

## DEFORMATIONS OF LIE ALGEBRAS

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### Abstract

The author considers general questions of deformations of Lie algebras over a field of characteristic zero, and the related problems of computing cohomology with coefficients in adjoint representations. The construction of a versal family, and the construction of obstructions to the extension of deformations, are also considered. Bibliography: 13 titles.

In this paper, we consider general questions on deformations of Lie algebras over a field of characteristic zero, and related problems of computing cohomology with coefficients in adjoint representations. We consider the construction of a versal family and the nature of obstructions to the extension of deformations. Our aim is to carry over general constructions of the modern theory of deformations and related properties of the cohomology of (local) commutative algebras to Lie algebras in parallel with the papers [3], [10], and [11].

1. We shall require some information on the Harrison cohomology of commutative rings (see [12] and [8]). Harrison cohomology is the cohomology in the category of commutative rings. We shall only require the 1-dimensional and 2-dimensional cohomology, and we shall restrict ourselves to their explicit definition. (In contrast with the traditional indexing, we consider Harrison cohomology with the indices increased by 1.)

Let  $A$  be a commutative  $\mathbf{k}$ -algebra, where  $\mathbf{k}$  is a field of characteristic zero, and let  $N$  be an  $A$ -module. We write down a cochain complex  $M \xrightarrow{d_0} K^1 \xrightarrow{d_1} K^2$ , where  $K^1 = \text{Hom}_k(A, N)$  and  $K^2$  is the subspace of  $\text{Hom}_k(S^2A, N)$  consisting of the maps  $\varphi$  for which

$$a\varphi(b, c) - \varphi(ab, c) - c\varphi(a, b) = 0$$

for any three elements  $a, b, c \in A$ . The differentials  $d_0$  and  $d_1$  are arranged so that

$$d_0(n)(a) = an, \quad a \in A, n \in N, \quad d_1\theta(a, b) = a\theta(b) - \theta(ab) + b\theta(a), \quad a, b \in A.$$

The spaces  $H_{\text{Har}}^1(A; N)$  and  $H_{\text{Har}}^2(A; N)$  of 1-dimensional and 2-dimensional cohomology are by definition  $\text{Ker } d_1 / \text{Im } d_0$  and  $K^2 / \text{Im } d_1$ , respectively.

From the definition one can see that 1-cocycles are derivations. Let  $A$  be an algebra,  $m$  a maximal ideal, and  $A/m \cong k$ . Then  $H_{\text{Har}}^1(A; \mathbf{k}) \cong (m/m^2)^*$ . In other words,  $H_{\text{Har}}^1(A; \mathbf{k})$  is isomorphic to the space of homomorphisms  $A \rightarrow \mathbf{k}[t]/(t^2)$  for which the kernel of the composition  $A \rightarrow \mathbf{k}[t]/(t^2) \rightarrow k$  is  $m$ .

The 2-dimensional cohomology is interpreted as extension (see [9]). An extension of the algebra  $A$  by a module  $N$  is an exact sequence  $0 \rightarrow N \xrightarrow{i} B \xrightarrow{\pi} A \rightarrow 0$ , where  $B$  is a commutative algebra and  $i(N)$  is an ideal in  $B$  with trivial multiplication such that  $bi(n) = \pi(b)n$  for  $b \in B$  and  $n \in N$ . To an extension we assign a cochain  $\varphi \in K^2$  in the following way. Let  $\eta : A \rightarrow B$  be a  $\mathbf{k}$ -linear map for which  $\pi\eta = \text{id}$ . We put  $i\varphi(a, b) = \eta(a)\eta(b) - \eta(ab)$ . It can easily be verified that  $\varphi \in K^2$  and that for any other choice of the map  $\eta$  the cochain  $\varphi$  is changed by a coboundary. Thus from the extension we have constructed an element of the space  $H_{\text{Har}}^2(A; N)$ . From the construction one can see that to every element of  $H_{\text{Har}}^2(A; N)$  there corresponds an extension, and the extension is trivial (that is,  $B$  is a semidirect product) if and only if the corresponding cohomology class is zero.

