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DEFORMATIONS OF NILPOTENT KAC–MOODY ALGEBRAS

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The main goal of this article is the calculation of the one and two-dimensional cohomology of maximal nilpotent subalgebras of affine Kac–Moody type Lie algebras. This calculation allows us to classify the exterior derivations and deformations of the indicated algebras.

The article consists of two sections: The first section contains basic definitions and statements of the results, while the second one contains the proofs.[1]

The author would like to thank Professor D. Fuchs for stimulating discussions and friendly help.[2]

1. Definitions and the statements of the results

1. Let $A = \|a_{ij}\|$ be an integer $n \times n$ matrix with $a_{11} = \cdots = a_{nn} = 2$ and $a_{ij} \leqq 0$ for $i \neq j$. Suppose that A is symmetrisable, i.e. there exist positive numbers $\rho_1, \ldots, \rho_n$ such that the matrix $\|\rho_i a_{ij}\| = \rho A$ is symmetric. From now on $\rho_1, \ldots, \rho_n$ denote the minimal positive integers with the property above. Define the *Kac–Moody Lie algebra* $\mathfrak{g}^A$ *with the Cartan matrix* A as a complex Lie algebra with the generators $e_1, \ldots, e_n,\ f_1, \ldots, f_n,$ $h_1, \ldots, h_n$ and the relations

$$[e_i, f_j] = \delta_{ij} h_j, \qquad [h_i, h_j] = 0,$$
$$[h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j,$$
$$[\underbrace{e_i, [e_i, \ldots, [e_i}_{-a_{ij}+1}, e_j] \ldots]] = 0, \qquad [\underbrace{f_i, [f_i, \ldots, [f_i}_{-a_{ij}+1}, f_j] \ldots]] = 0 \quad (i \neq j).$$

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[1]For another proof of a part of these results see in [7].

[2]The work was done during my stay in Moscow.

Define in $\mathfrak{g}^A$ a (multi-) gradation by

$$\deg h = (\underbrace{0,\dots,0}_{n}), \quad \deg e_i = (\underbrace{0,\dots,0,\overset{i}{1},0,\dots,0}_{n}),$$
$$\deg f_i = (\underbrace{0,\dots,0,-\overset{i}{1},0,\dots,0}).$$

Here n is called the rank of $\mathfrak{g}^A$.

Suppose that A is nondecomposable, i.e. it can not become of the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ under any simultaneous permutation of rows and columns.

The Weyl group $W = W^A$ of $\mathfrak{g}^A$ is defined as the subgroup of $GL(n, \mathbf{Z})$, generated by the matrices $\sigma_i = E - A_i$ where E is the unity and in A_i the ith row coincides with the ith row of A, while the other rows are zeros. (The elements of W may be considered as transformations of the "weight lattice" $\mathbf{Z}^n$, which grades $\mathfrak{g}^A$.)

Remind some facts about the Kac–Moody Lie algebras (see [1], [2], [3]).

(i) $\mathfrak{g}^A = \mathfrak{n}_+(A) + \mathfrak{h} + \mathfrak{n}_-(A)$, where $\mathfrak{n}_+(A)$ and $\mathfrak{n}_-(A)$ are subalgebras of $\mathfrak{g}^A$, generated by $e_1, \dots, e_n$ and $f_1 \dots, f_n$ respectively, while $\mathfrak{h}$ is n-dimensional (commutative) subalgebra, spanned by $h_1, \dots, h_n$.

(ii) The defining relation system for the generators $e_1, \dots, e_n$ of $\mathfrak{n}_+(A)$ consists of

$$\big[\underbrace{e_i, [e_i, \dots, [e_i}_{-a_{ij}+1}, e_j] \dots]\big] = 0.$$

The similar relations are true for $\mathfrak{n}_-(A)$.

It is natural to divide the Kac–Moody Lie algebras into three classes: algebras with positive definite matrix ρA, algebras with nonnegative definite matrices of rank $n-1$ and the remaining algebras.

(iii) The class of algebras $\mathfrak{g}^A$ with positive definite matrices ρA coincides with the class of simple finite-dimensional complex Lie algebras.

In this paper we restrict ourselves to the so called *affine algebras* of the second type. The nondecomposable matrices corresponding to these algebras are listed in Tables 1 and 2.

The vertices in Tables 1–2 correspond to the rows of A. The ith vertex is joined with the jth one by $a_{ij}a_{ji}$ edges; if $|a_{ij}| > |a_{ji}|$, these edges have an arrow, pointing towards the ith vertex. Numerical marks are the coefficients of linear dependence between the corresponding columns of the Cartan matrix A. Fix for these numbers the notation $\omega_1 \dots \omega_n$.

(iv) Let A be a positive definite Cartan matrix, corresponding to certain Dynkin diagram and $\widetilde{A}$ be the Cartan matrix of the extended Dynkin diagram from Table 1. Then $\mathfrak{g}^{\widetilde{A}}$ is the central extension of the *current algebra* $\mathfrak{g}^A \otimes \mathbf{C}[t, t^{-1}]$.

By this the canonical generators $e_1, \dots, e_n$ of $\mathfrak{g}^{\widetilde{A}}$ correspond to the products $e_1 \otimes 1, \dots, e_{n-1} \otimes 1$, $f \otimes t$, where $e_1, \dots, e_{n-1}$ are canonical generators of $\mathfrak{g}^A$ and f is the root vector of $\mathfrak{g}^A$, corresponding to the negative root of maximal length. Moreover, for $(m_1, \dots, m_n) \neq (0, \dots, 0)$

$$\mathfrak{g}^{\widetilde{A}}_{(m_1,\dots,m_n)} = \mathfrak{g}^A_{(m_1 - m_n\alpha_1, \dots, m_{n-1} - m_n\alpha_{n-1})} \otimes t^{m_n}$$

where $(\alpha_1,\dots,\alpha_{n-1})$ is the weight of f.

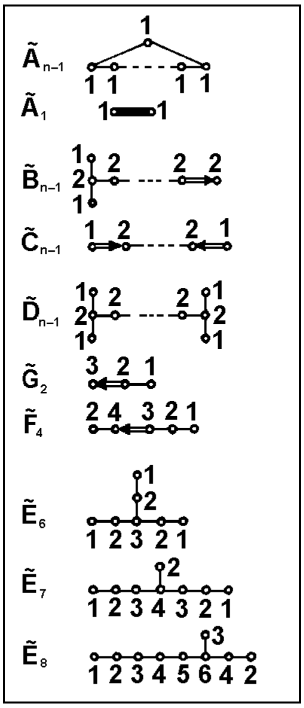


Table 1

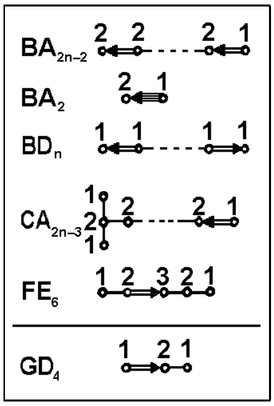


Table 2

We notice also that $\mathfrak{n}_+(\widetilde{A}) = (\mathfrak{n}_+(A)\otimes 1)\oplus\Big(\bigoplus_{m>0}(\mathfrak{g}^A\otimes t^m)\Big)$ and similar is true for $\mathfrak{n}_-(A)$.

Algebras, corresponding to matrices from Table 2 are defined by means of finite order exterior automorphisms of finite-dimensional simple algebras. Namely, if $\varphi:\mathfrak{g}\to\mathfrak{g}$ is such an automorphism and l is its order, then we define $\mathfrak{g}_\varphi$ as the subalgebra $\bigoplus_{\lambda=-\infty}^{\infty}\mathfrak{G}(\lambda)\otimes t^\lambda$ of $\mathfrak{g}\otimes\mathbf{C}[t,t^{-1}]$, where $\mathfrak{g}(\lambda)$ is the root subspace of the automorphism φ, corresponding to the eigenvalue $e^{2\pi i\lambda/l}$.

