

Formal Rigidity of the Witt and Virasoro Algebra

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ABSTRACT The formal rigidity of the Witt and Virasoro algebras was first established by the author in [4]. The proof was based on some earlier results of the author and Goncharowa, and was not presented there. In this paper we give an elementary proof of these facts.

I. Preliminaries

Consider the complexification \mathcal{W} of the Lie algebra of polynomial vector fields on the circle:

$$e_k \rightarrow e^{ik\varphi} \frac{d}{d\varphi},$$

where φ is the angular parameter. The bracket operation in this Lie algebra is

$$[e_n, e_m] = (m - n)e_{n+m}.$$

The Lie algebra \mathcal{W} is called the *Witt algebra*, and was first defined by E. Cartan [1]. The Lie algebra \mathcal{W} is infinite dimensional and graded with $\deg e_n = n$. It is well-known [6] that \mathcal{W} has a unique nontrivial one-dimensional central extension. It is generated by $e_n (n \in \mathbb{Z})$ and the central element c , and its bracket operation is defined by

$$[e_n, e_m] = (m - n)e_{n+m} + 1/12(m^3 - m)\delta_{n,-m}c, \quad [e_n, c] = 0.$$

The extended Lie algebra Vir is called the *Virasoro algebra*. It was first invented for characteristic 0 by Gelfand and Fuchs in [6].

In this paper we give an elementary proof of the formal rigidity of these algebras.

FORMAL DEFORMATIONS

Let L be a Lie algebra, A be a local finite dimensional algebra over a field \mathbb{K} .

Definition. (see [2, 3]) *A formal deformation L_A of L parametrized by a local finite dimensional algebra A is a Lie algebra structure over A on $A \otimes_{\mathbb{K}} L$ such that the Lie algebra structure on*

$$L = (L_A) \otimes_A \mathbb{K} = (L \otimes A) \otimes_A \mathbb{K}$$

is the given one on L .

Two deformations, L_A and L'_A , parameterized by A are called *equivalent*, if there exists a Lie algebra isomorphism over A of L_A on L'_A , inducing the identity of $L_A \otimes_A \mathbb{K} = L$ on $L'_A \otimes_A \mathbb{K} = L$. A deformation is *trivial* if $A \otimes L$ carries the trivially extended Lie structure.

Let now A be a complete local algebra over \mathbb{K} , so $A = \lim_{n \rightarrow \infty} (A/m^n)$, where m is the maximal ideal of A and we assume that $A/m \cong \mathbb{K}$.

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A formal deformation of L with base A is a Lie algebra structure over $A = \varprojlim_{n \rightarrow \infty} (A/m^n)$ on the completed tensor product $A \widehat{\otimes}_{\mathbb{K}} L = \varprojlim_{n \rightarrow \infty} ((A/m^n) \widehat{\otimes} L)$ such that

$$\varepsilon \widehat{\otimes} \text{id} : A \widehat{\otimes} L \rightarrow \mathbb{K} \otimes L = L$$

is a Lie algebra homomorphism.

There is an analogous definition for equivalence of deformations parameterized by a complete local algebra.

RIGIDITY.

Intuitively, rigidity of a Lie algebra means that we cannot deform the Lie algebra.

Definition. A Lie algebra L is formally *rigid*, if and only if every formal deformation of it is equivalent to the trivial deformation.

Proposition. The elements of the cohomology space $H^2(L; L)$ correspond bijectively to the nonequivalent infinitesimal deformations.

Corollary. The condition $H^2(L; L) = 0$ is sufficient for L to be rigid.

For details see the book of Fuchs, [5].

II. Rigidity of W and Vir

Theorem. The Witt and Virasoro algebra are formally rigid.

Proof. We will prove that $H^2(W; W) = 0$. It follows that $H^2(Vir; Vir) = 0$ as well. (Another, more complicated proof is in [7].)

We will present the proof in 9 steps. For the basics of Lie algebra cohomology see the book [5].

