

On the Cohomology of Infinite Dimensional Nilpotent Lie Algebras

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March 29, 2008

APPEARED IN: *Advances in Math.*, **97** (1993), 267–277

Abstract

In the paper one- and two-dimensional cohomology is compared for finite and infinite nilpotent Lie algebras, with coefficients in the adjoint representation. It turns out that, because the adjoint representation is not a highest weight representation in infinite dimension, the considered cohomology shows basic differences.

On my visit to M.I.T., B. Kostant asked the following question: What is the main difference between the cohomology of finite and infinite dimensional nilpotent Lie algebras with coefficients in the adjoint representation, at what points does the generalization of the finite dimensional situation fail?

Understanding this difference is especially important as the nilpotent Lie algebra cohomology is very hard to compute and in both finite and infinite dimensional cases only the one- and two-dimensional cohomology is known so far.

1 Comparison of the One-Dimensional Cohomology Spaces

Let us recall Kostant's result on the Lie algebra cohomology $H^1(\mathfrak{n}, \mathfrak{n})$ where \mathfrak{n} is the maximal nilpotent ideal of a Borel subalgebra of a finite dimensional simple Lie algebra \mathfrak{g} . The result can be obtained from Theorem 5.14 of [K], but Kostant never published it in an explicit form. He explained the structure of $H^1(\mathfrak{n}, \mathfrak{n})$ to Leger and Luks, who later deduced it from Theorem 5.14 of [K] and published the result in 1974.

Suppose that the dimension of the Cartan subalgebra \mathfrak{h} of \mathfrak{g} is l .

*The work was partly done during a fellowship of the Alexander von Humboldt-Stiftung at the Max-Planck-Institut für Mathematik, Bonn, Germany.

Theorem 1 ([L-L, Theorem 3.1]). *Except the Lie algebra sl_2 ,*

$$H^1(\mathfrak{n}, \mathfrak{n}) \cong \mathfrak{h} \oplus \mathfrak{h}.$$

For sl_2 , $\dim H^1(\mathfrak{n}, \mathfrak{n}) = 1$.

Let us note that in finite dimension, \mathfrak{n} is a highest weight representation. Consider the root space decomposition of \mathfrak{n} :

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

The Weyl group in this case is generated by reflections on the simple roots $s_{\alpha_1}, \dots, s_{\alpha_l}$. The one-dimensional cocycles arise from two different sources:

- (i) $D_h = \text{ad } h_{\alpha_i}$. The number of these cocycles is l .
- (ii) Let λ be a highest weight of \mathfrak{g} .

$$D'_\alpha(e_\beta) = \begin{cases} e_{s_\alpha(\lambda)} & \text{if } \beta = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

The number of such cocycles is l .

The bracket operation in the Lie algebra $H^1(\mathfrak{n}, \mathfrak{n})$ is

$$[D_h, D'_\alpha] = (s_\alpha(\lambda) - \alpha)D'_\alpha.$$

On the other hand, the result in the analogous nilpotent affine Kac–Moody cases is completely different. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the Cartan decomposition of an affine algebra $\hat{\mathfrak{g}}$. The second type (ii) of cocycles does not arise in infinite dimension, because $\hat{\mathfrak{n}}_+$ is not a highest weight representation. Instead, another algebra – now infinite dimensional – appears.

Theorem 2. *For an affine Lie algebra $\hat{\mathfrak{g}}$,*

$$H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+) \cong \hat{\mathfrak{h}} \oplus L_0,$$

where L_0 is a subalgebra of the Virasoro algebra, isomorphic to the Lie algebra of polynomial vector fields on the line, vanishing at the origin.

Remark. Theorem 2 without proof was stated in a previous work of the author with B. Feigin [F-F]. In [F] a proof was given by direct computation, counting explicitly the cocycles in $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$ in each affine case. Here we give another proof which shows more of the critical points of the difference between finite and infinite dimensional cases.

