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The moduli space of 4-dimensional nilpotent complex associative algebras [☆]

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In this paper, we study 4-dimensional nilpotent complex associative algebras. This is a continuation of the study of the moduli space of 4-dimensional algebras. The nonnilpotent algebras were analyzed in an earlier paper. Even though there are only 15 families of nilpotent 4-dimensional algebras, the complexity of their behavior warranted a separate study, which we give here.

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1. Construction of the algebras by extensions

The authors and collaborators have been carrying out a construction of moduli spaces of low dimensional complex and real Lie and associative algebras in a series of papers. Our method of constructing the moduli spaces of such algebras is based on the principle that algebras are either simple or can be constructed as extensions of lower dimensional algebras. There is a classical theory of extensions, which was developed by many contributors going back as early as the 1930s. In order to apply this theory, the authors interpreted the classical theory of extensions in the language of codifferentials [4], wherein we give a description of the theory of extensions of an algebra W by an algebra M. Consider the diagram

$$0 \to M \to V \to W \to 0$$

of associative $\mathbb{K}$-algebras, so that $V = M \oplus W$ as a $\mathbb{K}$-vector space, M is an ideal in the algebra V, and $W = V/M$ is the quotient algebra. Suppose that $\delta \in C^2(W)$ and $\mu \in C^2(M)$ represent the algebra structures on W and M respectively. We can view μ and δ as elements of $C^2(V)$. Let $T^n(V)$ be the n-th tensor power of V, so that $T^0(V) = \mathbb{K}$ and $T^{k+1}(V) = V \otimes T^n(V)$. Let $T^{k,l}$ be the subspace of $T^{k+l}(V)$ given recursively by

$$\begin{aligned} T^{0,0} &= \mathbb{K} \\ T^{k,l} &= M \otimes T^{k-1,l} \oplus V \otimes T^{k,l-1} \end{aligned}$$

Let $C^{k,l} = \operatorname{Hom}(T^{k,l}, M) \subseteq C^{k+l}(V)$. If we denote the algebra structure on V by d, we have

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda \in C^{1,1}$ and $\psi \in C^{0,2}$. Note that in this notation, $\mu \in C^{2,0}$. Then the condition that d is associative: $[d, d] = 0$ gives the following relations:

$$\begin{aligned} [\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] &= 0, && \text{The Maurer–Cartan equation} \\ [\mu, \lambda] &= 0, && \text{The compatibility condition} \\ [\delta + \lambda, \psi] &= 0, && \text{The cocycle condition} \end{aligned}$$

Since μ is an algebra structure, $[\mu, \mu] = 0$, so if we define D_μ by $D_\mu(\varphi) = [\mu, \varphi]$, then $D_\mu^2 = 0$. Thus D_μ is a differential on $C(V)$. Moreover $D_\mu : C^{k,l} \to C^{k+1,l}$. Let

$$Z_\mu^{k,l} = \ker\bigl(D_\mu : C^{k,l} \to C^{k+1,l}\bigr), \quad \text{the } (k,l)\text{-cocycles}$$

$$B^{k,l}_\mu = \operatorname{Im}\bigl(D_\mu : C^{k-1,l} \to C^{k,l}\bigr), \quad \text{the } (k,l)\text{-coboundaries}$$

$$H^{k,l}_\mu = Z^{k,l}_\mu / B^{k,l}_\mu, \quad \text{the } D_u \ (k,l)\text{-cohomology}$$

Then the compatibility condition means that $\lambda \in Z^{1,1}$. If we define $D_{\delta+\lambda}(\varphi) = [\delta + \lambda, \varphi]$, then it is not true that $D^2_{\delta+\lambda} = 0$, but $D_{\delta+\lambda} D_\mu = -D_\mu D_{\delta+\lambda}$, so that $D_{\delta+\lambda}$ descends to a map $D_{\delta+\lambda} : H^{k,l}_\mu \to H^{k,l+1}_\mu$, whose square is zero, giving rise to the $D_{\delta+\lambda}$-cohomology $H^{k,l}_{\mu,\delta+\lambda}$. If the pair (λ, ψ) gives rise to a codifferential d, and (λ, ψ') give rise to another codifferential d', then if we express $\psi' = \psi + \tau$, it is easy to see that $[\mu, \tau] = 0$, and $[\delta + \lambda, \tau] = 0$, so the image $\bar{\tau}$ of τ in $H^{0,2}_\mu$ is a $D_{\delta+\lambda}$-cocycle, and thus τ determines an element $\{\bar{\tau}\} \in H^{0,2}_{\mu,\delta+\lambda}$.