(v) The algebras from Table 2 are central extensions of the algebras $\mathfrak{g}_\varphi$. Namely, the first 5 cases correspond to two-order automorphisms, while the last one to three-order automorphism.

The homology of $\mathfrak{n}_+(A)$ with trivial coefficients is known [4], [5]. Let

$$Q_A(x_1,\dots,x_n) = -\frac{1}{2}\sum\rho_i a_{ij}x_ix_j + \sum\rho_i x_i.$$

(vi) If $Q_A(m_1,\dots,m_n)\neq 0$, then

$$H_k^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A)) = 0$$

for arbitrary k. If $Q_A(m_1,\dots,m_n)=0$, then there is a unique $k(m_1,\dots,m_n)$, for which

$$H_k^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A))=\begin{cases}\mathbf{C} & \text{for } k=k(m_1,\dots,m_n),\\ 0 & \text{for the others.}\end{cases}$$

For the practical computation of the number $k(m_1,\dots,m_n)$ it is convenient to use the transformations $s_i:\mathbf{Z}^n\to\mathbf{Z}^n$, defined by

$$s_i(m)=\sigma_i(m)+(0,\dots,0,\overset{i}{\rho_i},\dots,0).$$

(The transformations s_i also define an action of W in $\mathbf{Z}^n$.) It is easy to show that $Q_A\circ s_i=Q_A$ and that an arbitrary sequence $(m_1,\dots,m_n)$ with $Q_A(m_1,\dots,m_n)=0$ may be obtained from $(0,\dots,0)$ by means of finite number of transformations s_i. The minimal number of these transformations is $k(m_1,\dots,m_n)$.

In particular,

$$H_0(\mathfrak{n}_+(A))=H_0^{(0,\dots,0)}(\mathfrak{n}_+(A))=\mathbf{C},$$
$$H_1(\mathfrak{n}_+(A))=H_1^{(1,0,\dots,0)}(\mathfrak{n}_+(A))\oplus\dots\oplus H^{(0,\dots,0,1)}(\mathfrak{n}_+(A))=\mathbf{C}^n.$$

2. Let A be a Cartan matrix from Tables 1, 2. The main result of this paper is the computation of one- and two-dimensional cohomologies of $\mathfrak{n}_+(A)$ with coefficients in the adjoint representation. Remind that the computation of one-dimensional cohomology is equivalent to the classification of exterior derivations, and it is that language, in which we formulate here the result. The calculation of two-dimensional cohomology allows us to classify the deformations of the considered algebras.

Theorem 1. *The next derivations form a basis in the space of exterior derivations of* $\mathfrak{n}_+(A)$*:*

$$\overline{h}_i: g\to[h_i,g],\quad i=1,\dots,n-1;$$
$$\tau_i: t^{il+1}\frac{d}{dt},\quad i=0,1,2,\dots\ .$$

Here l and t have the same sense as in (iv) *and* (v) *of subsection 1.*

We describe now some concrete deformations of $\mathfrak{n}_+(A)$.

1°. Let $\alpha\in H^1(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$, $\beta\in H^1(\mathfrak{n}_+(A))$. The element α corresponds to the right extension

$$0\to\mathfrak{n}_+(A)\to\tilde{\mathfrak{n}}_+(A)\to\mathbf{C}\to 0$$

(the elements of $H^1(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$ may be interpreted not only as exterior derivations, but also as right extensions – see [5]), β to a functional $\varphi:\mathfrak{n}_+(A)\to\mathbf{C}$. For $t\in\mathbf{C}$ denote η_t the embedding $\mathfrak{n}_+(A)\to\tilde{\mathfrak{n}}_+(A)=\mathfrak{n}_+(A)\oplus\mathbf{C}$ defined by $\eta_t(g)=(g,t\varphi(g))$. It may be easily checked that $\eta_t(\mathfrak{n}_+(A))$ is a subalgebra of $\tilde{\mathfrak{n}}_+(A)$, that this subalgebra is connected with $\mathfrak{n}_+(A)$ by a natural linear isomorphism, and that for $t=0$ this isomorphism is compatible with the bracket operation. Thus we have a deformation of $\mathfrak{n}_+(A)$. The corresponding infinitesimal deformation is evidently the product

$$\alpha\beta\in H^2(\mathfrak{n}_+(A);\mathfrak{n}_+(A)).$$

(By all means, this construction may be applied to an arbitrary Lie algebra.)

2°. Let $1 \leqq i \leqq n$. The algebra $\mathfrak{n}_+(A)$ deforms inside $\mathfrak{g}^A$. The deformed algebra is spanned by the spaces $\mathfrak{g}^A_{(m_1,\dots,m_n)}$ with $(m_1,\dots,m_n) \neq (0,\dots,0,\overset{(i)}{1},\dots,0)$ and by the vector $e_i + tf_i$, where t is a parameter. (Informally speaking, e_i deforms into $e_i + tf_i$, while the other additive generators of $\mathfrak{n}_+(A)$ do not change.)

The number of such deformations is equal to the rank of $\mathfrak{g}^A$.

3°. Let $1 \leqq i,\ j \leqq n$; consider the entry $a_{ij} = -1$ and if $a_{ij} = a_{ji}$, then $i < j$. The algebra $\mathfrak{n}_+(A)$ deforms again inside $\mathfrak{g}^A$. The deformed algebra is generated by the spaces $\mathfrak{g}^A_{(m_1,\dots,m_n)}$ with

$$(m_1,\dots,m_n) \neq (0,\dots,0,\overset{(i)}{1},0,\dots,0),\ (0,\dots,0,\overset{(i)}{1},0,\dots,0,\overset{(j)}{1},0,\dots,0)$$

and the vectors $e_i + tf_j$ and $[e_i,e_j] - th_j$. (Informally speaking, e_i and $[e_i,e_j]$ deform into $e_i + tf_j$ and $[e_i,e_j] - th_j$, while the other additive generators of $\mathfrak{n}_+(A)$ are not deformed.)

The number of this type deformations is equal to the number of nonzero pairs (a_{ij},a_{ji}) with $i \neq j$; this number we denote below by p.

Remark that the equality $a_{ij} = -1$ is necessary for the verification of the fact that the deformed algebras are closed under the bracket and that with the only exception of the case $\widetilde{A}_1$, at least one of two nontrivial nondiagonal entries of the Cartan matrix a_{ij}, a_{ji} is equal to -1. This specific property of $\widetilde{A}_1$ compels us to consider the case $\mathfrak{n}_+(\widetilde{A}_1)$ separately.

Theorem 2. *Suppose that $A \neq \widetilde{A}_1$. Then*

(i) *All the homogeneous infinitesimal deformations of $\mathfrak{n}_+(A)$ may be extended to its real deformations.*

(ii) *The space of infinitesimal deformations $H^2(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$ is spanned by deformations, corresponding to the above types 1°, 2°, 3°. In other words, the mapping*

$$\psi : \left[H^1(\mathfrak{n}_+(A);\mathfrak{n}_-(A)) \otimes H^1(\mathfrak{n}_+(A))\right] \oplus \mathbf{C}^n \oplus \mathbf{C}^p \to H^2(\mathfrak{n}_+(A);\mathfrak{n}_+(A))$$

defined by the infinitesimal deformations listed above is epimorphism.