Proof. Consider the following exact sequence of $\hat{\mathfrak{n}}_+$ -modules:

$$0 \rightarrow \hat{\mathfrak{n}}_+ \rightarrow \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+ \rightarrow 0.$$

Here $\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+$ is isomorphic to $\hat{\mathfrak{h}} + \hat{\mathfrak{n}}_-$ as vector space and $\hat{\mathfrak{n}}_-$ is isomorphic to $\hat{\mathfrak{n}}_+^*$ by means of the Killing form. Consider the induced cohomology sequence:

$$\begin{aligned} H^0(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) &\rightarrow H^0(\hat{\mathfrak{n}}_+, \hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+^*) \rightarrow H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+) \rightarrow H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) \\ &\rightarrow H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+^*) \rightarrow \cdots \end{aligned}$$

As $H^0(\hat{\mathfrak{n}}_+, \hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+^*)$ is isomorphic to $\hat{\mathfrak{h}}$ and in $\hat{\mathfrak{g}}$ there are no invariant elements, we have

$$0 \longrightarrow \hat{\mathfrak{h}} \xrightarrow{\partial} H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+) \xrightarrow{\mu} H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) \xrightarrow{\nu} H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+^*) \longrightarrow \cdots$$

Each element $h_{\alpha_i} \in \hat{\mathfrak{h}}$ defines a nontrivial cohomology class in $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$ (just as in finite dimension) with the cocycle

$$D_h = \text{ad } h_{\alpha_i}, \quad h_{\alpha_i} \in \hat{\mathfrak{h}}.$$

The number of such cocycles is the number of simple roots ($= \text{rk } \hat{\mathfrak{g}}$). On the other hand, by the infinite dimensional analogue of the Bott–Kostant Theorem [F-F],

$$H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) \cong \mathbb{C}[t, t^{-1}] = \mathbb{C}[t] + t^{-1}\mathbb{C}[t^{-1}].$$

Each cohomology class can be represented by the cocycle of the form

$$f(t) \frac{\partial P}{\partial t},$$

where

$$f(t) \in \mathbb{C}[t, t^{-1}] \quad \text{and} \quad P(t) = x_0 + tx_1 + t^2x_2 + \cdots \in \hat{\mathfrak{n}}_+.$$

Proposition 1. *The kernel of the map $\nu : H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) \rightarrow H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+)$ is $t\mathbb{C}[t](\partial/\partial t)$.*

Proof. Any cocycle corresponding to a polynomial vector field $P(t)(\partial/\partial t)$ is of the form $\omega_p : f(t) \mapsto P(\partial f/\partial t)$. So, if $P \in \mathbb{C}[t](\partial/\partial t)$, then for any $f(t) \in \hat{\mathfrak{n}} = \mathfrak{n} + t\mathfrak{g} + t^2\mathfrak{g} + \cdots$, $P(t)(\partial f/\partial t)$ is in $\hat{\mathfrak{n}}_+$ and therefore $\nu(\omega_p) = 0$.

On the other hand, assume that $P \in \mathbb{C}[t^{-1}]$ and $\nu(\omega_p) = 0$. This means that for some P_0 from $\hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+$, ω_p is the differential of P_0 . By definition of the differential, for any $f \in \hat{\mathfrak{n}}_+$,

$$P \frac{\partial f}{\partial t} - [P_0, f] \in \hat{\mathfrak{n}}_+ = \mathfrak{n} + t\mathfrak{g} + t^2\mathfrak{g} + \cdots. \quad (1)$$

Let $P = \alpha_n t^{-n} + \cdots + \alpha_0$; $P_0 = A_n t^{-n} + \cdots + A_0$ where $\alpha_i \in \mathbb{C}$, $A_i \in \mathfrak{g}$. Apply (1) to $f = X_0 \in \mathfrak{n}$ (constant polynomial). Thus, for any $X_0 \in \mathfrak{n}$, $\sum_{i=0}^n [A_i, X_0] t^{-i} \in \hat{\mathfrak{n}}_+$; from this it follows that $[A_i, X_0] = 0$ for $i > 0$ and $[A_0, X_0] \in \mathfrak{n}$. So, A_i for $i > 0$ are the multiples of the highest weight vector $v_\lambda \in \mathfrak{g}$; $A_0 \in \mathfrak{h} + \mathfrak{n} \subset \mathfrak{g}$; $P_0(t) = (t^{-n}\beta_n + \cdots + t^{-1}\beta_1)v_\lambda + A_0$. Now, apply (1) to $f(t) = tX_1$ where $X_1 \in \mathfrak{g}$. We get $(\alpha_n t^{-n} + \cdots + \alpha_0)X_1 - [A_0, X_1]t - (\beta_n t^{-n} + \cdots + \beta_1 t^{-1})[v_\lambda, X_1]t \in \hat{\mathfrak{n}}_+$. Comparing coefficients near t^{-i} , we get