1. First assume that the 2-cochain c has weight d : $c \in C_d^2(W; W)$ which means that $c(e_i, e_j) = c_{i,j} e_{i+j+d}$. We will prove that if $d \neq 0$, and $\delta c = 0$, then $c = \delta b$ where $b(e_i) = \frac{c(e_i, e_0)}{d}$ (in particular, $b(e_0) = 0$). Indeed, let $c' = c - \delta b$. Then, first,

$$\begin{aligned} c'(e_i, e_0) &= c(e_i, e_0) - \delta b(e_i, e_0) \\ &= d \cdot b(e_i) - b([e_i, e_0]) + [e_i, b(e_0)] - [e_0, b(e_i)] \\ &= d \cdot b(e_i) + i \cdot b(e_i) - (i+d)b(e_i) = 0. \end{aligned}$$

Second, since $\delta c' = 0$, we have

$$\begin{aligned} \delta c'(e_i, e_j, e_0) &= (j-i)c'(e_{i+j}, e_0) + ic'(e_i, e_j) - jc'(e_j, e_i) \\ &= -[e_i, c'(e_j, e_0)] + [e_j, c'(e_i, e_0)] - [e_0, c'(e_i, e_j)] \\ &= (i+j - (i+j+d))c'(e_i, e_j) = -d \cdot c'(e_i, e_j) = 0. \end{aligned}$$

Hence, $c' = 0$. (See also Theorem 1.5.2 in [5].)

This shows that $H^2(W; W) = H_0^2(W; W)$. Indeed, if $c \in C^2(W; W)$ and $\delta c = 0$, then $c - \delta b \in C_0^2(W; W)$ where b is the following: if $c(e_i, e_j) = \sum_d c_{i,j,d} e_{i+j+d}$, then $b(e_i) =$

$$\sum_{d \neq 0} \frac{c_{i,0,d}}{d} e_{i+d}.$$

Thus, from now on, we consider a cocycle $c \in C_0^2(W; W)$, $c(e_i, e_j) = c_{i,j}e_{i+j}$, $\delta c = 0$. Denote c by $\{c_{i,j}\}$.

2. First, let us explore the possibility of adding δb , $b \in C_0^1(W; W)$, such that $b(e_i) = b_i e_i$, to c . Since $\{b_i = i\}$ is a coboundary and hence a cocycle, we can assume $b_1 = 0$.

PROPOSITION. *Varying b , we can achieve*

$$c_{i,1} = 0 \quad \text{for all } i, \quad \text{and } c_{-2,2} = 0. \quad (1)$$

Moreover, this is all that can be done by adding δb to c .

PROOF. Indeed,

$$\delta b(e_{-2}, e_2) = 4(b_0 - b_2 - b_{-2})e_0, \quad \delta b(e_i, e_1) = (1 - i)e_{i+1}. \quad (2)$$

The second of these formulas gives

$$\begin{array}{ll} \delta b(e_0, e_1) = & -b_0 e_1 & \delta b(e_2, e_1) = & -(b_3 - b_2)e_3 \\ \delta b(e_{-1}, e_1) = & 2(b_0 - b_{-1})e_0 & \delta b(e_3, e_1) = & -2(b_4 - b_3)e_4 \\ \delta b(e_{-2}, e_1) = & 3(b_{-1} - b_{-2})e_{-1} & \delta b(e_4, e_1) = & -3(b_5 - b_4)e_5 \\ \dots\dots & & \dots\dots & \end{array} \quad (3)$$

First, using the first column of (3) and choosing appropriate $b_0, b_{-1}, b_{-2}, \dots$, we can make $\delta b(e_i, e_1) = c(e_i, e_1)$ for $i \leq 0$. Second, using the left formula (2) and choosing an appropriate b_2 we can make $\delta b(e_{-2}, e_2) = c(e_{-2}, e_2)$. Third, using the second column of (3) and choosing appropriate b_3, b_4, b_5, \dots we can make $\delta b(e_i, e_1) = c(e_i, e_1)$ for $i \geq 2$.

Notice that this construction uses a unique choice of all b_i (excluding b_1 which is 0).