An automorphism of the extension  $0 \rightarrow N \xrightarrow{i} B \xrightarrow{\pi} A \rightarrow 0$  is an automorphism  $\mu$  of the algebra  $B$  such that  $\pi(\mu(b) - b) = 0$  for all  $b \in B$  and  $\mu(n) - n = 0$  if  $n \in i(N)$ . The map  $\theta : A \rightarrow N$ ,  $a \mapsto (\mu - 1)\eta a$  depends on the choice of  $\eta$ ; namely,  $\theta : A \rightarrow N$  is a cocycle which, as  $\eta$  changes, is changed by a coboundary. Thus the set of automorphisms of the extension  $0 \rightarrow N \rightarrow B \rightarrow A \rightarrow 0$  can be naturally identified with the space  $H_{\text{Har}}^1(A; N)$ .

2. Now we move to the theory of deformations of Lie algebras. We begin with a “naive” definition of deformation of Lie algebras. Let  $V$  be a linear space, and  $S(V)$  the set of all linear maps  $\Lambda^2 V \rightarrow V$  satisfying the Jacobi identity;  $S(V)$  is hence the set of common zeros of a certain system of second degree polynomials on the space  $\Lambda^2 V' \otimes V$ . This makes it possible to equip  $S(V)$  with the structure of affine algebraic variety. The group  $\text{GL}(V)$  acts on  $S(V)$ . The quotient  $L = S(V)/\text{GL}(V)$  is the set of pairwise nonisomorphic Lie algebra structures on the space  $V$ .

It is well known (see [6]) that in the category of algebraic varieties the quotient by a group action does not always exist. In particular,  $L$  is not an algebraic variety. However, one can define a functor assigning to each affine algebraic variety  $X$  a would-be set of morphisms  $\text{Mor}(X, L)$ . Namely, to  $X$  we assign  $D(X)$ , which is the quotient of the set  $\text{Mor}(X, S(V))$  by the action of the group of regular maps  $X \rightarrow \text{GL}(V)$ . If the functor  $D(X)$  could be represented in the category of algebraic varieties, then  $L$  would admit the structure of an algebraic variety, and  $D(X) \cong \text{Mor}(X, L)$ .

The study of the quotient  $D$  is the main problem of the theory of deformations. We are mainly interested in the local theory of deformations; that is, we restrict ourselves to the subcategory  $\Lambda$  of the category of affine algebraic varieties which consists of the varieties of the form  $\text{spec } A$ , where  $A$  is a local algebra. We recall that a local algebra is an algebra with a unique maximal ideal  $m$ ,  $A/m \cong \mathbf{k}$ . We now define a functor responsible for the structure of  $L$  in the neighborhood of a given point.

Let  $\mathcal{L}$  be a Lie algebra, and  $\text{spec } A$  an object of the category  $\Lambda$ . Then  $\text{Def}(\mathcal{L}, \text{spec } A)$  is by definition the preimage of  $\mathcal{L}$  under the map  $D(A) \rightarrow D(\mathbf{k})$  induced by the morphism  $\text{spec } \mathbf{k} \rightarrow \text{spec } A$ . Here we assume that  $V = \mathcal{L}$  and that  $\mathcal{L}$  itself is an element of the set  $L$ . The elements of the set  $\text{Def}(\mathcal{L}, \text{spec } A)$  are called *deformations* of the Lie

algebra  $\mathcal{L}$  with the base  $\text{spec}A$ .

One distinguishes especially the so-called formal deformations of a Lie algebra; that is, deformations over a ring of formal power series in one variable (see [13]).

Now we give a more explicit description of the set  $\text{Def}(\mathcal{L}, \text{spec}A)$ , where  $\text{spec}A \in \Lambda$ . It is the set of classes by isomorphism of the following pairs: a) the Lie  $A$ -algebra  $\overline{\mathcal{L}}(A)$ , which is free as an  $A$ -module, and b) the isomorphism  $\overline{\mathcal{L}}(A)/m\overline{\mathcal{L}}(A) \cong \mathcal{L}$ , where  $m$  is a maximal ideal in  $A$ . The coincidence of this definition with that of the preceding paragraph is obvious. Indeed, we choose in the algebra  $\overline{\mathcal{L}}(A)$  a basis (over  $A$ ). The Lie commutator in this basis gives us a map  $\text{spec}A \rightarrow S(V)$  ( $V \cong \mathcal{L}$ ). Conversely there exists a ‘‘tautological’’ algebra over the algebra  $\mathbf{k}[S(V)]$  (that is,  $\bigoplus_{s \in S(V)} V_s$ , where  $V_s$  is a Lie algebra with commutator  $s$ ), and by ‘‘change of base’’ in each morphism  $\text{spec}A \rightarrow S(V)$  we obtain a Lie  $A$ -algebra.