If $\beta \in C^{0,1}$, then $g = \exp(\beta) : \mathcal{T}(V) \to \mathcal{T}(V)$ is given by $g(m, w) = (m + \beta(w), w)$. Furthermore $g^* = \exp(-\operatorname{ad}_\beta) : C(V) \to C(V)$ satisfies $g^*(d) = d'$, where $d' = \delta + \mu + \lambda' + \psi'$ with $\lambda' = \lambda + [\mu, \beta]$ and $\psi' = \psi + [\delta + \lambda + \frac{1}{2}[\mu, \beta], \beta]$. In this case, we say that d and d' are equivalent extensions in the restricted sense. Such equivalent extensions are also equivalent as codifferentials on $\mathcal{T}(V)$. Note that λ and λ' differ by a D_μ-coboundary, so $\bar{\lambda} = \bar{\lambda}'$ in $H^{1,1}_\mu$. If λ satisfies the MC-equation for some ψ, then any element λ' in $\bar{\lambda}$ also gives a solution of the MC equation, for the ψ' given above. The cohomology classes of those λ for which a solution of the MC equation exists, determine distinct restricted equivalence classes of extensions.

Let $G_{M,W} = \mathbf{GL}(M) \times \mathbf{GL}(W) \subseteq \mathbf{GL}(V)$. If $g \in G_{M,W}$ then $g^* : C^{k,l} \to C^{k,l}$, and $g^* : C^k(W) \to C^k(W)$, so $\delta' = g^*(\delta)$ and $\mu' = g^*(\mu)$ are codifferentials on $\mathcal{T}(M)$ and $\mathcal{T}(W)$ respectively. The group $G_{\delta,\mu}$ is the subgroup of $G_{M,W}$ consisting of those elements g such that $g^*(\delta) = \delta$ and $g^*(\mu) = \mu$. Then $G_{\delta,\mu}$ acts on the restricted equivalence classes of extensions, giving the equivalence classes of general extensions. Also $G_{\delta,\mu}$ acts on $H^{k,l}_\mu$, and induces an action on the classes $\bar{\lambda}$ of λ giving a solution to the MC equation.

Next, consider the group $G_{\delta,\mu,\lambda}$ consisting of the automorphisms h of V of the form $h = g \exp(\beta)$, where $g \in G_{\delta,\mu}$, $\beta \in C^{0,1}$ and $\lambda = g^*(\lambda) + [\mu, \beta]$. If $d = \delta + \mu + \lambda + \psi + \tau$, then $h^*(d) = \delta + \mu + \lambda + \psi + \tau'$ where

$$\tau' = g^*(\psi) - \psi + \left[\delta + \lambda - \frac{1}{2}[\mu, \beta], \beta\right] + g^*(\tau).$$

Thus the group $G_{\delta,\mu,\lambda}$ induces an action on $H^{0,2}_{\mu,\delta+\lambda}$ given by $\{\bar{\tau}\} \to \{\overline{\tau'}\}$.

The general group of equivalences of extensions of the algebra structure δ on W by the algebra structure μ on M is given by the group of automorphisms of V of the form $h = \exp(\beta) g$, where $\beta \in C^{0,1}$ and $g \in G_{\delta,\mu}$. Then we have the following classification of such extensions up to equivalence.

Theorem 1.1. *(See [4].) The equivalence classes of extensions of δ on W by μ on M are classified by the following:*

(1) *Equivalence classes of* $\bar{\lambda} \in H^{1,1}_{\mu}$ *which satisfy the MC equation*

$$[\delta, \lambda] + \frac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0$$

for some $\psi \in C^{0,2}$, *under the action of the group* $G_{\delta,\mu}$.

(2) *Equivalence classes of* $\{\bar{\tau}\} \in H^{0,2}_{\mu,\delta+\lambda}$ *under the action of the group* $G_{\delta,\mu,\lambda}$.

Equivalent extensions will give rise to equivalent codifferentials on V, but it may happen that two codifferentials arising from nonequivalent extensions are equivalent. This is because the group of equivalences of extensions is the group of invertible block upper triangular matrices on the space $V = M \oplus W$, whereas the equivalence classes of codifferentials on V are given by the group of all invertible matrices, which is larger.

