ON SINGULAR FORMAL DEFORMATIONS

ALICE FIALOWSKI AND MICHAEL PENKAVA

1. Introduction

In [3], an algebraic notion of a *singular* deformation was introduced in relation to an example of a certain infinite dimensional Lie algebra which has a very surprising deformation pattern. The algebra in question is the subalgebra L_1 of the Virasoro algebra.

Consider the complex infinite-dimensional Lie algebra L_1 of polynomial vector fields in $\mathbb{C}$ with trivial 1-jet at 0. This Lie algebra is spanned by the vector fields $e_i = z^{i+1}d/dz, i = 1, 2, 3, ...$ and the commutator is defined by the standard formula

$$[e_i, e_j] = (j - i)e_{i+j}.$$

This Lie algebra is especially interesting from the point of view of deformation theory. Its deformations were studied by the first author in [1, 2]. In [1] three one-parameter deformations of the Lie algebra L_1 were considered:

$$[e_i, e_j]^1_t =(j - i)(e_{i+j} + te_{i+j-1}),$$

$$[e_i, e_j]^2_t = \begin{cases} (j - i)e_{i+j} & \text{if } i \neq 1, j \neq 1, \\ (j - 1)e_{j+1} + tje_j & \text{if } i = 1, j \neq 1. \end{cases}$$

$$[e_i, e_j]^3_t = \begin{cases} (j - i)e_{i+j} & \text{if} i \neq 2, j \neq 2, \\ (j - 2)e_{j+2} + tje_j & \text{if} i = 2, j \neq 2. \end{cases}$$

All these three deformations are pairwise not equivalent. It was proved in [2].

Theorem 1.1. ([2]) *Any formal one-parameter deformation of the Lie algebra L_1 may be reduced by a formal parameter change to one of the three deformations above.*

1991 *Mathematics Subject Classification.* 14D15,13D10,14B12,16S80,16E40, 17B55,17B70.

Key words and phrases. Formal deformation, singular deformation, Lie algebra.

The second of these deformations is singular in the following sense. Let $\mathfrak{g}$ be a Lie algebra with the commutator $[,]$. Consider a formal one-parameter deformation

$$[g,h]_t = [g,h] + \sum_{k\geq 1} \alpha_k(g,h)t^k \tag{1}$$

of $\mathfrak{g}$.

Definition 1.2. A formal deformation is called *non-singular*, if there exists a formal one-parameter family of linear transformations

$$\phi_t(g) = g + \sum_{l\geq 1} \beta_l(g)t^l \tag{2}$$

of $\mathfrak{g}$ and a formal (not necessarily) invertible parameter change $u = u(t)$ which transforms the deformation $[g,h]_t$ into a deformation

$$[g,h]'_u = [g,h] + \sum_{k\geq 1} \alpha'_k(g,h)u^k, \tag{3}$$

$$\phi_t^{-1}[\phi_t(g), \phi_t(h)]_t = [g,h]'_{u(t)}. \tag{4}$$

with the cocycle $\alpha'_1 \in C^2(\mathfrak{g};\mathfrak{g})$ being not cohomologous to 0.

Otherwise, the formal deformation is called *singular*.

In [3] the following was proved.

Theorem 1.3. *The base of the miniversal deformation of L_1 is the union of two smooth curves and one cuspidal curve passing through 0 with the common tangent.*

Hence not only the base of the minivarsal deformation of the Lie algebra L_1 is a singular variety, but also one of its irreducible components is singular. Such examples have not been studied before.

In [2] the cohomology $H^2(L_1;L_1)$ was computed with coefficients in the adjoint representation. This space is three dimensional and has weight $-2,-3,-4$. As we know, the representative cocycles are responsible for nonequivalent infinitesimal deformations. Let us study now the three deformations above. Each of them is of the form $[,]_t = [,] + t\varphi_i$, where $[,]_t$ represents the deformed algebra, $[,]$ represents the original bracket, and φ_i is a certain 2-cocycle. If it were true that φ_1, φ_2 and φ_3 were a basis of H^2, then the picture would be quite unsurprising. However, it turns out that while φ_3 is a non cohomologous cocycle, the first two are actually coboundaries. But they both define nontrivial deformations.