**Example.** Let  $A \rightarrow \mathbf{k}[t]/(t^2)$ . We describe  $\text{Def}(\mathcal{L}, \text{spec}A)$ .

**Proposition 1.**  $\text{Def}(\mathcal{L}, \text{spec}A) \cong H^2(\mathcal{L}; \mathcal{L})$ .

*Proof.* An element of the set  $\text{Def}(\mathcal{L}, \text{spec}A)$  is a Lie algebra  $\overline{\mathcal{L}}$  equipped with an endomorphism  $t, \overline{\mathcal{L}}/t\overline{\mathcal{L}} \cong \mathcal{L}$ ;  $t\overline{\mathcal{L}}$  is an abelian ideal. Thus  $0 \rightarrow t\overline{\mathcal{L}} \rightarrow \overline{\mathcal{L}} \rightarrow \mathcal{L} \rightarrow 0$  is an extension of the algebra  $\mathcal{L}$  by the adjoint representation. It is easy to show that, conversely, every such extension defines an element of  $\text{Def}(\mathcal{L}, \text{spec}A)$ . On the other hand, the extensions can be classified by the elements of the linear space  $H^2(\mathcal{L}; \mathcal{L})$ . The proposition is proved.

Hence  $H^2(\mathcal{L}; \mathcal{L})$  is the tangent space to  $L$  at the point  $\mathcal{L}$ . It is natural to refer to its elements as infinitesimal deformations of the Lie algebra  $\mathcal{L}$ . To every deformation of  $\mathcal{L}$  there corresponds a unique infinitesimal deformation, which is also called the characteristic class of this deformation.

The functor  $\text{Def}$  is, generally speaking, not representable; that is, there does not exist a universal element. However, there exists a so-called versal (more precisely, mini-versal) element, whose definition we shall now give.

Let  $X$  be an object of the category  $\Lambda$  and  $\tau \in \text{Def}(\mathcal{L}, X)$ . The pair  $(X, \tau)$  defines a morphism of functors  $\theta: \text{Mor}(Y, X) \rightarrow \text{Def}(\mathcal{L}, Y)$  (both functors act from the category  $\Lambda$  into the category of sets). The map  $\theta$  relates to the morphism  $\varphi: Y \rightarrow X$  the element  $\varphi^*(\tau)$ . We recall that the pair  $(X, \tau)$  is called universal if  $\theta$  is an isomorphism for any object  $Y$ . The pair  $(X, \tau)$  is called (*mini*)*versal* if a) the map  $\theta$  is surjective for any  $Y$ , and b)  $\theta$  is an isomorphism if  $Y = \text{spec} \mathbf{k}[t]/t^2$ .

Conditions a) and b) have a simple geometric meaning. Namely, if  $(X, \tau)$  is a versal deformation, then the corresponding map  $X \rightarrow S(V)$  is an embedding, and the image of  $X$  intersects the  $\text{GL}(V)$ -orbit of  $\mathcal{L}$  in a unique point  $p$ , where the tangent space at  $p$  is the sum of the tangent space to the orbit and the tangent space to the image of  $X$ .

In [1] and [7] there appear general theorems which imply the existence of a versal deformation. We give an inductive construction of such a deformation. For this we require some information from the theory of obstructions.

**3.** Let  $\varepsilon \in \text{Def}(\mathcal{L}, \text{spec}A)$ , let  $A$  be a commutative algebra, and let  $m$  be a maximal ideal in  $A$ ,  $A/m \cong \mathbf{k}$ . We choose an element  $f \in H_{\text{Har}}^2(A; \mathbf{k})$  and let  $0 \rightarrow \mathbf{k} \rightarrow B \rightarrow A \rightarrow 0$  be the corresponding extension. We denote by  $\overline{m}$  a maximal ideal in  $B$ . We attempt to extend the deformation  $\varepsilon$  to a deformation with base  $\text{spec}B$ .