The fundamental theorem of finite dimensional algebras allows us to restrict our consideration of extensions to two cases. First, we can consider those extensions where δ is a semisimple algebra structure on W, and μ is a nilpotent algebra structure on M. In this case, because we are working over $\mathbb{C}$, we can also assume that $\psi = \tau = 0$. Thus the classification of extension reduces to considering equivalence classes of λ.

Secondly, we can consider extensions of the trivial algebra structure $\delta = 0$ on a 1-dimensional space W by a nilpotent algebra μ. This is because a nilpotent algebra has a codimension 1 ideal M, and the restriction of the algebra structure to M is nilpotent. However, in this case, we cannot assume that ψ or τ vanishes, so we need to use the classification theorem above to determine the equivalence classes of extensions. In many cases, solving the MC equation for a particular λ, if there is any ψ yielding a solution, then $\psi = 0$ also gives a solution, so the action of $G_{\delta,\mu,\lambda}$ on $H^{0,2}_{\mu}$ takes on a simpler form than the general action we described above.

In addition to the complexity which arises because we cannot take the cocycle term ψ in the extension to be zero, there is another issue that complicates the construction of the extensions. If an algebra is not nilpotent, then it has a maximal nilpotent ideal which is unique, and it can be constructed as an extension of a semisimple algebra by this unique ideal. Both the semisimple and nilpotent parts in this construction are completely determined by the algebra. Therefore, a classification of extensions up to equivalence of extensions will be sufficient to classify the algebras. This means that the equivalence classes of the module structure λ determine the algebras up to isomorphism.

For nilpotent algebras, we don't have this assurance. The same algebra structure may arise by extensions of the trivial algebra structure on a 1-dimensional space by two different nilpotent algebra structures on the same $n-1$-dimensional space.

In addition, the deformation theory of the nilpotent algebras is far more involved than the deformation theory of the nonnilpotent algebras. Thus, we decided to discuss the nilpotent 4-dimensional complex algebras in a separate paper from the discussion of the nonnilpotent ones.

In this paper, we study the complex 4-dimensional nilpotent algebras. In [1], the following idea for construction of nilpotent algebras was discussed, and since it is simpler than the general construction, we outline the idea here.

First, if a nonzero algebra is nilpotent, then it has a nontrivial ideal M which has the property that the product of any element in M with an arbitrary element is zero. We call such an ideal completely trivial. There is a unique ideal which is maximal in the set of completely trivial ideals, which we call the kernel of the algebra. If we call the quotient of the algebra by its kernel the core of the algebra, then we see that any nilpotent algebra determines a unique kernel and core, and the algebra is given by an extension of its core by the kernel in a particularly simple manner. In the language we introduced above, we have $\lambda = 0$ and $\mu = 0$, so that the compatibility relation and the Maurer Cartan equation are satisfied trivially, and in terms of the coboundary operator D_δ given by $D_\delta(\varphi) = 0$, we have that the cocycle ψ is actually a D_δ-cocycle, and equivalent cocycles determine equivalent algebras, so that the algebras are classified by the action of the group $G_{\delta,\mu,\lambda}$ on the D_δ-cohomology classes.

All of this holds for any completely trivial ideal, and it is more convenient to study the case when we don't assume M is the kernel, even though this means that some of the algebras will be constructed in a nonunique manner. In particular, let us consider the case where δ vanishes as well as μ and λ. Then the group $G_{\mu,\delta,\lambda}$ coincides with the group $G_{M,W}$. If we express $M = \langle e_1 \rangle$, $W = \langle e_2, \dots, e_n \rangle$, then we can express $\psi = \psi_1^{i+1,j+1} c_{i,j}$, and an element g in $G_{M,W}$ is given in block form by a matrix $G = \left[\begin{smallmatrix} g_{1,1} & 0 \\ 0 & G_2 \end{smallmatrix}\right]$. If $C = [c_{i,j}]$ is the $n \times n$ matrix determined by the coefficients of ψ, and $\psi' = G^*(\psi)$ is the cocycle determined by the action of G on ψ, and $C' = [c'_{i,j}]$ is the corresponding matrix of coefficients of ψ', then $C' = g_{1,1}^{-1} G^T C G$, so that C' and C are *cogredient* matrices. Therefore, the classification of cogredient matrices, or equivalently, complex bilinear forms, is a useful component of the construction.