(iii) *The kernel of the mapping ψ is contained in*

$$H^1(\mathfrak{n}_+(A);\mathfrak{n}_+(A)) \otimes H^1(\mathfrak{n}_+(A))$$

and its dimension is n. It is spanned by the elements $\kappa_1,\dots,\kappa_n$defined as follows. Let $1 \leqq i \leqq n$. Choose the numbers $\beta_1,\dots,\beta_{n-1}$ so that $\sum\limits_1^{n-1} \beta_j a_{kj} = 1$ for $k \neq i$ (such numbers can be found, because the rank of the Cartan matrix with one column removed equals to $n-1$). Then

$$\kappa_i = \overline{h}_i \otimes \overline{e}_i, \qquad i = 1,\dots,n-1,$$

$$\kappa_n = \Big(\sum_1^{n-1} \beta_j \overline{h}_j\Big) \otimes \overline{e}_i,$$

where $\overline{e}_i$ is the class of the cocycle from $\mathbf{C}^1(\mathfrak{n}_+(A))$, assigning 1 to e_i and 0 to other e_k's, while the $\overline{h} - s$ were introduced in Theorem 1.

Now turn to the case $A = \widetilde{A}_1$. In this case the Cartan matrix is $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, and this excludes the possibility of applying the construction 3°. Mention also that it is not true for this case that all infinitesimal deformations may be extended to real deformations.

Theorem 3. (i) *Infinitesimal deformations, corresponding to deformations of type* 1°, 2° *span in* $H^2\big(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1)\big)$ *a codimension* 2 *subspace. The complementary subspace is spanned by elements from* $H^2_{(-1,-2)}$ *and* $H^2_{(-2,-1)}$ *respectively. These elements can not be extended to the deformation of* $\mathfrak{n}_+(\widetilde{A}_1)$. (Cocycles representing these two classes are given in subsection 2.2).

(ii) *The kernel of the mapping*

$$\big[H^1(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1)) \otimes H^1(\mathfrak{n}_+(\widetilde{A}_1))\big] \oplus \mathbf{C}^2 \to H^2(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1))$$

may be described just as the kernel of ψ *in part* (iii) *of Theorem 2.*

2. Proofs

1. Let $\mathfrak{g} = \bigoplus\limits_{i>0} \mathfrak{g}_i$ be a nilpotent graded Lie algebra and $B = \oplus B_j$ be a graded $\mathfrak{g}$-module. The space $C_k^{(m)}(\mathfrak{g};B)$ is spanned by "monomials", i.e. by the chains

$$g_1 \wedge \cdots \wedge g_k \otimes b, \;\; \text{where } g_s \in \mathfrak{g}_{i_s}, \quad b \in B_j, \; i_1 + \ldots i_k + j = m.$$

Denote by $F_pC_k^{(m)}(\mathfrak{g};B)$ the subspace of $C_k^{(m)}(\mathfrak{g};B)$, generated by monomials with $i_1 + \cdots + i_k \leqq p$. Evidently, $\{F_p\}$ is a decreasing filtration in $C_*^{(m)}(\mathfrak{g};B)$. The spectral sequence corresponding to this filtration we will call *Feigin–Fuchs spectral seguence* and denote it by $\mathcal{E}(\mathfrak{g},B,m)$. Here $E^0_{p,q} = C^{(p)}_{p+q}(\mathfrak{g};B_{m-p})$, where B_{m-p} is considered as trivial $\mathfrak{g}$-module and $d^0_{p,q}$ is the differential

$$d_{p+q} : C^{(p)}_{p+q}(\mathfrak{g};B_{m-p}) \to C^{(p)}_{p+q-1}(\mathfrak{g};B_{m-p})$$

hence

$$E^1_{p,q} = H^{(p)}_{p+q}(\mathfrak{g};B_{m-p}) = H^{(p)}_{p+q}(\mathfrak{g}) \otimes B_{m-p}.$$

For the algebra L_1 of polynomial vector fields on the line with trivial 1-jets in the point 0 this spectral sequence was considered in [6]. In the cases interesting for us the algebra $\mathfrak{g}$ has multigradation $\mathfrak{g} = \bigoplus\limits_{(i_1,\ldots,i_k)>(0,\ldots,0)} \mathfrak{g}_{(i_1,\ldots,i_k)}$. In this case the spectral sequence $\mathcal{E}(\mathfrak{g},B,m)$ decomposes into the sum of spectral sequences $\mathcal{E}(\mathfrak{g},B,m_1,\ldots,m_k)$, $m_1 + \cdots + m_k = m$. The initial term of the last spectral sequence is given by the formula

$$E^1_{p,q} = \bigoplus_{p_1+\cdots+p_k=p} H^{(p_1,\ldots,p_k)}_{p+q}(\mathfrak{g}) \otimes B_{m_1-p_1,\ldots,m_k-p_k}.$$

We apply the above spectral sequence to the computation of the one- and two-dimensional homology of the algebra $\mathfrak{n}_+(A)$ with coefficients in the coadjoint representation $\mathfrak{n}_+(A)'$.

(This is equivalent to the computation of the cohomology of $\mathfrak{n}_+(A)$ with coefficients in the adjoint representation.) For each of the matrices from Tables 1, 2 the terms and differentials of the spectral sequence $\mathcal{E}(\mathfrak{n}_+(A), \mathfrak{n}_+(A)', m)$ may be explicitly determined, and this leads to the calculation of the indicated homology. All computations are similar, and we shall give details only for the cases $\widetilde{A}_{n-1}$ and BA_2.

2. Let us begin with $\widetilde{A}_1$. There is a convenient explicit description of the quotient algebra of $\mathfrak{g}^{\widetilde{A}_1}$ by its (one-dimensional) center. Namely, it contains an additive basis ε_i $(i \in \mathbf{Z})$ such that

$$[\varepsilon_i, \varepsilon_j] = \alpha_{ij}\varepsilon_{i+j}, \quad \text{where } \alpha_{ij} \begin{cases} = -1, 0, 1, \\ \equiv (j-i) \bmod 3. \end{cases}$$

(In this notation $\varepsilon_1, \varepsilon_2, \varepsilon_{-1}, \varepsilon_{-2}$ correspond to e_1, e_2, f_1, f_2, defined in Section 1.) (Bi-)-gradation in this basis is given by

$$\deg \varepsilon_{3m} = (m, m), \quad \deg \varepsilon_{3m-1} = (m, m-1), \quad \deg \varepsilon_{3m+1} = (m, m+1).$$

The subspace $\mathfrak{n}_+(\widetilde{A}_1)$ of $\mathfrak{g}^{\widetilde{A}_1}$ is spanned by ε_i, where $i > 0$.

According to (vi) in Section 1, for $k > 0$

$$H_k(\mathfrak{n}_+(\widetilde{A}_1)) = H_k^{((k(k-1))/2,(k(k+1))/2)} \oplus H_k^{((k(k+1))/2,(k(k-1))/2)} = \mathbf{C} \oplus \mathbf{C}$$

(see Fig. 1) moreover, nontrivial elements of the spaces

$$H_k^{((k(k-1))/2,(k(k+1))/2)}, H_k^{((k(k+1))/2,(k(k-1))/2)}$$

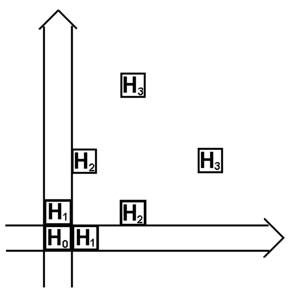


Fig. 1

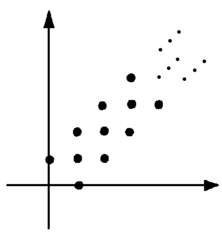


Fig. 2

are represented by cycles $\varepsilon_1 \wedge \varepsilon_4 \wedge \cdots \wedge \varepsilon_{3k-2}$, $\varepsilon_2 \wedge \varepsilon_5 \wedge \cdots \wedge \varepsilon_{3k-1}$ (see [5]). Since

$$\dim(\mathfrak{n}_+(\widetilde{A}_i))_{(m_1,m_2)} = \begin{cases} 1 & \text{if } |m_2 - m_1| \leqq 1, \quad m_2 + m_1 > 0, \\ 0 & \text{in all other cases} \end{cases}$$

(see Fig. 2), in the spectral sequence

$$\mathcal{E}(m_1, m_2) = \mathcal{E}(\mathfrak{n}_+(\widetilde{A}_1), \mathfrak{n}_+(\widetilde{A}_1)', m_1, m_2)$$

$$\dim E^1_k = \begin{cases} 2 & \text{if } k=1,\ m_1=m_2 \leqq 0, \\ 1 & \text{if } k-1 \leqq |m_2-m_1| \leqq k+1, \quad m_1+m_2<k^2, \\ 0 & \text{in all other cases.} \end{cases}$$

(See Fig. 3; the circles and points show the degrees of the homology with trivial coefficients and the degrees of the nontrivial spaces E^1_k, respectively.)