$\alpha_n = 0$; $\alpha_i X_1 = \beta_{i+1}[v_\lambda, X_1]$ for $i > 0$; but this may be so for any $X_1 \in \mathfrak{g}$ only when $\alpha_i = \beta_{i+1} = 0$. Also we get $\alpha_0 X_1 \equiv \beta_1[v_\lambda, X_1] \pmod{n}$ for any $X_1 \in \mathfrak{g}$, which implies $\alpha_0 = \beta_1 = 0$. So, we proved that $\ker \nu$ is exactly $t\mathbb{C}[t](\partial/\partial t)$.

Now we have the following cohomology sequence:

$$0 \longrightarrow \hat{\mathfrak{h}} \xrightarrow{\partial} H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+) \longrightarrow \mathbb{C}[t] \longrightarrow 0.$$

The second type of nontrivial cocycles from $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$ have the form

$$P \mapsto f(t)P'(t), \quad \text{where } f(t) \in \mathbb{C}[t].$$

The nonequivalent cocycles of this type form a Lie algebra, isomorphic to L_0 .

Note. The difference between the finite and infinite dimensional case is that in finite dimension, by the Bott–Kostant Theorem [K] the dimension of $H^1(\mathfrak{n}, \mathfrak{g})$ is equal to the elements of length l in the Weyl group, while here we have

$$H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) \cong \mathbb{C}[t, t^{-1}].$$

2 Method of Computation for the Two-Dimensional Cohomology

Using Kostant’s results [K], Leger and Luks [L-L] computed $H^2(\mathfrak{n}, \mathfrak{n})$ for finite dimensional simple Lie algebras \mathfrak{g} . Their main idea is the following. Consider the next exact sequences of \mathfrak{n} -modules:

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & \mathfrak{n} & & & \\ & & & \downarrow & & & \\ & & & \mathfrak{g} & & & \\ & & & \downarrow & & & \\ 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g}/\mathfrak{n} & \longrightarrow & \mathfrak{n}^* \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

These induce the following exact cohomology sequences:

$$\begin{array}{ccccccc}
& & & & H^1(\mathfrak{n}, \mathfrak{n}) & & \\
& & & & \downarrow & & \\
& & & & H^1(\mathfrak{n}, \mathfrak{g}) & & \\
& & & & \downarrow & & \\
& & & & H^2(\mathfrak{n}, \mathfrak{n}) & & \\
& & & & \downarrow & & \\
& & & & H^2(\mathfrak{n}, \mathfrak{g}) & & \\
& & & & \downarrow & & \\
& & & & H^2(\mathfrak{n}, \mathfrak{n}/\mathfrak{h}) & & \\
\end{array}$$

$H^0(\mathfrak{n}, (\mathfrak{h} + \mathfrak{n})^*) \rightarrow H^0(\mathfrak{n}, \mathfrak{n}^*) \xrightarrow{i} H^1(\mathfrak{n}, \mathfrak{h}) \xrightarrow{\partial} H^1(\mathfrak{n}, \mathfrak{h} + \mathfrak{n}^*) \xrightarrow{2l-1} H^1(\mathfrak{n}, \mathfrak{n}^*) \rightarrow H^2(\mathfrak{n}, \mathfrak{h}) \quad (*)$

$B_{\alpha\beta} = 0 \quad \left. \begin{array}{l} \text{if } \alpha \text{ or } \beta \text{ not} \\ \text{simple} \end{array} \right\}$
or
 $\left. \begin{array}{l} \text{if } \alpha \neq \beta \text{ and} \\ \alpha + \beta \notin \Delta \end{array} \right\}$

The dimensions of the cohomology spaces are marked above the spaces.