In the next section, we classify complex bilinear forms on a 2-dimensional complex vector space and then in the following section, we use this classification to give a construction of the nilpotent 3-dimensional complex algebras, which agrees with the classification in [6]. Then in the succeeding sections we use the same idea to construct the nilpotent 4-dimensional complex algebras.

2. Classification of bilinear forms on a 2-dimensional complex vector space

A complete classification of complex bilinear forms on a finite dimensional space was given in [8]. However, this classification does not involve a stratification of the moduli space by complex projective orbifolds, which we believe is the right way to understand this classification. In this paper, we will give a stratification of the moduli space of 2 and 3 dimensional complex bilinear forms by projective orbifolds.

Let β be a bilinear form on $\mathbb{C}^2$ which is given by the matrix $B = (b_{ij})$ where $b_{ji} = \beta(e_i, e_j)$ in terms of some basis $\langle e_1, e_2 \rangle$. We say that two matrices B and C are said to be cogredient if there is a nonsingular matrix G such that $C = P^T BP$. This is precisely the condition that B and C represent β with respect to different bases.

Then every nontrivial bilinear form is given up to equivalence by either a matrix of the form $B(p : q) = \left[\begin{smallmatrix} 1 & p \\ q & 0 \end{smallmatrix}\right]$ or $C = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Moreover, the matrices of the form $B(p : q)$ form a projective family parameterized by the orbifold $\mathbb{P}^1/\Sigma_2$, where the action of the symmetric group Σ_2 on $\mathbb{P}^1$ is given by interchanging the projective coordinates $(p : q)$, so that $B(p : q) \sim B(q : p)$ where $\sim$ stands for the equivalence of cogredient matrices.

To see why this is true, we note first that if β is nontrivial and $\beta(u, u) = 0$ for all $u \in \mathbb{C}^2$, then $\beta(v, u) = -\beta(u, v)$ for all $u, v \in \mathbb{C}^2$, since $0 = \beta(u + v, u + v) = \beta(u, v) + \beta(v, u)$. Since β is nontrivial, there must be some u and v such that $\beta(v, u) = 1$, and the matrix of β with respect to this basis is C.

Thus we may assume there is some u in $\mathbb{C}^2$ such that $\beta(u, u) = 1$. We claim that there is a nonzero vector v such that $\beta(v, v) = 0$. For suppose that we choose any vector v which is linearly independent of u. If $\beta(v, v) \neq 0$, let $w = u + xv$, and then we compute

$$\beta(u, w) = 1 + \big(\beta(u, v) + \beta(v, u)\big)x + \beta(v, v)x^2.$$

This is a quadratic equation in x which has a nontrivial solution $x \neq 0$. In terms of the basis u, w, the matrix of β has the form $B(p : q)$ for some $(p : q)$. Moreover, it is easy to check that $B(p : q) \sim B(x : y)$ precisely when $(p : q) = (x : y)$ or $(p : q) = (y : x)$.

If one studies the miniversal deformation of the moduli space of bilinear forms, one discovers that elements of the form $B(p : q)$ have smooth deformations in a neighborhood of the element $B(p : q)$ and do not have jump deformations, while the element C has a jump deformation to $B(1 : -1)$ and smooth deformations in a neighborhood of $B(1 : -1)$. We shall see that the deformations of the moduli space of bilinear forms are reproduced in the deformations of the moduli space of three dimensional complex algebras which are determined by these bilinear forms. For basic notions of deformations see [2,3].

3. Nilpotent 3-dimensional complex associative algebras

The moduli space of complex 3-dimensional associative algebras was constructed in [6]. Here we wish to construct the nilpotent algebras using the following observations. In [1], the author discusses what he calls central extensions of an associative algebra, which are extensions such that the ideal in the extended algebra has trivial multiplication with the entire algebra. This language derives from the similar construction of central extensions of Lie algebras. It is pointed out that if a finite dimensional associative algebra is nilpotent, then there is a unique ideal consisting of all elements which have trivial product with any element of the algebra. Let us call this ideal the kernel of the algebra, and the quotient of the algebra by this ideal the core of the algebra. It follows that the core and the kernel of an algebra are uniquely determined. This gives a fairly reasonable process of constructing the nilpotent algebras without overlaps. In this section, we show how to use the classification of bilinear forms as a tool in constructing the 3-dimensional nilpotent algebras.