The deformation  $\varepsilon$  is a Lie  $A$ -algebra structure on the space  $\mathcal{L} \otimes_{\mathbf{k}} A$ . We need to define a Lie  $B$ -algebra structure on the space  $\mathcal{L} \otimes_{\mathbf{k}} B = \overline{\mathcal{L}}(B)$  such that the map  $\chi : \mathcal{L} \otimes B \rightarrow \mathcal{L} \otimes A$  induced by the homomorphism  $B \rightarrow A$  would be a Lie algebra homomorphism. We identify the kernel of  $\chi$  with the algebra  $\mathcal{L}$ . Let  $\phi : \overline{\mathcal{L}}(B) \rightarrow \mathcal{L}$  be the map induced by the homomorphism  $B \rightarrow B/\overline{m} = \mathbf{k}$ .

On  $\overline{\mathcal{L}}(B)$  we can define a  $B$ -linear skew-symmetric operation  $\{ , \}$  for which a)  $\chi(\{l_1, l_2\}) = [\chi(l_1), \chi(l_2)]$ ,  $l_i \in \overline{\mathcal{L}}$ , and b)  $\{l, l_1\} = [l, \phi(l)]$ ,  $l \in \ker \chi$ ,  $l_1 \in \overline{\mathcal{L}}(B)$ . The operation  $\{ , \}$  “partially” satisfies the Jacobi identity; that is,

$$\{l_1, \{l_2, l_3\}\} - \{\{l_1, l_2\}, l_3\} - \{l_2, \{l_1, l_3\}\} = \Phi(l_1, l_2, l_3) \in \ker \chi.$$

We remark that the function  $\Phi$  is multilinear and skew-symmetric.

Furthermore, if  $l_1 \in \overline{m}\overline{\mathcal{L}}(B)$ , then  $\Phi(l_1, l_2, l_3) = 0$ . In fact if  $l_1 = nl$ ,  $n \in \overline{m}$ , then  $\Phi(nl, l_2, l_3) = n\Phi(l, l_2, l_3) = 0$ . This means that  $\Phi$  defines a multilinear skew-symmetric form  $\Phi$  on  $\mathcal{L} = \overline{\mathcal{L}}(B)/\overline{m}\overline{\mathcal{L}}(B)$  with values in  $\ker \chi \cong \mathcal{L}$ . We shall view  $\overline{\Phi}$  as an element of the space  $C^3(\mathcal{L}; \mathcal{L})$ . It is easy to show by direct calculation that  $\delta\overline{\Phi} = 0$ .

If we replace the operation  $\{ , \}$  by another one which also satisfies conditions a) and b), then the cocycle  $\overline{\Phi}$  is changed by a coboundary. Moreover, by changing  $\{ , \}$  we can obtain any element from the cohomology class of  $\overline{\Phi}$ . So we have assigned to an element  $f$  of  $H_{\text{Har}}^2(A; \mathbf{k})$  a cohomology class of  $H^3(\mathcal{L}; \mathcal{L})$ . One can see from the construction that this correspondence is linear in  $f$ ; that is, we have obtained a homomorphism  $\mathcal{P}_2 : H_{\text{Har}}^2(A; \mathbf{k}) \rightarrow H^3(\mathcal{L}; \mathcal{L})$ .  $\square$

**Proposition 2.** *The deformation  $\varepsilon$  can be extended to a deformation with base  $\text{spec} B$  if and only if  $\mathcal{P}_2(f) = 0$ .*

In fact, a deformation can be extended in the case when one can choose the operation  $\{ , \}$  so that the form  $\Phi$  is zero. From this it follows that if  $\mathcal{P}_2(f) \neq 0$ , then the deformation cannot be extended. However, if  $\mathcal{P}_2(f) = 0$ , the operation  $\{ , \}$  can be modified in such a way that the Jacobi identity holds.