We approach the space $H^2(\mathfrak{n}, \mathfrak{n})$ step by step, studying the above diagram.

The elements of $H^0(\mathfrak{n}, \mathfrak{n}^*)$ are the invariant elements of $\mathfrak{n}^* : \{e_{-\alpha}\}$, where α is a simple root.

Cocycles in $H^1(\mathfrak{n}, \mathfrak{h})$ have the form

$$\begin{aligned}
\varphi(e_\gamma) &= 0, & \gamma &\neq \alpha \\
\varphi(e_\alpha) &= h_\beta, & \text{if } \alpha, \beta &\text{ are simple roots.}
\end{aligned}$$

There are l^2 such cocycles. Their images look the same in $H^1(\mathfrak{n}, \mathfrak{h} + \mathfrak{n}^*)$. Obviously among them the ones with the form

$$\begin{aligned}
\varphi(e_\alpha) &= h_\alpha \\
\varphi(e_\gamma) &= 0, & \gamma &\neq \alpha
\end{aligned}$$

are zero cocycles.

It is easy to see that i is embedding into $H^1(\mathfrak{n}, \mathfrak{h})$.

The space $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is generated by $\langle e_\alpha \rangle$, α simple, and $\mathfrak{h} \cong \langle h_\alpha \rangle$, α simple and the rank of such a homomorphism is l^2 .

Let us compute $\text{im } \partial$: take $L : (\langle e_\alpha \rangle, \alpha \text{ simple}) \rightarrow (\langle h_\alpha \rangle, \alpha \text{ simple})$, $L(e_\alpha) = \sum_\beta L_{\alpha\beta} h_\beta$. Then as $\text{im } \partial$ we have to take the factor

$$\begin{aligned}
&\langle L \rangle / \langle L : L_{\alpha\beta} = 0, \alpha \neq \beta \rangle. \\
\dim &= l^2 & \dim &= l
\end{aligned}$$

So from the left we get $l^2 - l$ cocycles.

3 The Spaces $H^1(\mathfrak{n}, \mathfrak{n}^*)$ and $H^2(\mathfrak{n}, \mathfrak{n})$

The main problem is to compute $H^1(\mathfrak{n}, \mathfrak{n}^*)$. The cocycles representing the cohomology classes in $H^1(\mathfrak{n}, \mathfrak{n}^*)$ are bilinear forms ϕ on \mathfrak{n} such that

$$\phi([X, Y], Z) = -\phi(Y, [X, Z]) + \phi(X, [Y, Z]).$$

Leger and Luks state [Theorem 4.1] that the cocycles are exactly the ones obtained with the help of a symmetric invariant form B ,

$$\phi(X, Y) = B(TX, Z) - B(TY, X),$$

where T is a derivation.

Note that Leger and Luks prove it for any finite dimensional Lie algebra, but the statements are true for any Lie algebra.

Now we have to count the symmetric bilinear forms on \mathfrak{n} . Their explicit form is

$$\begin{aligned} B(e_\alpha, e_\beta) &= 1, \quad \alpha, \beta \text{ simple,} \\ B(e_\gamma, e_\delta) &= 0, \quad (\gamma, \delta) \neq (\alpha, \beta) \text{ } (\gamma \text{ or } \delta \text{ is not simple).} \end{aligned}$$

The number of such forms in finite dimension is

$$l(\alpha = \alpha) + \frac{l(l-1)}{2}(\alpha \neq \beta \text{ simple}) + \frac{l(l+1)}{2} = \frac{l^2+l}{2}.$$

By Theorem 4.1 of Leger and Luks, in $H^1(\mathfrak{n}, \mathfrak{n}^*)$ we have $(l^2 + l)/2$ classes. Consider the differential

$$H^1(\mathfrak{n}, \mathfrak{n}^*) \xrightarrow{\partial} H^2(\mathfrak{n}, \mathfrak{h}) = H^2(\mathfrak{n}) \otimes \mathfrak{h}.$$

Let us compute $\ker \partial$.