The deformation $[,]_t^1$ is non-singular (see [3]). To see this, apply a sequence of appropriate transformations $\varphi_t(g) = g + \lambda_3(g)t^3$, $\varphi_t(g) = g + \lambda_5(g)t^5, ...,$ and apply the parameter change $u(t) = t^2$, we get the deformation

$$[e_i, e_j]_u^1 = [e_i, e_j] + \sum_{l \geq 1} \gamma_{2l}(e_i, e_j)u^l, \tag{5}$$

with γ_2 being not cohomologous to 0.

Let us suppose in general that $[,]_t = [,] + t\varphi$ is a deformation of $\mathfrak{g}$ and that $\varphi = D(\beta)$ is a coboundary. We must have $[\varphi, \varphi] = 0$. Let $g = \exp(-t\beta)$, so that g is a formal automorphism. Then

$$g^*([,]_t) = [,] + \frac{1}{2}[\beta, \phi]t^2 + \frac{1}{3}[\beta, [\beta, \phi]]t^3 + \text{ho}.$$

It may turn out that $[\beta, \phi]$ is a nontrivial cocycle, in which case the deformation cannot be trivial. In fact, in the case of L_1, if we follow the construction above with a certain β, then in the resulting transformed deformation both the t^2 and t^3 terms are nontrivial cocycles, and they are not cohomologous.

The notion of a singular deformation as arising from a singularity in the base is more fundamental than the definition given in [3], and the purpose of this article is to explore the differences in these notions.

2. Definitions

Let $\mathcal{A}$ be a complete local $\mathbb{K}$-algebra, in other words, a commutative, unital algebra over $\mathbb{K}$ with a unique maximal ideal $\mathfrak{m}$, such that $\mathcal{A}$ is isomorphic to the inverse limit of the sequence of algebras $\mathcal{A}/\mathfrak{m}^k$. We will call such an algebra *formal* if $\mathcal{A}/\mathfrak{m} \cong \mathbb{K}$.

Such an algebra $\mathcal{A}$ is called a *formal algebra* if $\mathcal{A}/\mathfrak{m} \cong \mathbb{K}$. There is a canonical decomposition $\mathcal{A} = \mathfrak{m} \rtimes \mathbb{K}$. If $\mathfrak{m}/\mathfrak{m}^2 = \langle t_1, \cdots, \rangle$, then every element a in $\mathcal{A}$ can be represented as a formal power series of the form

$$a = a_0 + a^i t_i + a^{i,j} t_i t_j + \cdots .$$

Moreover, $\mathcal{A} \cong \mathbb{K}[[t_1, \cdots,]]/I$ where I is a proper ideal, and every such quotient is a formal algebra. A *formal automorphism* of a $\mathbb{K}$-vector space V is a map $g = Id + \lambda$, where $\lambda : V \to V \otimes \mathfrak{m}$. A formal automorphism of V extends to an $\mathcal{A}$-linear automorphism $g : V \otimes \mathcal{A} \to V \otimes \mathcal{A}$, where we use the decomposition $V \otimes \mathcal{A} = V \oplus V \otimes \mathfrak{m}$, using the decomposition $\mathcal{A} = \mathbb{K} \oplus \mathfrak{m}$ as a $\mathbb{K}$-vector space. In terms of a basis $\langle t_i \rangle$ of $\mathfrak{m}$, we can express a formal automorphism g of V as a series

$$g = I + t_i \lambda^i + t_i t_j \lambda^{ij} + \cdots ,$$

where $\lambda^i, \lambda^{ij}, \ldots$ are endomorphisms of V.

Definition 2.1. A *formal deformation* of a Lie algebra $\mathfrak{g}$ on V with formal base $\mathcal{A}$ is a map $\mathfrak{g}_{\mathcal{A}} = [,] + \psi$ where $\psi : V \wedge V \to V \otimes \mathfrak{m}$, such that $\mathfrak{g}_{\mathcal{A}}$ has the structure of a Lie algebra over $\mathcal{A}$ ($[,]$ is the bracket in $\mathfrak{g}$).