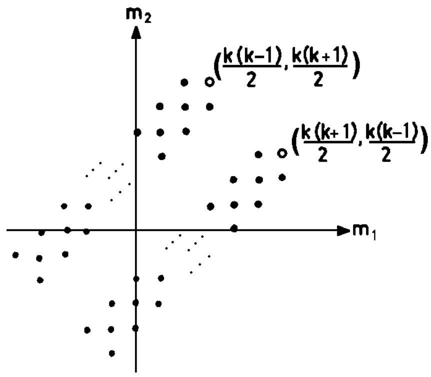


Fig. 3

So, the term E^1 of the spectral sequence $\mathcal{E}(m_1,m_2)$ is constructed in the following way. Let $l=|m_2-m_1|$ and $m=\min(m_1,m_2)$. If $l>0$, then the dimensions of the spaces E^1_k are given by the table

$k=$	$\dots$	$l-2$	$l-1$	l	$l+1$	$l+2$	$\dots$	
	$\dots$	0	1	1	1	0	$\dots$	for $m \leqq \frac{l^2-3l}{2}$,
	$\dots$	0	0	1	1	0	$\dots$	for $\frac{l^2-3l}{2} < m < \frac{l^2-l}{2}$,
	$\dots$	0	0	0	1	0	$\dots$	for $\frac{l^2-l}{2} \leqq m \leqq \frac{l^2+l}{2}$,
	$\dots$	0	0	0	0	0	$\dots$	for $\frac{l^2+l}{2} < m$,

and if $l=0$, then by the table

$k=$	0	1	2	$\dots$	
	1	2	0	$\dots$	for $m<0$
	0	2	0	$\dots$	for $m=0$
	0	0	0	$\dots$	for $m>0$.

Lemma. *The non-trivial differentials d^1_k are the following ones:*

$$d^1_l : E^1_l \to E^1_{l-1}, \quad \text{if } l \neq 0,\ m \leqq \frac{l^2-3l}{2},$$
$$d^1_1 : E^1_1 \to E^1_0, \quad \text{if } l=0,\ m<0;$$

the differentials d_k^r with $r > 1$ are all trivial.

From this lemma it follows

Proposition.

$$H_0(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1)') = 0;$$

$$\dim H_1^{(m_1,m_2)}(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_1)') = \begin{cases} 2 & \text{if } m_1 = m_2 = 0, \\ 1 & \text{if } m_1 = m_2 < 0, \\ 0 & \text{in the other cases;} \end{cases}$$

if $k > 1$, then

$$\dim H_1^{(m_1,m_2)})(\mathfrak{n}_+(\widetilde{A}_1);\mathfrak{n}_+(\widetilde{A}_i)') =$$

$$= \begin{cases} 1 & \text{if } |m_1 - m_2| = k-1,\ m_1 + m_2 < k^2 - 1 \\ & \text{and if } |m_1 - m_2| = k,\ (k-1)^2 < m_1 + m_2 \leqq k^2 - 2, \\ 0 & \text{in the other cases.} \end{cases}$$

(See Fig. 4, on which there are shown the weights of one- and two-dimensional homologies.)

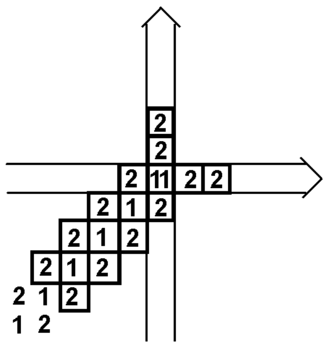


Fig. 4

Lemma may be proved by performing a straight but not particularly short calculation. Since we are interested only in homology of dimension one and two, we give the proof here only for the cases $k \leqq 2$. We have to show that the differentials

(i) d_1^1 for $m_2 = m_1 < 0$,

(ii) d_1^1 for $|m_2 - m_1| = 1, \min(m_1, m_2) \leqq -1$,

(iii) d_2^1 for $m_2 - m_1| = 2, \min(m_1, m_2) \leqq -1$,

(iv) d_3^1 for $|m_2 - m_1| = 3, \min(m_1, m_2) \leqq 0$

are non-trivial and

(v) d_3^1 for $m_1 = 2, m_2 = 0$ and $m_1 = 0, m_2 = 2$

is trivial. Since the roles of m_1 and m_2 are symmetric, we may consider only the case $m_1 \leqq m_2$. The differential d_k^1 in the spectral sequence $\mathcal{E}(m_1,m_2)$ is non-trivial if there exists a chain

$$c \in C_k^{(m_1,m_2)}(\mathfrak{n}_+(\widetilde{A}_1),\mathfrak{n}_+(\widetilde{A}_1)') \text{ such that}$$
$$c = \varepsilon_1 \wedge \dots \wedge \varepsilon_{3k-2} \otimes \varepsilon'_1 + \dots$$
$$\partial c = \mu\varepsilon_1 \wedge \dots \wedge \varepsilon_{3k-5} \otimes \varepsilon'_j + \dots$$

where $\mu \neq 0$ and dots in the general case stand for terms of smaller filtration. We find such chains for the cases (i)–(iv), putting $m = -m_2$.

(i) $$c = \varepsilon_1 \otimes \varepsilon'_{3m+1};\ \partial c = \varepsilon'_{3m},$$

(ii) $$c = \varepsilon_1 \otimes \varepsilon'_{3m};\ \partial c = -\varepsilon'_{3m-1},$$

(iii) $$c = \varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon'_{3m} - \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon'_{3m-1};\ \partial c = 2\varepsilon_1 \otimes \varepsilon'_{3m-4},$$

(iv) $$c = \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon'_{3m} - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 \otimes \varepsilon'_{3m-1} - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6 \wedge \otimes \varepsilon'_{3m-1} + \\ + \frac{3}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 \otimes \varepsilon'_{3m-4};\ \partial c = -\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon'_{3m-7}.$$

The differential d_k^1 is trivial, if there is a chain c of the above form, for which $\partial c = 0$. For the case (v) such a chain is the following:

(v) $$c = \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon'_{10} + \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 \otimes \varepsilon'_9 + \frac{1}{2}\varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6 \otimes \varepsilon'_9 - \\ - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_6 \otimes \varepsilon'_8 - \varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_4 \otimes \varepsilon'_6 - \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_4 \otimes \varepsilon'_5.$$

Now we describe cycles, representing bases in $H_k(\mathfrak{n}_+(A_1);\mathfrak{n}_+(A_1)')$ for $k = 1,2$.
In $C_1^{(0,0)}$: $\varepsilon_1 \otimes \varepsilon'_1$, $\varepsilon_2 \otimes \varepsilon'_2$.
In $C_1^{(m,m)}$, $m < 0$: $\varepsilon_1 \otimes \varepsilon'_{-3m+1} + \varepsilon_2 \otimes \varepsilon'_{-3m+2}$.
In $C_2^{(0,2)}$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon'_3 - \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon'_2$.
In $C_2^{(1,2)}$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon'_1$.
In $C_2^{(m,m+1)}$, $m \leqq 0$: $\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon'_{-3m+4} + \varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon'_{-3m+3} + \varepsilon_1 \wedge \varepsilon_2 \otimes \varepsilon'_{-3m+2}$.
Cycles in $C_2^{(2,0)}, C_2^{(m+1,m)}$ are given similarly, by substituting $\varepsilon_1 \leftrightarrow \varepsilon_2$, $\varepsilon_4 \leftrightarrow \varepsilon_5, \dots$.