COROLLARY. *Every cohomology class in $H^2(W; W)$ is represented by a unique cocycle $\{c_{i,j}\}$ satisfying relations (1).*

3. The cocycle condition. It remains to prove that if $c = \{c_{i,j}\}$ as above satisfies the condition $\delta c = 0$, then $c = 0$. Remark that the condition $\delta c(e_i, e_j, e_k) = 0$ takes the form

$$\begin{aligned} (j-i)c_{i+j,k} + (k-j)c_{j+k,i} + (i-k)c_{k+i,j} \\ + (j-i+k)c_{k,j} + (j-i-k)c_{k,i} - (i+j-k)c_{i,j} = 0. \end{aligned} \quad (4)$$

4. Set $k = 1$. Because of the results of Section 2, three of the six terms in (4) vanish, and what remains is

$$(j-1)c_{i,j+1} + (i-1)c_{i+1,j} - (i+j-1)c_{i,j} = 0. \quad (5)$$

We will use abbreviated notation $a_j = c_{2,j}$. We already know that $a_{-2} = a_1 = a_2 = 0$.

5. PROPOSITION. *If $i \leq 0$, then*

$$c_{ij} = 0 \quad \text{for } j \leq 1, \quad \text{and } c_{ij} = (i-1)a_0 \quad \text{for } j \geq -i+2.$$

PROOF. We proceed by induction with respect to $-i$. First, take $i = 0$. By the Proposition in Section 2, equation (5) loses one more term, and becomes $(j-1)(c_{0,j+1} - c_{0,j}) = 0$. This gives

$$\cdots = c_{0,-2} = c_{0,-1} = c_{0,0} = c_{0,1}; c_{0,2} = c_{0,3} = c_{0,4} = \dots$$

Since $c_{0,2} = -b_0$ and $c_{0,1} = 0$, this is precisely what we need in this case.

Assume now that $i < 0$ and that the formulas in our Proposition hold for $c_{i+1,j}$. Then for $j \leq 0$ the middle term in (5) vanishes and we have $(i+j-1)c_{i,j} = (j-1)c_{i,j+1}$, and, since $c_{i,1} = 0$, this formula with $j = 0, -1, -2, \dots$ successively gives $c_{i,0} = 0, c_{i,-1} = 0, c_{i,-2} = 0, \dots$. For $j \geq -i+1$, formula (5) becomes

$$(j-1)c_{i,j+1} = (i+j-1)c_{i,j} - i(i-1)a_0.$$

If $j = -i+1$, this becomes $-ic_{i,-i+2} = -i(i-1)a_0$ which implies $c_{i,-i+2} = (i-1)a_0$. If we already know that $c_{i,j} = (i-1)a_0$, then

$$(j-1)c_{i,j+1} = ((i+j-1)(i-1) - i(i-1))a_0 = (j-1)(i-1)a_0.$$

Consequently, $c_{i,j+1} = (i-1)a_0$.

All that we know now about the cocycle $\{c_{i,j}\}$ is presented in the following table:

	$j = -4$	$j = -3$	$j = -2$	$j = -1$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 5$		$4a_0$	$3a_0$	$2a_0$	a_0	0	$-a_5$			0
$i = 4$			$3a_0$	$2a_0$	a_0	0	$-a_4$		0	
$i = 3$				$2a_0$	a_0	0	$-a_3$	0		
$i = 2$	a_{-4}	a_{-3}	0	a_{-1}	a_0	0	0	a_3	a_4	a_5
$i = 1$	0	0	0	0	0	0	0	0	0	0
$i = 0$	0	0	0	0	0	0	$-a_0$	$-a_0$	$-a_0$	$-a_0$
$i = -1$	0	0	0	0	0	0	$-a_{-1}$	$-2a_0$	$-2a_0$	$-2a_0$
$i = -2$	0	0	0	0	0	0	0		$-3a_0$	$-3a_0$
$i = -3$	0	0	0	0	0	0	$-a_{-3}$			$-4a_0$
$i = -4$	0	0	0	0	0	0	$-a_{-4}$			

Nothing is known, so far, about the values in the empty cells.