Moreover, a formal automorphism of V acts on formal deformations by $\mathfrak{g} \mapsto g^*(\mathfrak{g}) = g^{-1}\mathfrak{g}g$, where in the last expression g represents the natural extension of g to an automorphism of $W = \bigwedge V$ by

$$g(v_1 \wedge \cdots \wedge v_n) = g(v_1) \wedge \cdots \wedge g(v_n).$$

Definition 2.2. Two deformations $\mathfrak{g}_{\mathcal{A}}$ and $\mathfrak{g}'_{\mathcal{A}}$ of $\mathfrak{g}$ with base $\mathcal{A}$ are equivalent if there is a formal automorphism g of $W = \bigwedge V$ such that $g^*(\mathfrak{g}_{\mathcal{A}}) = \mathfrak{g}'_{\mathcal{A}}$.

For simplicity, let us fix the base as $\mathcal{A} = \mathbb{K}[[t]]$, so that we are considering 1-parameter formal deformations $\mathfrak{g}_t$ of an algebra $\mathfrak{g}$, with the bracket operation

$$[,]_t = [,] + t\varphi_1 + t^2\varphi_2 + \cdots$$

Here $[,]$ denotes the original bracket in $\mathfrak{g}$.

An n-cochain $\varphi \in C^n(V)$ is a map $\bigwedge^n V \to V$, and there is a natural structure of a graded Lie algebra on the space $C(V) = \bigoplus C^n(V)$. We have that $[\mathfrak{g}, \mathfrak{g}] = 0$. Define $D : C(W) \to C(W)$ by $D(\varphi) = [d\mathfrak{g}, \varphi]$. The homology of this differential on $C(W)$ is called the (Chevalley-Eilenberg) cohomology of $\mathfrak{g}$. Denote by $\bar{\varphi}$ the cohomology class of a cocycle φ.

The singular deformation of L_1 which we discussed above arises from a singular curve in the base of the versal deformation of L_1. This algebraic geometric definition makes it difficult to determine whether a deformation is singular. In [3], a more algebraic definition of a singular deformation was given. Recall that the singular deformation of L_1 is of the form $[,]_t = [,] + t\varphi$, where φ is a coboundary, meaning $\varphi = D(\beta)$ for some 1-cochain β.

Recall that if $\beta : V \to V$ is an endomorphism, then

$$\exp(t\beta) = 1 + t\beta + \tfrac{1}{2}t^2\beta^2 + \cdots$$

is a formal automorphism of V, and moreover, if $g = \exp(t\beta)$ then $g^* = \exp(-\operatorname{ad}_{t\beta})$, so that

$$g^*(\psi) = \psi + t[\psi, \beta] + \tfrac{1}{2}t^2[[\psi, \beta], \beta] + \cdots ,$$

for any 2-cochain ψ. In particular, we obtain that

$$\exp(-t\beta)^*([,]_t) = [,] + t^2\psi_2 + t^3\psi_3 + \cdots$$

Moreover, in this case, both the ψ_2 and ψ_3 which arise are nontrivial, noncohomologous cocycles. At first glance, it might seem that we have arrived at a reasonable condition for a deformation to be singular, but we also must take into account that a change of variables may render a seemingly singular deformation into a nonsingular form. A parameter change is given by a morphism of formal algebras $\mathbb{K}[[u]] \to \mathbb{K}[[t]]$, which is given by a formal series $u = a_1 t + a_2 t^2 + \cdots$. The following definition of a nonsingular deformation takes into account the possibility of a parameter change rendering a seemingly singular deformation into a nonsingular form. et $S(W)$ denote the symmetric algebra of the space W.