Since $\dim H^k_{(m_1,m_2)} = \dim H_k^{(-m_1,-m_2)}$, the cohomology needed for us is completely computed. It is easy to see that the above result agrees with the corresponding parts of Theorems 1, 3.

Cocycles, representing basis elements of the cohomology spaces are indicated in the next table.

weight	cocycle	
$(0,-2)$	$(\varepsilon_1,\varepsilon_{3j} \mapsto \varepsilon_{3j-1}, (\varepsilon_1,\varepsilon_{3j+1}) \mapsto -\varepsilon_{3j}$ the rest $\mapsto 0$.	for $j>0$,
$(-2,0)$	$(\varepsilon_2,\varepsilon_{3j}) \mapsto \varepsilon_{3j-2}$, $(\varepsilon_2,\varepsilon_{3j+2}) \mapsto -\varepsilon_{3j}$ the rest $\mapsto 0$.	for $j>0$,
$(m,m-1)$ $m \geqq 0$	$(\varepsilon_1,\varepsilon_j) \mapsto j\varepsilon_{j+3m}$ the rest $\mapsto 0$.	for $j \neq 1$,
$(m-1,m)$ $m \geqq 0$	$(\varepsilon_2,\varepsilon_j) \mapsto j\varepsilon_{j+3m}$ the rest $\mapsto 0$.	for $j \neq 1$,
$(-1,-2)$	$(\varepsilon_1,\varepsilon_4) \mapsto 9\varepsilon_1$, $(\varepsilon_1,\varepsilon_j) \mapsto j\varepsilon_{j-3}$ $(\varepsilon_3,\varepsilon_{3j}) \mapsto 2\varepsilon_{3j-1}$, $(\varepsilon_3,\varepsilon_{3j-2}) \mapsto -2\varepsilon_{3j-3}$ $(\varepsilon_4,\varepsilon_{3j-1}) \mapsto 5\varepsilon_{3j-1}$, $(\varepsilon_4,\varepsilon_{3j+4}) \mapsto -5\varepsilon_{3j+4}$ the rest $\mapsto 0$.	for $j \geqq 5$, for $j \geqq 2$, for $j \geqq 1$,
$(-2,-1)$	$(\varepsilon_2,\varepsilon_5) \mapsto 9\varepsilon_2$, $(\varepsilon_2,\varepsilon_j) \mapsto j\varepsilon_{j-3}$ $(\varepsilon_3,\varepsilon_{3j}) \mapsto \varepsilon_{3j-2}$, $(\varepsilon_3,\varepsilon_{3j+2}) \mapsto -\varepsilon_{3j}$ $(\varepsilon_5,\varepsilon_{3j-2}) \mapsto 4\varepsilon_{3j-2}$, $(\varepsilon_5,\varepsilon_{3j+5}) \mapsto -4\varepsilon_{3j+5}$ the rest $\mapsto 0$.	for $j=4,6,7,\dots$, for $j \geqq 1$, for $j \geqq 1$,

We can easily verify that the indicated cochains are really cocycles and they do not vanish on the above cycles.

It remained to show that infinitesimal deformations, determined by two-dimensional cocycles of weight $(0,-2)$, $(-2,0)$ and $(m+1,m)$, $(m,m+1)$ with $m \geqq -1$ can be extended to real deformations, while infinitesimal deformations of weight $(-1,-2)$, $(-2,-1)$ can not. The extensions in question are explicitly given in Section 1. On the other hand, the cocycles of weight $(-1,-2)$, $(-2,-1)$ have nontrivial squares; for instance the first of them takes the value 135 at the cycle

$$\varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_7 \otimes \varepsilon_4' + \frac{1}{2}(\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_7 + \varepsilon_1 \wedge \varepsilon_4 \wedge \varepsilon_6) \otimes \varepsilon_3' - \frac{1}{2}\varepsilon_1 \wedge \varepsilon_3 \wedge \varepsilon_6 \otimes \varepsilon_2'.$$

3. Let us now consider the case BA_2. The corresponding Cartan matrix is $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. The quotient algebra of $\mathfrak{g}^{BA_2}$ by its center has explicit description. Namely, it contains an additive basis ε_i $(i \in \mathbf{Z})$ with $[\varepsilon_i,\varepsilon_j] = \alpha_{ij}\varepsilon_{i+j}$, where α_{ij} depends only on i, j mod 8, $\alpha_{i,j} + \alpha_{i',j'} = 0$ if $i+i'$ and $j+j'$ are multiples of 8, and for $0 \leqq i$, $j \leqq 7$ it is given in the following table:

i mod 8 \ j mod 8	1	2	3	4	5	6	7
0	1	−2	−1	0	1	2	−1
1		1	−1	3	−2	0	1
2			0	0	1	−1	
3				−3	−1		

Gradation is given by formulas

$$\deg \varepsilon_{8m} = (2m, 4m), \quad \deg \varepsilon_{8m+1} = (2m, 4m+1), \quad \deg \varepsilon_{8m+2} = (2m+1, 4m),$$
$$\deg \varepsilon_{8m+3} = (2m+1, 4m+1), \quad \deg \varepsilon_{8m+4} = (2m+1, 4m+2),$$
$$\deg \varepsilon_{8m+5} = (2m+1, 4m+3)$$
$$\deg \varepsilon_{8m+6} = (2m+1, 4m+4), \quad \deg \varepsilon_{8m+7} = (2m+2, 4m+3).$$

The subalgebra $\mathfrak{n}_+(BA_2)$ is spanned by ε_i; with $i > 0$.

By (vi) from Section 1 for $k > 0$

$$H_{2k-1}(\mathfrak{n}_+(BA_2)) = H_{2k-1}^{((3k^2-k)/2, 3k^2-4k+1)} \oplus H_{2k-1}^{((3k^2-5k+2)/2, 3k^2-2k)} = \mathbf{C} \oplus \mathbf{C},$$

$$H_{2k}(\mathfrak{n}_+(BA_2)) = H_{2k}^{((3k^2+k)/2, 3k^2-2k)} \oplus H_{2k}^{((3k^2-k)/2, 3k^2+2k)} = \mathbf{C} + \mathbf{C}$$

(see Fig. 5) and nontrivial elements of the homology in question are represented with the cycles

$$(\varepsilon_2 \wedge \varepsilon_{10} \wedge \cdots \wedge \varepsilon_{8k-6}) \wedge (\varepsilon_3 \wedge \varepsilon_7 \wedge \cdots \wedge \varepsilon_{4k-5}),$$
$$(\varepsilon_6 \wedge \varepsilon_{14} \wedge \cdots \wedge \varepsilon_{8k-10}) \wedge (\varepsilon_1 \wedge \varepsilon_5 \wedge \cdots \wedge \varepsilon_{4k-3}),$$
$$(\varepsilon_2 \wedge \varepsilon_{10} \wedge \cdots \wedge \varepsilon_{8k-6}) \wedge (\varepsilon_3 \wedge \varepsilon_7 \wedge \cdots \wedge \varepsilon_{4k-1}),$$
$$(\varepsilon_6 \wedge \varepsilon_{14} \wedge \cdots \wedge \varepsilon_{8k-2}) \wedge (\varepsilon_1 \wedge \varepsilon_5 \wedge \cdots \wedge \varepsilon_{4k-3}).$$

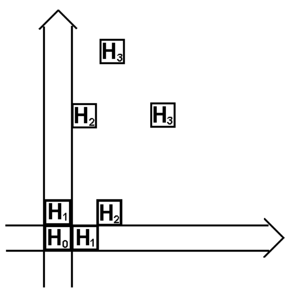


Fig. 5

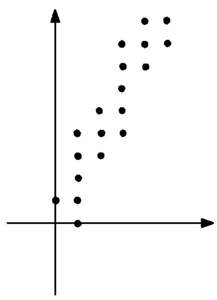


Fig. 6

The dimensions of the spaces $\mathfrak{n}_+(BA_2)_{(m_1,m_2)}$ equal to 0 and 1; the points (m_1, m_2)

corresponding to spaces of dimension 1 are shown on Fig. 6.