Suppose that we have a functional φ on \mathfrak{n} defined by the cochain ϕ ,

$$\varphi(X)(Y) = B(TX, Y) - B(TY, X) \quad \text{if } Y \in \mathfrak{n}$$

and assume that $\varphi(X)(H) = 0$ if $H \in \mathfrak{h}$. Let us continue it onto $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{n}^*$. Then its differential is

$$\begin{aligned} \partial\varphi &= X\varphi(Y)(H) - Y\varphi(X)(H) = \varphi(Y)([X, H]) - \varphi(X)([Y, H]) \\ &= -B(TY, [X, H]) + B(TX, [Y, H]) + B(T[X, H], Y) \\ &\quad - B(T[Y, H], X). \end{aligned}$$

Put $X = e_\alpha, Y = e_\beta$, where α, β are simple positive roots.

Then $TY = l(\beta)e_\alpha$ where $l(\beta)$ is the length of the root β and $[e_\alpha, H] = \alpha[H]e_\alpha$. Then

$$\begin{aligned} -l(\beta)\alpha(H)B(e_\alpha, e_\beta) + l(\alpha)\beta(H)B(e_\alpha, e_\beta) - l(\alpha)\alpha(H)B(e_\alpha, e_\beta) \\ + l(\beta)\beta(H)B(e_\alpha, e_\beta). \end{aligned}$$

We get

$$\partial\varphi_{B_{\alpha,\beta}}(H) = (l(\alpha) + l(\beta))(\beta - \alpha)(H)B(e_\alpha, e_\beta).$$

As a consequence, if $\alpha = \beta$ then $\partial\varphi = 0$.

Let us assume $\alpha \neq \beta$. Then

$$\partial\varphi_{B_{\alpha,\beta}}(e_\alpha, e_\beta) = (l(\alpha) + l(\beta))(\check{\beta} - \check{\alpha}), \quad \text{where } \check{\alpha}, \check{\beta} \in \mathfrak{h}.$$

It is easy to see that the cocycle

$$\omega(e_\alpha, e_\beta) = 1, \quad \omega(e_\gamma, e_\delta) = 0$$

is trivial. The space $H^2(\mathfrak{n})$ is isomorphic to $\oplus\{\mathbb{C}w, l(w) = 2, w \in W\}$:

$$\partial\varphi_{B_{\alpha,\beta}} = 0 \Leftrightarrow B(e_\alpha, e_\beta) = 0 \Leftrightarrow \alpha + \beta \in \Delta.$$

So we get $\ker \partial = \{\alpha + \beta \in \Delta \text{ and } \alpha = \beta\}$.

Now we have the exact sequence

$$0 \rightarrow \{B_{\alpha\beta}\} \rightarrow H^1(\mathfrak{n}, \mathfrak{n}^*) \rightarrow H^2(\mathfrak{n}, \mathbb{C}) \rightarrow 0.$$

The basis for the image of $H^2(\mathfrak{n}, \mathbb{C})$ in $H^1(\mathfrak{n}, \mathfrak{n}^*)$ is represented by cocycles obtained by symmetric bilinear forms with $\alpha + \beta \in \Delta$, or $\alpha < \beta$ and $\alpha + \beta \notin \Delta$. So we can write

$$\begin{array}{ccc} 0 \rightarrow & (\alpha, \beta) \rightarrow H^1(\mathfrak{n}, \mathfrak{n}^* + \mathfrak{h}) & \rightarrow B_{\alpha\beta} \\ & \alpha \neq \beta & \left\{ \begin{array}{l} \alpha + \beta \notin \Delta \\ \text{or} \\ \alpha = \beta \\ \alpha, \beta \text{ simple} \end{array} \right\} \\ & & \\ & H^1(\mathfrak{n}, \mathfrak{h}) \rightarrow H^1(\mathfrak{n}, \mathfrak{n}^* + \mathfrak{h}) & \rightarrow H^1(\mathfrak{n}, \mathfrak{n}^*) \end{array}$$

from which it immediately follows that

$$\dim H^1(\mathfrak{n}, \mathfrak{n}^*) = \frac{l^2 + l}{2} + \frac{l}{2}(l + 2)(l - 1) = l^2 + l - 1$$

[L-L, Theorem 5.4].