Definition 2.3. A formal deformation $[,]_t = [,] + t^i\varphi_i$ is *algebraically non-singular* if there is a formal automorphism g_t of $S(W)\hat{\otimes}\mathbb{K}[[t]]$ such that if $g_t^*([,]_t) = [,]'_t$, where $[,]'_t = [,] + t^i\varphi'_i$, then there is a formal parameter change $u = u(t)$ such that $[,]'_t = [,] + u^k\alpha_k$ where the cohomology class $\bar{\alpha}_1 \neq 0$.

In the opposite case we call the deformation *algebraically singular.*

Note that if $[,]_t = [,] + t^2\varphi_2 + t^3\varphi_3 + \text{ho}$, then both φ_2 and φ_3 are cocycles.

Theorem 2.4. *Suppose that*

$$[,]_t = [,] + t^2\varphi_2 + t^3\varphi_3 + ho$$

is a formal deformation of $\mathfrak{g}$ *such that*

(1) $\bar{\varphi}_2 \neq 0$*;*
(2) *there is no cohomology class* $\bar{\beta}$ *such that*

$$\bar{\varphi}_3 = a\bar{\varphi}_2 + [\bar{\varphi}_2, \bar{\beta}],$$

for any constant a*.*

Then the deformation $[,]_t$ *is algebraically singular.*

Proof. Write $u = q_i t^i$ $(i \geq 1)$. Then

$$\mathfrak{g}_u = \mathfrak{g} + tq_1\alpha_1 + t^2(q_2\alpha_1 + q_1^2\alpha_2) + t^3(q_3\alpha_1 + 2q_1q_2\alpha_2 + q_1^3\alpha_3) + \text{ho}.$$

Assume that $g_t^*(\mathfrak{g}_u) = \mathfrak{g}_t$, where $g_t = \exp(t^i\beta_i)$ is a formal automorphism. Let $\beta = t^i\beta_i$. Note this is inverse to the one in the definition. Then

$$g_t^*(\mathfrak{g}_u) = \mathfrak{g}_u + [\mathfrak{g}_u, \beta] + \tfrac{1}{2}[[\mathfrak{g}_u, \beta], \beta] + \tfrac{1}{6}[[[\mathfrak{g}_u, \beta], \beta], \beta] + \cdots.$$

It is easily computed that the t-term in this deformation is $t(q_1\alpha_1 + [\mathfrak{g}, \beta_1])$. Since $\bar{\alpha}_1 \neq 0$, we must have $q_1 = 0$ and $[\mathfrak{g}, \beta_1] = 0$. Thus β_1

is a cocycle. Ignoring the q_1 terms in the expression for L_u, we obtain, up to third order, that

$$g_t^*(\mathfrak{g}_u) = \mathfrak{g}+t^2(q_2\alpha_1+[\mathfrak{g},\beta_2])+t^3(q_3\alpha_1+[\mathfrak{g},\beta_3]+\tfrac{1}{2}[[\mathfrak{g},\beta_2],\beta_1]+[q_2\alpha_1,\beta_1].$$

It follows that $\bar{\varphi}_2 = q_2\bar{\alpha}_1$. Since we are assuming that $\bar{\varphi}_2 \neq 0$, this implies that $q_2 \neq 0$. Next, we have

$$\bar{\varphi}_3 = q_3\bar{\alpha}_1 + [\bar{\varphi}_2, \bar{\beta}_1] = \frac{q_3}{q_2}\bar{\varphi}_2 + [\bar{\varphi}_2, \bar{\beta}_1].$$

Since this contradicts the second assumption, it follows that $\mathfrak{g}_t$ is singular. □

3. A puzzling example

Consider the 3-dimensional nontrivial nilpotent Lie algebra $\mathfrak{n}_3$ with basis $\langle e_1, e_2, e_3\rangle$ and nontrivial bracket $[e_2, e_3] = e_1$. For this Lie algebra $\dim H^2(\mathfrak{n}_3, \mathfrak{n}_3) = 5$.