**Example.** Let  $A = \mathbf{k}[t]/t^2$  and  $B = \mathbf{k}[t]/t^3$ , and let  $f$  be a class of extension  $0 \rightarrow \mathbf{k} \rightarrow B \rightarrow A \rightarrow 0$ . A deformation  $\varepsilon$  with base  $\text{spec} A$  is nothing more than an infinitesimal deformation; that is, an element of  $H^2(\mathcal{L}; \mathcal{L})$ . The obstruction  $\mathcal{P}_2(f)$  to the extension of  $\varepsilon$  to a deformation with base  $\text{spec} B$  is directly computed. This is equivalent to the cohomology class of the cocycle

$$(l_1, l_2, l_3) \rightarrow e(e(l_1, l_2), l_3) + e(e(l_2, l_3), l_1) + e(e(l_3, l_1), l_2),$$

where  $e$  is the cocycle representing  $\varepsilon$ . This class is called the Lie square of the class  $\varepsilon$  and is denoted by  $[\varepsilon, \varepsilon]$ . Thus the infinitesimal deformation  $\varepsilon$  can be extended to a deformation with base  $\text{spec} \mathbf{k}[t]/t^3$  if and only if the class  $[\varepsilon, \varepsilon] \in H^3(\mathcal{L}; \mathcal{L})$  is 0.

Obstructions to the further extension of a deformation onto  $\text{spec} \mathbf{k}[t]/t^q$ ,  $q = 4, 5, \dots$ , which also lie in  $H^3(\mathcal{L}; \mathcal{L})$  can be described using Massey Lie operations (see Section 5 below, and also [13]).

The map  $\mathcal{P}_2$  is characteristic of the deformation  $\varepsilon$ . We require one more homomorphism  $\mathcal{P}_1 : H_{\text{Har}}^1(A; \mathbf{k}) \rightarrow H^2(\mathcal{L}; \mathcal{L})$ , also characteristic of the deformation. This

homomorphism relates to an element of  $H_{\text{Har}}^1(A; \mathbf{k})$ ; that is, to a homomorphism  $q : A \rightarrow \mathbf{k}[t]/t^2$ , a characteristic class of the deformation  $q^*\varepsilon$  over  $\mathbf{k}[t]/t^2$ .

We suppose that  $\mathcal{P}_2(f) = 0$ . Then the deformation  $s$  can be extended to a deformation with base  $\text{spec} B$ , and in many ways. Let  $\{ , \}_1$  and  $\{ , \}_2$  be two brackets on  $\overline{\mathcal{L}}(B)$ , and let us consider the difference  $\{ , \}_1 - \{ , \}_2$ . This is a skew-symmetric bilinear function  $\rho$  which relates to a pair of elements  $l_1, l_2 \in \overline{\mathcal{L}}(B)$  an element of  $\ker \chi$  and moreover  $\rho(l_1, l_2) = 0$  if  $l_1 \in \overline{m}\overline{\mathcal{L}}(B)$ . From this it follows that  $\rho$  defines a 2-form

$$\overline{\rho} : \Lambda^2(\overline{\mathcal{L}}(B)/\overline{m}\overline{\mathcal{L}}(B)) \cong \Lambda^2 \mathcal{L} \rightarrow \ker \chi \cong \mathcal{L}.$$

It can be verified directly that  $\overline{\rho}$  is a closed form. The cohomology class of the form  $\overline{\rho}$  is the ‘‘distinguisher’’ for the two brackets  $\{ , \}_1$  and  $\{ , \}_2$ . The endomorphisms of the extension  $0 \rightarrow \mathbf{k} \rightarrow B \rightarrow A \rightarrow 0$  act on the set of brackets on  $\overline{\mathcal{L}}(B)$ . We now study this action.

**Proposition 3.** *Let  $a$  be an automorphism of the extension  $0 \rightarrow \mathbf{k} \rightarrow B \rightarrow A \rightarrow 0$ , and let  $\overline{a}$  be the corresponding element of the space  $H_{\text{Har}}^1(A, \mathbf{k})$ . Furthermore, let  $\{ , \}$  be a bracket on  $\overline{\mathcal{L}}(B)$ . Then the difference between the bracket  $a\{ , \}$  and  $\{ , \}$  is  $\mathcal{P}_1(\overline{a})$ .*

The proof of this Proposition is obvious.