6. Formula (5) with $i = 2, 3, \dots$ gives an expression of $c_{i,j}$, $i \geq 3$ in terms of a_k ; thus, we can fill in the remaining cells in the last table. In particular,

$$\begin{aligned} c_{3,j} &= (j+1)a_j - (j-1)a_{j+1}; \\ c_{4,j} &= \frac{(j+1)(j+2)}{2}a_j - (j-1)(j+2)a_{j+1} + \frac{j(j-1)}{2}a_{j+2}; \\ c_{5,j} &= \frac{(j+1)(j+2)(j+3)}{6}a_j - \frac{(j-1)(j+2)(j+3)}{2}a_{j+1} \end{aligned}$$

$$+ \frac{(j-1)j(j+3)}{2}a_{j+2} - \frac{(j-1)j(j+1)}{6}a_{j+3};$$

etc. (It is not hard to write a general formula, but we will not need it.) We see that all the c_{ij} -s are expressed as linear combinations of a_k , so all we need is to prove that all a_k are zero. Notice also that the equalities $c_{i,i} = 0$ provide some new relations between a_k :

$$\begin{aligned} c_{3,3} = 0 &\Rightarrow 2a_3 - a_4 = 0 \Rightarrow a_4 = 2a_3; \\ c_{4,4} = 0 &\Rightarrow 5a_4 - 6a_5 + 2a_6 = 0 \Rightarrow a_6 = 3a_5 - 5a_3; \\ c_{5,5} = 0 &\Rightarrow 14a_5 - 28a_6 + 20a_7 - 5a_8 \Rightarrow a_8 = 4a_7 - 14a_5 + 28a_3; \\ c_{6,6} = 0 &\Rightarrow \dots \Rightarrow a_{10} = 5a_9 - 30a_7 + 117a_5 - 255a_3; \end{aligned}$$

etc. This can be continued to provide good-looking formula expressing a_{2k} , $k \geq 2$ in terms of $a_{2\ell-1}$, $\ell \geq 2$; however, in our further computations we will only use the first of the relations above ($a_4 = 2a_3$).

7. Put $k = 2$. Formula (5) becomes

$$\begin{aligned} (i+j-2)c_{i,j} - (i-2)c_{i+2,j} - (j-2)c_{i,j+2} \\ = (i-j)a_{i+j} - (i-j+2)a_i - (i-j-2)a_j; \end{aligned} \quad (6)$$

this will be a source of relations between a_k which will kill all of them.

8. Let $i = -2$. Formula (5) becomes

$$(j-4)c_{-2,j} + 4c_{0,j} - (j-2)c_{-2,j+2} = -(j+2)a_{j-2} + (j+4)a_j. \quad (7)$$

We know that

$$c_{0,j} = \begin{cases} 0, & \text{if } j \leq 1, \\ -a_0, & \text{if } j \geq 2; \end{cases} \quad c_{-2,j} = \begin{cases} 0, & \text{if } j \leq 2, \\ -3a_{-1}, & \text{if } j = 3, \\ -3a_0, & \text{if } j \geq 4 \end{cases}$$

Note that we used a formula from Section 6:

$$c_{-2,3} = -c_{3,-2} = -[(-2+1)a_{-2} - (-2-1)a_{-1}] = -3a_{-1}.$$

Thus, for $j \leq 0$, the left hand side of (7) is zero, and we have $(j+4)a_j = (j+2)a_{j-2}$. Thus,

$$a_0 = 0, \quad 0 = a_{-6} = a_{-8} = a_{-10} = \dots$$

and

$$3a_{-1} = a_{-3} = -a_{-5} = -3a_{-7} = -5a_{-9} = -7a_{-11} = \dots$$

Since $a_0 = 0$, the left hand side of formula (7) is zero also for $j \geq 4$. Thus,

$$(j+4)a_j = (j+2)a_{j-2} \text{ also for } j \geq 4.$$

Hence,

$$0 = a_2 = a_4 = a_6 = \dots, \quad a_3 = a_5 = a_7 = \dots$$

But $a_4 = 2a_3$ (see the end of Section 6), so actually, $a_k = 0$ for all $k > 0$. Thus, there remain two unknown values: a_{-4} and a_k for any one odd negative k .

9. Last Step. Consider formula (6) for $i = -3$:

$$\begin{aligned} (j-5)c_{-3,j} + 5c_{-1,j+2} - (j-2)c_{-3,j+2} \\ = -(j+3)a_{j-2} + (j+1)a_{-3} + (j+5)a_j. \end{aligned}$$

If $j = -6$ (or $-8, -10$, etc.), we get $a_{-3} = 0$; if $j = -4$, we get $a_{-4} = 0$. This completes the calculation.

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