In terms of the basis $\langle e_1e_2, e_1e_3, e_2e_3\rangle$ of $\bigwedge^2(\mathbb{C}^3)$, a matrix of the versal deformation of this algebra is $A = \begin{bmatrix} 0 & 0 & 1 \\ t_3 & t_1 & t_5 \\ t_4 & -t_3 & t_2 \end{bmatrix}$ (see [4]).

The relations on the base are

$$t_1t_2 + t_3t_5 = 0, \qquad t_2t_3 + t_4t_5 = 0.$$

Clearly, if we set $t_2 = t_5 = 0$, then the relations are satisfied. In fact, to construct a singular deformation, we set $t_1 = t^2$, $t_4 = t^3$ and $t_3 = 0$. Then the matrix of $[,]_t$ is simply $\begin{bmatrix} 0 & 0 & 1 \\ 0 & t^2 & 0 \\ t^3 & 0 & 0 \end{bmatrix}$. Evidently, this matrix is nonsingular for $t \neq 0$, which means that this is a jump deformation from our Lie algebra to $\mathfrak{sl}_2(\mathbb{C})$. (Jump deformation means that $\mathfrak{g}_t \cong \mathfrak{g}_s$ for every s, t, and not isomorphic to $\mathfrak{g}$.) Moreover, if we consider the linear transformation g_t given by the matrix $G = \begin{bmatrix} 0 & 0 & t^{-2} \\ 0 & t^{-1} & 0 \\ -1 & 0 & 0 \end{bmatrix}$, then it is easily computed that $[,]'_t = g_t^*([,]_t)$ has matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & t & 0 \\ t & 0 & 0 \end{bmatrix}$. Of course, g_t is not a formal automorphism of $S(W)$, in fact, it is not an automorphism of $S(W)$ over $\mathbb{C}[[t]]$. The fact that $[,]'_t$ is non-singular is immediate from $[,]'_t = [,] + t\delta_1 + t\delta_4$, where δ_i is the basis of $H^2(\mathfrak{n}_3; \mathfrak{n}_3)$ corresponding to our parameterization. Note that $[,]_t = [,]+t^2\delta_1+t^3\delta_4$. Now, suppose that we consider a 1-cocycle β, which can always be represented by a matrix of the form $B = \begin{bmatrix} b_1+b_3 & 0 & 0 \\ 0 & b_1 & b_4 \\ 0 & b_2 & b_4 \end{bmatrix}$. It is easily computed that

$$[\delta_1, \beta] = 2b_3\delta_1 + b_2\delta_3,$$

where $\delta_3 = \psi_2^{12} - \psi_3^{13}$ is the third element in the basis of $H^2(\mathfrak{n}_3; \mathfrak{n}_3)$. Therefore, it is impossible that δ_4 can be expressed in the form

$$\bar{\delta}_4 = a\bar{\delta}_1 + [\bar{\delta}_1, \bar{\beta}].$$

It follows that the deformation $[,]_t$ is singular. This means that there can be singular deformations, when there is a perfectly good nonsingular deformation which is even equivalent to it in a larger class of equivalences. Somehow, this gives us the feeling that the definition of singular deformation may need to be modified, if it is to have the flavor of giving a deformation in a direction unobtainable by ordinary means. Note that g_t has coefficients in the field $\mathbb{C}((t))$.

4. Essentially singular deformations

Based on the example we constructed, it is clear that one can construct singular deformations that fail to meet the strong condition of singularity that is satisfied by the singular deformation of L_1. A natural way of dealing with this situation is to introduce a stronger notion of singular deformation.

Definition 4.1. We say that a deformation

$$[,]_t = [,] + t\psi_1 + t^2\psi_2 + \cdots$$

is *essentially singular* if it is given by a singular curve in the base of the versal deformation.

Here we have to be careful what is meant by a *singular curve* in the base. For example, suppose that for some two parameter family $[,]_{t_1,t_2} = [,] + t_1\psi_1 + t_2\psi_2 + \mathrm{ho}$, we have that $[,]_{t_1,t_2} \sim \mathfrak{g}'$ for some algebra $\mathfrak{g}'$, in other words, we have a two-parameter jump deformation from $\mathfrak{g}$ to $\mathfrak{g}'$. Let us suppose that there is a neighborhood of the origin in the (t_1, t_2) space for which this deformation is defined. Then certainly one can find singular curves in this parameter space which determine 1-parameter deformations, but this type of singularity is nonessential, because we can smooth this singular curve to a nonsingular one which gives a jump deformation to the same algebra $\mathfrak{g}'$.