**Corollary.** *Let  $\varepsilon \in \text{Def}(\mathcal{L}, A)$  be a deformation for which the homomorphism  $\mathcal{P}_1$  is a surjection, and let  $0 \rightarrow \mathbf{k} \rightarrow B \rightarrow A \rightarrow 0$  be an extension for which  $\varepsilon$  can be extended to a deformation over  $\text{spec} B$ . Then the automorphism group of the extension acts transitively on the set of all extensions of  $\varepsilon$ .*

**4.** We now move to the construction of a versal deformation. Let  $\Sigma$  be the subcategory of the category of local algebras which consists of algebras  $A$  for which  $m^2 = 0$ ,  $m$  is a maximal ideal, and  $A/m \cong \mathbf{k}$ . Then the functor  $\text{Def}(\mathcal{L}, \text{spec} A)$  can be represented on the category  $\Sigma$ ; that is, it admits a universal pair  $(X, \varepsilon)$   $\varepsilon \in \text{Def}(\mathcal{L}, X)$  (see [7]). We construct such a pair. We put  $X = \text{spec} A$ , where  $A = \mathbf{k} \otimes H_2(\mathcal{L}; \mathcal{L})$  (here  $H_2(\mathcal{L}; \mathcal{L})$  is an ideal in  $A$  with zero multiplication). We now remark that the space

$$H^2(\mathcal{L}; H_2(\mathcal{L}; \mathcal{L}) \cdot \mathcal{L}) = H^2(\mathcal{L}; \mathcal{L}) \otimes H_2(\mathcal{L}; \mathcal{L}),$$

like every tensor product of dual spaces, has a distinguished element. Let  $v$  be a cochain representing this element. We now define a Lie  $A$ -algebra structure on the space

$$\mathcal{L} \otimes A = \mathcal{L} \otimes 1 \oplus \mathcal{L} \otimes H^2(\mathcal{L}; \mathcal{L}),$$

by putting

$$[l_1 \otimes 1, l_2 \otimes 1] = [l_1, l_2] \otimes 1 + v(l_1, l_2), \quad l_1, l_2 \in \mathcal{L}.$$

This is the Lie  $A$ -algebra that corresponds to the deformation  $\varepsilon$ .

If  $C = \mathbf{k} \oplus m$ ,  $m^2 = 0$ , is an object of the category  $\Sigma$ , then to each element of the set  $\text{Def}(\mathcal{L}, \text{spec} C)$  there corresponds naturally (as in Proposition 1) an element of  $H^2(\mathcal{L}; m \otimes \mathcal{L})$ . This correspondence is one-to-one. We note that

$$H^2(\mathcal{L}; m \otimes \mathcal{L}) \cong \text{Hom}(H_2(\mathcal{L}; \mathcal{L}), m) \cong \text{Hom}(A, C) \cong \text{Def}(\mathcal{L}, \text{spec} C).$$

But this implies that the pair  $(X, \varepsilon)$  is universal.

Let  $A$  be a local algebra, let  $m$  be a maximal ideal, and let  $N = (H^2(A; A/m))^* \otimes_{\mathbf{k}} A/m$  be an  $A$ -module. We identify the latter with the space  $H_{\text{Har}}^2(A; \mathbf{k})^*$ . The space

$$H_{\text{Har}}^2(A; N) = H_{\text{Har}}^2(A; \mathbf{k}) \otimes (H_{\text{Har}}^2(A; \mathbf{k}))^*$$

contains a canonical element  $u$ . We construct the corresponding extension  $0 \rightarrow N \rightarrow F(A) \rightarrow A \rightarrow 0$ . Let  $A = \mathbf{k}[x_1, \dots, x_n]/m^2$ ,  $m = (x_1, \dots, x_n)$ . Then the projective limit of the system  $A \leftarrow F(A) \leftarrow F(F(A)) \leftarrow \dots$  is an algebra of formal series in  $n$  variables.

