Mathematics **99** (1974), 257–276.
9. A.A. Kirillov and Y.A. Neretin, *The variety A_n of n-dimensional Lie algebra structures*, Amer. Math. Soc. Transl. **137** (1987), no. 2, 21–30.
10. B. Komrakov and A. Tchourioumov, *Small dimensional and linear Lie algebras*, International Sophus Lie Centre Press, 2000.
11. A. Nijenhuis and R. Richardson, *Deformations of Lie algebra structures*, Jour. Math. Mech. **17** (1967), 89–105.
12. A.L. Onishik and and E.B. Vinberg (ed.) *Lie Groups and Lie Algebras III.(Structure of Lie groups and Lie algebras)*, Encyclopaedia of Math. Sci., 1994.
13. R.O. Popovich, V.M. Boyko, M.O. Nesterenko, and M.W. Lutfullin, *Realization of real low-dimensional Lie algebras*, Journal of Phys. A Math Gen. **36** (2003), 7337–7360.
14. M. Rhomdani, *Classification of real and complex nilpotent Lie algebras of dimension 7*, Linear and Multilinear Algebra **24** (1989), 167–189.
15. P. Turkowski, *Literature on the structure of low-dimensional nonsemisimple Lie algebras and its applications to cosmology*, Acta Cosmologica **20** (1994), 147–153.

Alice Fialowski, *email*: `fialowsk@cs.elte.hu`
Michael Penkava, *email*: `penkavmr@uwec.edu`