Now it is easy to compute the nontrivial cocycles in $H^2(\mathfrak{n}, \mathfrak{n})$. The differential from $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$ to $H^2(\mathfrak{n}, \mathfrak{n})$ is a monomorphism. We know that, in finite dimension, the map

$$H^2(\mathfrak{n}, \mathfrak{n}) \rightarrow H^2(\mathfrak{n}, \mathfrak{g})$$

is an epimorphism and we also know the space $H^2(\mathfrak{n}, \mathfrak{g})$;

$$\dim H^2(\mathfrak{n}, \mathfrak{g}) = \#\{w : w \in W, l(w) = 2\},$$

and the image is equal to the kernel of the differential. The space $H^2(\mathfrak{n}, \mathfrak{g})$ is represented by the cocycles

$$f_{\alpha\beta}(e_\sigma, e_\tau) = \begin{cases} s_\beta s_\alpha(\lambda) & \text{if } (\sigma, \tau) = (\alpha + r\beta, \beta) \\ 0 & \text{otherwise,} \end{cases}$$

where α, β are simple positive roots such that

$$\alpha + \beta \in \Delta$$

or

$$\alpha < \beta \quad \text{and} \quad \alpha + \beta \notin \Delta.$$

The number of those cocycles is $\frac{1}{2}(l+2)(l-1)$.

From this and the previous considerations it follows:

Theorem 3 ([L-L, Theorem 6.4]). *If \mathfrak{g} is not of type A_1, A_2 , or B_2 then*

$$H^2(\mathfrak{n}, \mathfrak{n}) \approx H^2(\mathfrak{n}, \mathfrak{g}) \oplus H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n}).$$

Here

$$\begin{aligned} \dim H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n}) &= (2l-1) + l^2 - l, \\ \dim H^2(\mathfrak{n}, \mathfrak{g}) &= \frac{1}{2}(l+1)(l-1). \end{aligned}$$

4 The Spaces $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+^*)$ and $H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$

In the affine situation there is a different picture for $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$. It is true that all the $B_{\alpha, \beta}$ symmetric bilinear forms define a cocycle, but not only those. There is an infinite series of cocycles, namely each polynomial without constant term defines one (see [F-F]). But they do not lie in the kernel. The differential acts by the following:

$$\partial \varphi_{B_{\alpha, \beta}}(e_\alpha, e_\beta)(H) = (l(\beta) + l(\alpha))(\beta - \alpha)(H)B(e_\alpha, e_\beta)$$

with

$$B(e_\alpha, e_\beta) = \left\langle P(t^{-1}) \frac{\partial e_\alpha}{\partial t}, e_\beta \right\rangle + \left\langle P(t^{-1}) \frac{\partial e_\beta}{\partial t}, e_\alpha \right\rangle.$$

If $e_{\beta_0} = t \cdot e_{-\lambda}$ where λ is the highest weight of a representation, then

$$B(e_\alpha, t e_{-\lambda}) = \langle P(t^{-1}) e_{-\lambda}, e_\alpha \rangle = \text{Res}_t P(t^{-1}) \cdot (e_{-\lambda}, e_\alpha).$$

If $-\lambda = \alpha$ then the Killing form is nonzero.

It follows immediately that the image of the infinite series from $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}})$ to $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{h}} + \mathfrak{n}_+^*)$, and the preimage of the infinite series of cocycles in $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+^*)$ cancel each other in the affine cases, and in the diagram (*) from above and from the right we get no other cocycles in $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{h}} + \hat{\mathfrak{n}}_+^*)$ but the ones in the finite dimensional cases.

Let us summarize once more what they are. In finite dimension, the differential of the cocycles of $H^1(\mathfrak{n}, \mathfrak{g})$ is zero, while from the right in the diagram (*) we get additional nontrivial cocycles in $H^1(\mathfrak{n}, \mathfrak{g}/\mathfrak{n})$. Their number is $\frac{1}{2} \# \{ \alpha, \beta \text{ simple and } \alpha + \beta \text{ is a root} \} +$ the number of diagonal elements in the Cartan matrix of \mathfrak{g} .