Let  $(X, \varepsilon)$  be a universal pair in the category  $\Sigma$ , and let  $A = \mathbf{k}[x]$ . The deformation  $\varepsilon$  gives us the homomorphism

$$P_2 : H_{\text{Har}}^2(A; \mathbf{k}) \rightarrow H^3(\mathcal{L}; \mathcal{L}).$$

We consider the dual map

$$P_2^* : H_3(\mathcal{L}; \mathcal{L}) \rightarrow (H_{\text{Har}}^2(A; \mathbf{k}))^*,$$

and let  $\bar{u}$  be the image of the class  $u \in H^2(A; (H_{\text{Har}}^2(A; \mathbf{k}))^*)$  under the homomorphism

$$H^2(A; (H_{\text{Har}}^2(A; \mathbf{k}))^*) \rightarrow H^2(A; (H_{\text{Har}}^2(A; \mathbf{k}))^* / \text{Im } P_2^*).$$

We construct the extension corresponding to  $\bar{u}$ :

$$0 \rightarrow H_{\text{Har}}^2(A; \mathbf{k})^* / \text{Im } P_2^* \rightarrow \bar{F}(A) \rightarrow A \rightarrow 0.$$

From the constructions of Section 3 it follows that the family of  $\varepsilon$  can be extended to a family with base  $\text{spec } \bar{F}(A)$ . From the corollary to Proposition 3 it follows that the automorphism group of the extension acts transitively on the set of extensions. This means that the Lie algebra  $\bar{\mathcal{L}}(\bar{F}(A))$  is uniquely defined up to isomorphism. We apply the same construction to the algebra  $\bar{F}(A)$  again, then once more, etc. We obtain a projective system of algebras:

$$\dots \rightarrow \bar{\bar{F}}(A) \rightarrow \bar{F}(A) \rightarrow A.$$

By  $v(\mathcal{L})$  we denote the projective limit of this system of algebras. From the above it follows that  $v(\mathcal{L})$  is the quotient of the algebra  $\mathbf{k}[[H_2(\mathcal{L}; \mathcal{L})]]$  by a certain ideal. So, in the obvious way, there is a deformation of  $\bar{\mathcal{L}}(v(\mathcal{L}))$  with base  $\text{spec } v(\mathcal{L})$ .

**Proposition 4.** *The deformation with base  $\text{spec } v(\mathcal{L})$  just constructed is versal.*

This proposition is proved by standard means. The proof of a similar result for local commutative algebras can be found in a paper by Schlessinger [7].

The algebra  $v(\mathcal{L})$  is the quotient of  $\mathbf{k}[[H_2(\mathcal{L}; \mathcal{L})]]$  by an ideal  $J$ . We assume that  $\dim H_2(\mathcal{L}; \mathcal{L}) < \infty$ . The algebra of formal power series in a finite number of variables is Noetherian, and hence the ideal  $J$  has finitely many generators. The space of generators for  $J$  can be identified (see [7]) with the space  $(H_{\text{Har}}^2(v(\mathcal{L}), \mathbf{k}))^*$ . From the construction of  $v(\mathcal{L})$  it follows that the map  $H_3(\mathcal{L}; \mathcal{L}) \rightarrow (H_{\text{Har}}^2(v(\mathcal{L}); \mathbf{k}))^*$  is surjective. Thus the

coordinate ring of the base of the versal deformation is the quotient of  $\mathbf{k}[[H_2(\mathcal{L}; \mathcal{L})]]$  by the ideal generated by the relators corresponding to the elements of  $H_3(\mathcal{L}; \mathcal{L})$ .

**5.** In this section, following [4], we introduce certain cohomology operations which serve as the main means for computing the versal deformation. For this we require the standard homology complex of a Lie superalgebra. We do not quote its definition (see [2] or [5]).