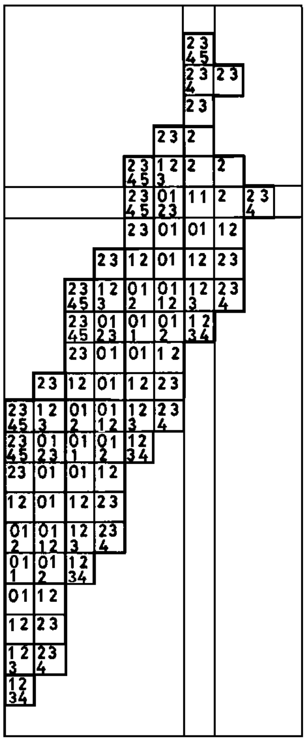


Fig. 7

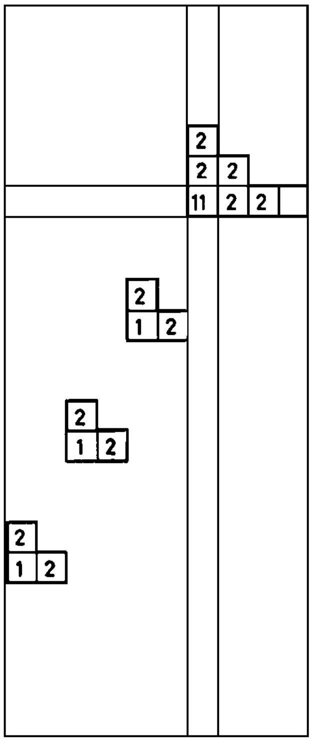


Fig. 8

In this way we can determine the dimensions of the spaces, forming the initial terms of the spectral sequences $\mathcal{E}_{(m_1,m_2)} = \mathcal{E}(\mathfrak{n}_+(BA_2), \mathfrak{n}_+(BA_2)', m_1, m_2)$. We restrict ourselves to (m_1, m_2) such that the space $\bigoplus_{k=0}^{2} E_k^1$ is nontrivial. These (m_1, m_2) are represented by small cells on Fig. 7. On this figure the cell (m_1, m_2) contains as many k's as the dimension of E_k^1 (for instance, in the spectral sequence $\mathcal{E}(-1,-3)$ the dimensions of E_k^1 are $1, 2, 1, 0, 0, \ldots$). We remark that the left half plane on Fig. 7 is periodic with period 2 on the abscissa axis and with period 4 on the ordinate axis. The action of the differentials in these spectral sequences may be calculated in the same way as in subsection 2.2. The result of the computations is shown on Fig. 8: the number of the 1's and 2's in the cell (m_1, m_2) equals to the dimension of $H_1^{(m_1,m_2)}$ and $H_2^{(m_1,m_2)}$, respectively.

Now we describe the cycles, which represent the basis in $H_k(BA_2, BA_2')$, $k = 1, 2$.

In $C_1^{(0,0)}$: $\varepsilon_1 \otimes \varepsilon_1'$, $\varepsilon_2 \otimes \varepsilon_2'$.

In $C_1^{(2m,4m)}$, $m < 0$: $2\varepsilon_1 \otimes \varepsilon'_{-8m+1} + \varepsilon_2 \otimes \varepsilon'_{-8m+2}$.

In $C_2^{(0,2)}$: $\varepsilon_1 \wedge \varepsilon_6 \otimes \varepsilon_5' + \frac{2}{3}\varepsilon_1 \wedge \varepsilon_5 \otimes \varepsilon_4' + \frac{2}{9}\varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon_3' - \frac{2}{9}\varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon_2'$.

In $C_2^{(1,1)}$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon_2'$.

In $C_2^{(2,0)}$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon'_1$.
In $C_2^{(2m,4m+1)}$, $m \leqq 0$: $\varepsilon_1 \wedge \varepsilon_6 \otimes \varepsilon'_{-8m+6} - \varepsilon_1 \wedge \varepsilon_5 \otimes \varepsilon'_{-8m+5} + \varepsilon_1 \wedge \varepsilon_4 \otimes \varepsilon'_{-8m+4} -$

$$-\varepsilon_1 \wedge \varepsilon_3 \otimes \varepsilon'_{-8m+3} + \varepsilon_1 \wedge \varepsilon_2 \otimes \varepsilon'_{-8m+2}.$$

In $C_2^{(2m+1,4m)}$, $m \leqq 0$: $\varepsilon_2 \wedge \varepsilon_3 \otimes \varepsilon'_{-8m+3} - \varepsilon_2 \wedge \varepsilon_1 \otimes \varepsilon'_{-8m+1}$.

So, the cohomology needed for us is computed. It is easy to see that the above result agrees with Theorems 1, 2.

Cocycles, representing basis elements of the cohomology spaces are indicated in the next table.

weight	cocycle
$(0,-2)$	$(\varepsilon_1,\varepsilon_{8j} \mapsto -\varepsilon_{8j-1}$, $(\varepsilon_1,\varepsilon_{8j+1}) \mapsto -\varepsilon_{8j}$ $(j \geqq 0)$, $\left.\begin{matrix}(\varepsilon_1,\varepsilon_{8j+3}) \mapsto -2\varepsilon_{8j+2},\ (\varepsilon_1,\varepsilon_{8j+4}) \mapsto 3\varepsilon_{8j+3} \\ (\varepsilon_1,\varepsilon_{8j+5}) \mapsto -\varepsilon_{8j+4},\ (\varepsilon_1,\varepsilon_{8j+6}) \mapsto \varepsilon_{8j+5}\end{matrix}\right\\} j \geqq 0$ the rest $\mapsto 0$.
$(-1,-1)$	$\left.\begin{matrix}(\varepsilon_1,\varepsilon_{8j-1}) \mapsto \varepsilon_{8j-3},\ (\varepsilon_1,\varepsilon_{8j}) \mapsto -2\varepsilon_{8j-2} \\ (\varepsilon_1,\varepsilon_{8j+2}) \mapsto \varepsilon_{8j},\ (\varepsilon_1,\varepsilon_{8j+3}) \mapsto -\varepsilon_{8j+1} \\ (\varepsilon_3,\varepsilon_{j+1}) \mapsto \varepsilon_{8j+1}\end{matrix}\right\\} j \geqq 1$ $\left.\begin{matrix}(\varepsilon_3,\varepsilon_{8j+2}) \mapsto -2\varepsilon_{8j+2},\ (\varepsilon_3,\varepsilon_{8j+5}) \mapsto \varepsilon_{8j+5} \\ (\varepsilon_3,\varepsilon_{8j+6}) \mapsto 2\varepsilon_{8j+6},\ (\varepsilon_3,\varepsilon_{8j+7}) \mapsto -\varepsilon_{8j+7}\end{matrix}\right\\} j \geqq 0$ $(\varepsilon_1,\varepsilon_3) \mapsto -2\varepsilon_1$, the rest $\mapsto 0$.
$(-2,0)$	$(\varepsilon_2,\varepsilon_{8j}) \mapsto 2\varepsilon_{8j-2}$, $(\varepsilon_2,\varepsilon_{8j+2}) \mapsto -\varepsilon_{8j}$ $(j \geqq 1)$ $(\varepsilon_2,\varepsilon_{8j+3}) \mapsto \varepsilon_{8j+1}$, $(\varepsilon_2,\varepsilon_{8j+7}) \mapsto -\varepsilon_{8j+5}$ $(j \geqq 0)$ the rest $\mapsto 0$.
$(2m,4m-1)$ $m \geqq 0$	$(\varepsilon_1,\varepsilon_j) \mapsto j\varepsilon_{j+8m}$ for $j \neq 1$, the rest $\mapsto 0$.
$(2m-1,4m)$ $m \geqq 0$	$(\varepsilon_2,\varepsilon_j) \mapsto j\varepsilon_{j+8m}$ for $j \neq 2$, the rest $\mapsto 0$.