The space $H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}})$ again differs in infinite dimension. Here we have

$$H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) = H^1(\hat{\mathfrak{n}}_+) \otimes \mathbb{C}[t, t^{-1}] \quad (\text{see [F-F]}).$$

Proposition 2. $H^1(\hat{\mathfrak{n}}_+) \otimes t\mathbb{C}[t]$ is in the kernel of the differential map. The sequence

$$H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+) \rightarrow H^1(\hat{\mathfrak{n}}_+) \otimes \mathbb{C}[t] \rightarrow 0$$

is exact.

Proof. This follows easily from the corresponding statement for $H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+^*)$ and from the fact that the maps

$$H^*(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+) \rightarrow H^*(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) \rightarrow H^*(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+)$$

are homomorphisms of $H^*(\hat{\mathfrak{n}}_+)$ -modules.

Now we are able to compute $H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$. We have the following cohomology sequences:

$$\begin{array}{ccccc}
& & H^1(\mathfrak{n}_+, \hat{\mathfrak{g}}) \Pi_\infty & & \\
& & \downarrow & & \\
H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{h}}) & \longrightarrow & H^1((\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}/\hat{\mathfrak{n}}_+)) & \longrightarrow & H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+^*) \Pi \begin{array}{l} \textcircled{\text{II}}_f \\ \text{II}_\infty \end{array} \\
\textcircled{\text{I}}_f & & \downarrow & & \\
& & H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+) & & \\
\textcircled{\text{III}}_\infty & & \downarrow & & \\
& & H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{g}}) & &
\end{array}$$

Theorem 4. With the exception of $\hat{\mathfrak{sl}}(2, \mathbb{C})$, the space $H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$ is the direct sum of three subspaces, coming from three kinds of cocycles I–III. The cocycles of type I_f and II_f are the same as for finite dimensional algebras. The cocycles of type II_∞ coming from above and from the right cancel each other. Cocycles of type III_∞ only appear in the affine cases. They form a space isomorphic to $H^1(\hat{\mathfrak{n}}_+) \otimes L_0$.

Remark. Cocycles of type I_f and III_∞ form the space

$$H^1(\hat{\mathfrak{n}}_+) \otimes H^1(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+).$$

(Comparing this result with the ones in [F], there they are the cocycles of type (1°).)

Cocycles of type II_f give infinitesimal deformations of type 2° and 3° in [F]:

(2°) Let $1 \leq i \leq n$. The algebra $\hat{\mathfrak{n}}_+$ deforms inside $\hat{\mathfrak{g}} = \mathfrak{g}^A$ where A is the Cartan matrix. The deformed algebra is spanned by the spaces $\mathfrak{g}_{(m_1, \dots, m_n)}^A$ with

$$(m_1, \dots, m_n) \neq (0, \dots, 0, \overset{(i)}{1}, \dots, 0)$$

and by the vector $e_i + t f_i$ where t is a parameter. The number of such cocycles is equal to the rank of $\hat{\mathfrak{g}}$.

(3°) Let $1 \leq i \leq n$; consider the entry $a_{ij} = -1$ in the Cartan matrix A , and if $a_{ji} = a_{ij}$, then $i < j$. The algebra $\hat{\mathfrak{n}}_+$ deforms again inside $\hat{\mathfrak{g}}$. The deformed algebra is generated by the spaces

$$\mathfrak{g}_{(m_1, \dots, m_n)}^A \text{ with } (m_1, \dots, m_n) \\ \neq (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0), (0, \dots, 0, \overset{(i)}{1}, 0, \dots, 0, \overset{(j)}{1}, 0, \dots, 0)$$

and the vectors $e_i + tf_j$ and $[e_i, e_j] - th_j$. The number of this type of cocycles is equal to the number of nonzero pairs (a_{ij}, a_{ji}) in the Cartan matrix with $i \neq j$.

The exceptional case $\hat{\mathfrak{sl}}(2, \mathbb{C})$ is also discussed in [F]. In this case, cocycles of type 3° do not exist. Instead, there are two additional cocycles in $H^2(\hat{\mathfrak{n}}_+, \hat{\mathfrak{n}}_+)$. This Lie algebra is really exceptional in the deformation sense also: the two above-mentioned additional cocycles cannot define an extendible deformation of $\hat{\mathfrak{n}}_+$, while all the other types of infinitesimal deformations for any affine Lie algebra are extendible.

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