Let  $A = C^*(\mathcal{L})$  be the standard cochain complex consisting of cochains of the algebra  $\mathcal{L}$ . Let  $A$  be a differential  $\mathbf{Z}$ -graded algebra. By  $\text{Der}A$  we denote the set of superderivations of the algebra  $A$ . The space  $\text{Der}A$  is equipped in the usual way with a Lie superalgebra structure. We note that  $\text{Der}A \cong C^*(\mathcal{L}; \mathcal{L}) \cong \Lambda^*(\mathcal{L}^*) \otimes \mathcal{L}$  (an element  $\omega_1 \otimes l \in \Lambda^*(\mathcal{L}^*) \otimes \mathcal{L}$  is assigned the derivation  $\omega_1 \partial / \partial l$ ,  $l \in \mathcal{L}$ ,  $\omega_1 \in \Lambda^*(\mathcal{L}^*)$ ). The space  $\text{Der}A$  has a distinguished element  $\delta$  to which there corresponds a commutation operator  $\Lambda^2 \mathcal{L} \rightarrow \mathcal{L}$ . The differential in the complex  $\text{Der}A$  is given by the formula  $u \rightarrow [u\delta]$ ; so  $\text{Der}A$  is turned into a differential graded Lie superalgebra. The space  $H^*(\mathcal{L}; \mathcal{L})$  inherits the Lie superalgebra structure.

Let  $K$  be the standard complex of the superalgebra  $\text{Der}A$ ; that is,  $K = \{\text{Der}A \leftarrow \Lambda^2 \text{Der}A \leftarrow \dots\}$ ; we recall that the exterior power is here understood in the super sense. We specify on this complex the filtration

$$K_i = \text{Der}A \oplus \Lambda^2 \text{Der}A \oplus \dots \oplus \Lambda^i \text{Der}A.$$

The first term of the associated spectral sequence, called the Quillen spectral sequence for superalgebras  $\text{Der}A$  (see [5]), is isomorphic to the standard complex of the superalgebra  $H^*(\mathcal{L}; \mathcal{L})$ . According to [5], if a Massey operation is defined on the elements  $\alpha_1, \dots, \alpha_n \in H^*(\mathcal{L}; \mathcal{L})$ , then a boundary differential is also defined in the spectral sequence under consideration. Hence the images of the boundary differentials in this spectral sequence can naturally be called generalized Massey Lie operations.

We shall choose in  $H^2(\mathcal{L}; \mathcal{L})$  an element  $\alpha$ . This element is even, and hence in the standard complex  $H^*(\mathcal{L}; \mathcal{L})$  the elements  $\alpha^n$  are defined. As computations show, the first differential is  $d_1 \alpha^2 = \langle \alpha, \alpha \rangle \in H^2(\mathcal{L}; \mathcal{L})$ . If  $\langle \alpha, \alpha \rangle = 0$ , then the differential

$$d_2 \alpha^3 \in H^3(\mathcal{L}; \mathcal{L}) / d_1(S^2 H^2(\mathcal{L}; \mathcal{L}))$$

is defined (we assume that  $d_2 \alpha^3 = \langle \alpha, \alpha, \alpha \rangle$ ), and so on.

The equation  $\langle \alpha, \alpha \rangle = 0$  defines a quadratic cone in  $H^2(\mathcal{L}; \mathcal{L})$ . If  $\langle \alpha, \alpha \rangle = 0$ , then the condition  $\langle \alpha, \alpha, \alpha \rangle \ni 0$  defines a subset of this quadratic cone. Iterating this procedure, we obtain a certain homogeneous subvariety  $V$  in  $H^2(\mathcal{L}; \mathcal{L})$ . Let  $v(\mathcal{L})$  be the algebra constructed in the preceding section, let  $m$  be a maximal ideal in  $v(\mathcal{L})$ , and let  $\bar{v}(\mathcal{L})$  be the graded algebra  $\mathbf{k} \oplus m/m^2 \oplus m^2/m^3 \oplus \dots$ .

**Proposition 5 (see [4]).** *The spectrum of the algebra  $\bar{v}(\mathcal{L})$  is  $V$ .*

**Corollary.** *If for every  $\alpha \in H^2(\mathcal{L}; \mathcal{L})$*

$$\langle \alpha, \alpha \rangle = 0, \quad \langle \alpha, \alpha, \alpha \rangle \ni 0, \dots,$$

*then  $\text{spec } v(\mathcal{L}) = H^2(\mathcal{L}; \mathcal{L})$ ; that is, the base of a versal deformation of the algebra  $\mathcal{L}$  is  $H^2(\mathcal{L}; \mathcal{L})$ .*

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