Remark 1. Essentially singular deformations are clearly singular in the algebraic sense (see Definition 2.2).

Example 1. Consider the 4-dimensional nilpotent Lie algebra with nonzero brackets in terms of the standard basis $[e_2, e_3] = e_1, [e_3, e_4] = e_2$. It has a 6-parameter versal deformation with 5 nontrivial relations on the parameters, some of which are given by polynomials of degree 5. (The details of this construction are given in [5], p.34.) One of the solutions to the relations depends on three parameters, and generically the deformation given by this solution is isomorphic to the algebra which is the direct sum of two copies of the 2-dimensional nontrivial solvable Lie algebra. (In terms of the standard basis of $\mathbb{C}^4$, the algebra is given by the rules $[e_1, e_3] = e_1$, $[e_2, e_4] = e_2$.) In particular, we can construct a 1-parameter deformation

$$[,]_t = [,] + \psi_1 t^2 + \psi_2 t^2 + \psi_3 t^3 + \psi_4 t^5 + \psi_5 t^5,$$

such that $[,]_t \sim [,]'$ whenever $t \neq 0$. Here $\psi_1(e_1, e_4) = e_1, \psi_2(e_2, e_4) = e_2, \psi_3(e_2, e_4) = e_3, \psi_4(e_1, e_2) = e_2$ and $\psi_5(e_1, e_4) = e_4$. It is not difficult to show that for $\psi_3' = \psi_3$ and $\psi_2' = \psi_1 t^2 + \psi_2$, there is no number a and 1-cocycle β such that $\overline{\psi_3'} = a\overline{\psi_2'} + [\overline{\psi_2'}, \overline{\beta}]$. Thus by Theorem 2.4, the deformation $[,]_t$ is singular. But clearly, this deformation can be desingularized, so it is not essentially singular.

We found it rather easy to construct singular deformations by simply looking at versal deformations where there was a more than 1-parameter family of jump deformations to a certain algebra. On the other hand, we also investigated whether this phenomenon could extend to smooth deformations, that is deformations $[,]_t$ such that $[,]_t \not\sim [,]_s$ unless $t = s$ in some neighborhood of $t = 0$, but were not able to find any such examples, so we suspect that smooth deformations are not singular. However, we do not have a proof of this fact.

It is interesting that the only case we have been able to find an essentially singular deformation is infinite dimensional. It may be the case that essential singularities do not arise in finite dimensional Lie algebras.

References

1. A. Fialowski, *Deformations of the Lie algebra of vector fields on the line*, Russian Math. Surveys **38** (1983), 185–186.
2. A. Fialowski, *An example of formal deformations of Lie algebras*, NATO Conference on Deformation Theory of Algebras and Appl., Il ciocco, Italy. Proceedings Kluwer 1988. 375–401.
3. A. Fialowski, D.B. Fuchs, *Singular Deformations of Lie algebras. Example: Deformations of the Lie algebra L_1*, Topics in Singularity Theory: V.I. Arnold's 60th Anniv. Coll., Transl. Amer. Math. Soc. Ser 2 **180** (1997), 77–92.

4. A. Fialowski, M. Penkava, *Versal deformations of three-dimensional Lie algebras, as L_∞ algebras*, Comm. in Contemp. Math. **7** (2005), 145–165.
5. A. Fialowski, M. Penkava, *Versal deformations of four dimensional Lie algebras*, Comm. in Contemp. Math. **9** (2007), 41–79.

Eötvös Loránd University, Budapest, Hungary
E-mail address: fialowsk@cs.elte.hu

University of Wisconsin, Eau Claire, WI 54702-4004
E-mail address: penkavmr@uwec.edu