4. Now consider the case $\widetilde{A}_{n-1}$ with $n \geqq 3$. The case $n = 3$ is somewhat different from the general case (the main difference, from our point of view, is in the structure of the three-dimensional homology with trivial coefficients). Nevertheless, the final formula is the same, and the differences in the proofs are not essential. Therefore from now on we shall ignore the specific case $n = 3$, nondirectly assuming that $n \geqq 4$.

The Cartan matrix of $\mathfrak{g}^{\widetilde{A}_{n-1}}$ is:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}.$$

By (vi) in Section 1

$$\dim H_*^{(m_1,\dots,m_n)}(\mathfrak{n}_+(\widetilde{A}_{n-1})) = \begin{cases} 1, & \text{if } P(m_1,\dots,m_n)=0,\\ 0 & \text{in the other cases,}\end{cases}$$

where $P(m_1,\dots,m_n) = m_1^2+\cdots+m_n^2-(m_1m_2+\cdots+m_{n-1}m_n+m_nm_1)-(m_1+\cdots+m_n)$. In more details, if $k=1,2,3$ then the space $H_k^{(m_1\dots,m_n)}(\mathfrak{n}_+(\widetilde{A}_{n-1}))$ has dimension 1 for the following sequences $(m_1,\dots,m_n)$:

$$k=0:\ (0,\dots,0);\ \ k-1:=(1,0,\dots,0);$$
$$k=2:\ (2,1,0,\dots,0),\ (1,\underbrace{0,\dots,0}_{>0},1,\underbrace{0,\dots,0}_{>0});$$

$$\begin{aligned} k=3:\ &(2,2,0,\dots,0),(2,1,2,0,\dots,0),\\ &(3,2,1,0,\dots,0),(1,3,1,0,\dots,0),\\ &(2,1,\underbrace{0,\dots,0}_{>0},1,\underbrace{0,\dots,0}_{>0}),(1,\underbrace{0,\dots,0}_{>0},1,\underbrace{0,\dots,0}_{>0},1,\underbrace{0,\dots,0}_{>0}),\end{aligned}$$

and also for the cases, obtained from these by cyclic permutation and reflection; for the remaining $(m_1,\dots,m_n)$ the named homology is 0.

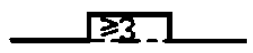
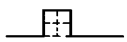
Next we give cycles which represent generators of the above homology (ε_{ij} here and below stand for the matrix with 1 in the section of ith row and jth column and 0 elsewhere).

$$\begin{array}{ll} 1, & \varepsilon_{12},\\ \varepsilon_{12}\wedge\varepsilon_{i,i+1}, & \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{23},\\ \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{23}, & \varepsilon_{12}\wedge\varepsilon_{14}\wedge\varepsilon_{34},\\ \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{14}, & \varepsilon_{13}\wedge\varepsilon_{23}\wedge\varepsilon_{24},\\ \varepsilon_{12}\wedge\varepsilon_{13}\wedge\varepsilon_{i,i+1}, & \varepsilon_{12}\wedge\varepsilon_{i,i+1}\wedge\varepsilon_{j,j+1}, \end{array}$$

where $\varepsilon_{n,n+1}=\varepsilon_{n,1}t$ by definition. Similarly, if as the result of cyclic permutation, we find the first index to be larger than the second one, we have to multiply ε by t.

Now we can determine the dimensions of the space which form the initial terms of the spectral sequences

$$\mathcal{E}(m_1,\dots,m_n)=\mathcal{E}(\mathfrak{n}_+(\widetilde{A}_{n-1}),\mathfrak{n}_+(\widetilde{A}_{n-1})',m_1,\dots,m_n).$$

$(m_1,\dots,m_n)$	dim E_0^1	dim E_1^1	dim E_2^1	dim E_3^1	
	n−1	n	0	0	*
	0	n	0	0	(m=0)*
	1	n−1	n−1	0	*
	0	0	n−1	0	(m=0)*
	1	2	2	1	
	0	0	2	1	(m=0)*
	1	2	1	0	
	0	2	n−1	n−2	
	0	1	2	1	
	0	1	2	1	
	0	0	2	1	(m=0)
	0	1	n−1	n−1	
	0	1	2	1	
	0	1	2	1	
	0	0	2	≥2	

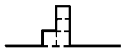
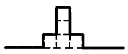
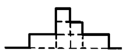
$(m_1,\dots,m_n)$	dim E_0^1	dim E_1^1	dim E_2^1	dim E_3^1
	0	0	1	≥1
	0	0	1	≥1
	0	0	2	2
	0	0	1	≥1
	0	0	1	1
	0	0	2	≥2
	0	0	1	≥1
	0	0	1	≥1
	0	0	1	≥1
	0	0	1	≥1
	0	0	3	≥3

Table 3

We restrict ourselves to such $m_1,\dots,m_n$ that $\bigoplus_{k=0}^{2} E_k^1$ are nontrivial. The dimensions of E_k^1 for these sequences are presented in Table 3.

In this table the sequence $(m_1,\dots,m_n)$ is presented as a graph: the thick broken line is the graph of the step function with equally long steps and $m_1, \dots, m_n$ sequence of values. The left end of the line corresponds to the level $-m(m_1=-m)$. Whenever $m=0$ it is written at the end of the row. All calculations and dimensions are the same for those $(m_1,\dots,m_n)$ which can be obtained by reflection and cyclic permutation from those ones in the table.

It is easy to compute the differentials of the spectral sequences and it turns out that homologies with dimension 1, 2 occur only in the cases which are marked in the table by stars. We calculate the differentials in these cases.

1° $$(m_1,\dots,m_n)=(-m,\dots,-m).$$

In this case E_0^1 is trivial for $m=0$; and for $m>0$ it is spanned by the classes of the chains
$$\alpha_i=\big((\varepsilon_{i,i}-\varepsilon_{i+1,i+1})t^m\big)',$$
and E_1^1 is always spanned by classes of the chains
$$\beta_i=\varepsilon_{i,i+1}\otimes(\varepsilon_{i,i+1}t^m)',\quad i=1,\dots,n-1,\quad \beta_n=\varepsilon_{n,1}t\otimes(\varepsilon_{n,1}t^{m+1})'.$$
Evidently, $d\beta_i=\alpha_i$ for $i=1,\dots,n-1$ and $d\beta_n=-\alpha_1-\dots-\alpha_{n-1}$. So,
$$\dim H_1^{(-m,\dots,-m)}=\begin{cases}1 & \text{for } m>0\\ n & \text{for } m=0,\end{cases}\qquad H_2^{(-m,\dots,-m)}=0.$$

One-dimensional cohomologies for $m>0$ are spanned by the class of the chain $\beta_1+\dots+\beta_n$, and for $m=0$ by classes of the chains $\beta_1,\dots,\beta_n$.

2°. $$(m_1,\dots,m_n)=(\underbrace{-m,\dots,-m}_{i-1},-m+1,-m,\dots,-m),\quad 1\leqq i\leqq n.$$

In this case E_0^1 is trivial for $m=0$, and for $m>0$ it is spanned by the class of the chain
$$\alpha=(\varepsilon_{i+1,i}t^m)';$$
E_1^1 is trivial for $m=0$, and for $m>0$ it is spanned by the classes of the chains
$$\beta_j=\varepsilon_{i,i+1}\otimes\big((\varepsilon_{j,j}-\varepsilon_{j+1,j+1})t^m\big)',\quad j=1,\dots,n-1;$$
E_2^1 is always spanned by the classes of the chains
$$\begin{aligned}\gamma_j&=\varepsilon_{i,i+1}\wedge\varepsilon_{j,j+1}\otimes(\varepsilon_{j,j+1}t^m)',\quad j=1,\dots,i-2,i+2,\dots,n-1\\ \gamma_n&=\varepsilon_{i,i+1}\wedge\varepsilon_{n,1}t\otimes(\varepsilon_{n,1}t^{m+1})',\\ \gamma_{i-1}&=\varepsilon_{i-1,i+1}\wedge\varepsilon_{i,i+1}\otimes(\varepsilon_{i-1,i+1}t^m)',\\ \gamma_{i+1}&=\varepsilon_{i,i+1}\wedge\varepsilon_{i,i+2}\otimes(\varepsilon_{i,i+2}t^m)'\end{aligned}$$
(γ_i is absent). The differential $d=d^1$ acts by
$$d\beta_j=\begin{cases}-2\alpha & \text{for } j=i\\ \alpha & \text{for } j=i\pm1,\\ 0 & \text{in the other cases;}\end{cases}$$
$$d\gamma_j=\begin{cases}\beta_j & \text{for } j\neq i,\ i\pm1,\\ -\beta_1-\dots-\beta_{n-1} & \text{for } j=n,\\ -2\beta_{i-1}-\beta_i & \text{for } j=i-1,\\ \beta_i+2\beta_{i+1} & \text{for } j=i+1.\end{cases}$$
So,
$$H_1^{(-m,\dots,-m+1,\dots,-m)}=0,$$
$$\dim H_2^{(-m,\dots,-m+1,\dots,-m)}=\begin{cases}1 & \text{for } m>0\\ n-1 & \text{for } m=0.\end{cases}$$
The two-dimensional homologies for $m>0$ are spanned by the class of the cycle $\gamma_1+\dots+\gamma_{i-2}+\frac{1}{2}\gamma_{i-1}+\frac{1}{2}\gamma_{i+1}+\gamma_{i+2}+\dots+\gamma_n$, while for $m=0$ by the classes of the cycles $\gamma_1,\dots,\gamma_{i-1},\gamma_{i+1},\dots,\gamma_n$.

3°. $$(m_1,\dots,m_n)=(\underbrace{0,\dots,0}_{i-1},1,1,0,\dots,0),\quad 1\leqq i\leqq n-1.$$

In this case $E_0^1=E_1^1=0$, E_2^1 is spanned by the classes of the chains
$$\gamma_1=\varepsilon_{i,i+1}\wedge\varepsilon_{i,i+2}\otimes(\varepsilon_{i,i+1})',\quad \gamma_2=\varepsilon_{i,i+2}\wedge\varepsilon_{i+1,i+2}\otimes(\varepsilon_{i+1,i+2})',$$

E_3^1 is spanned by the class of

$$\delta = \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \wedge \varepsilon_{i+1,i+2} \otimes (\varepsilon_{i,i+2})';$$

the differential acts by $d\delta = \gamma_1 - \gamma_2$. That means,

$$H_1^{(0,\dots,0,1,1,0,\dots,0)} = 0$$
$$\dim H_2^{(0,\dots,0,1,1,0,\dots,0)} = 1.$$

The two-dimensional homologies are spanned by the class of γ_1 (or γ_2).

The case $(m_1,\dots,m_n) = (1,0,\dots,0,1)$ is similar to the above one.

4°. $$(m_1,\dots,m_n) = (\underbrace{0,\dots,0}_{i-1},2,0,\dots,0), \quad 1 \leqq i \leqq n.$$

In this case $E_0^1 = E_1^1 = 0$, E_2^1 is spanned by the classes of

$$\gamma_i = \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \otimes (\varepsilon_{i+1,i+2})', \quad \gamma_2 = \varepsilon_{i-1,i+1} \wedge \varepsilon_{i,i+1} \otimes (\varepsilon_{i-1,i})',$$

E_3^1 is spanned by the class of the chain

$$\delta = \varepsilon_{i-1,i+1} \wedge \varepsilon_{i,i+1} \wedge \varepsilon_{i,i+2} \otimes (\varepsilon_{i-1,i+2})';$$

the differential acts by $d\delta = \gamma_1 - \gamma_2$. That means,

$$H_1^{(0,\dots,0,2,0,\dots,0)} = 0,$$
$$\dim H_2^{(0,\dots,0,2,0,\dots,0)} = 1.$$

The two-dimensional homologies are spanned by the class of γ_1 (or γ_2).

As usually, we have isomorphism between the cohomology and homology. As it is clear from the list of deformations given before Theorem 1 in Section 1, all classes of two-dimensional cohomologies are represented by deformations of $\mathfrak{n}_+(\widetilde{A}_{n-1})$.

5. The general case of an affine algebra $\mathfrak{g}^A$ for $A \neq \widetilde{A}_1$ is quite similar to the above case. We restrict ourselves to formulate the final result.

$$\dim H_1^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A);\mathfrak{n}_+(A)') =$$
$$= \begin{cases} n & \text{for } (m_1,\dots,m_n) = (0,\dots,0), \\ 1 & \text{for } (m_1,\dots,m_n) = (-ml\omega_1,\dots,-ml\omega_n), \quad m > 0, \\ 0 & \text{in all other cases,} \end{cases}$$

where $\omega_1,\dots,\omega_n$ are the coefficients of linear dependence between the columns of the Cartan matrix, while l equals to 1 for the current algebras (Table 1) and for the matrices from Table 2 is indicated in (v), Section 1.

$$\dim H_2^{(m_1,\dots,m_n)}(\mathfrak{n}_+(A);\mathfrak{n}_+(A)') =$$

$$= \begin{cases} n-1 & \text{for } (m_1,\dots,m_n) = (\underbrace{0,\dots,0}_{i-1},1,0,\dots,0), \quad 1 \leqq i \leqq n \\ 1 & \text{for } (m_1,\dots,m_n) = \\ & \quad = (-ml\omega_1,\dots,-ml\omega_{i-1},-ml\omega_i+1,-ml\omega_{i+1},\dots,-ml\omega_n) \\ & \qquad 1 \leqq i \leqq n,\ m > 0 \\ 1 & \text{for } (m_1,\dots,m_n) = (\underbrace{0,\dots,0}_{i-1},2,0,\dots,0), \quad 1 \leqq i \leqq n \\ 1 & \text{for } (m_1,\dots,m_n) = (\underbrace{\underbrace{0,\dots,0}_{i-1},1,0,\dots,0}_{j-1},1,0,\dots,0) \\ & \qquad 1 \leqq i < j \leqq n,\ a_{ij} \neq 0 \\ 0 & \text{in all other cases.} \end{cases}$$

As in the previous case, $H_k^{(m_1,\dots,m_n)} = H^k_{(-m_1,\dots,-m_n)}$, and all the two-dimensional cohomologies are represented as deformations.

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