There is only one nontrivial nilpotent 2-dimensional complex algebra, represented by the codifferential $\delta = \psi_2^{33}$, on the space $W = \langle e_2, e_3 \rangle$. If we extend the codifferential δ by a completely trivial ideal $M = \langle e_1 \rangle$, then this extension is given by a cocycle $\psi = \psi_1^{22} c_1 + \psi_1^{23} c_2 + \psi_1^{32} c_3 + \psi_1^{33} c_4$. The cocycle condition $[\delta, \psi] = 0$ gives $c_1 = 0$ and $c_2 = c_3$. Moreover, if $\beta = \varphi_1^2 b_1 + \varphi_1^3 b_2 \in C^{1,0}$, then $[\delta, \beta] = -\psi_1^{33} b_1$, which means that up to a coboundary term, we can assume $\psi = (\psi_1^{2,3} + \psi_1^{3,2}) c_3$. Applying an element of the group $G_{\delta,\mu,\lambda}$ to ψ replaces ψ by an arbitrary nonzero multiple, so this means we only have two cases to study, when $c_3 = 1$ or $c_3 = 0$. The first case gives the codifferential $d = \psi_2^{33} + \psi_1^{23} + \psi_1^{32}$, which is equivalent to the algebra d_{19} on our list of 3-dimensional algebras. The second case gives $d = \psi_2^{33}$, which is equivalent to $d_{20}(0 : 0)$ on our list, but note that this algebra has a kernel which is 2-dimensional, so this algebra would arise in a different fashion as well.

Next, consider the trivial nilpotent 2-dimensional complex algebra given by $\delta = 0$. In this case the cocycle condition on $\psi = \psi_1^{22} c_1 + \psi_1^{23} c_2 + \psi_1^{32} c_3 + \psi_1^{33} c_4$ is trivial. Let $C = \left[\begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix}\right]$. An element g of the group $G_{\delta,\mu,\lambda}$ is given by an arbitrary invertible matrix G of the block diagonal form $G = \operatorname{diag}(g_1, G_2)$ where $g_1 \neq 0$ and G_2 is an invertible 2×2 matrix. The action of g on ψ transforms ψ into the element ψ' whose matrix is $g_1^{-1} G_2^T C G_2$, which means that if the matrices representing ψ and ψ' are cogredient then they determine equivalent algebras. Using our classification of cogredient matrices, we find that we obtain a family $d_{20}(p : q)$ given by the matrix $B(p : q)$ in our classification of bilinear forms on a 2-dimensional complex vector space, and the algebra d_{21}, given by the matrix C in this classification.

This gives all of the nontrivial 3-dimensional nilpotent complex associative algebras in a very simple fashion.

4. Classification of bilinear forms on a 3-dimensional complex vector space

If β is a bilinear form on an n-dimensional space U, then we say that β is decomposable if $U = V \oplus W$, where V and W are nontrivial subspaces of U satisfying $\beta(V, W) = \beta(W, V) = 0$. Using the classification of 2-dimensional complex bilinear forms, it is easy to see that every matrix B representing a nontrivial bilinear form on a complex 3-dimensional space is cogredient to a matrix of one of the six types given below.

$$B_1(p:q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & q & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$B_3(p:q) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & q & 0 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$B_5 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad B_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The first four matrices correspond to the decomposable bilinear forms, and it is easy to see that every bilinear form is cogredient to one of these four matrix types. It is not as easy to see that any indecomposable bilinear form is cogredient to one of the last two matrices.

The matrix $B_1(0:0)$ is cogredient to the matrix $B_3(1:1)$. Moreover, the families $B_1(p:q)$ and $B_3(p:q)$ are parametrized by $\mathbb{P}^1/\Sigma_2$, so they determine projective orbifolds. Other than these identifications, all of the matrices represent distinct equivalence classes. As a consequence, we see that the moduli space of complex bilinear forms on a 3-dimensional vector space is stratified by projective orbifolds. It is also true that the deformations of the elements in the moduli space are either given by jumps between the strata, by smooth deformations which factor through a jump deformation, or by smooth deformations along a stratum. This pattern is consistent with the patterns that we have observed in moduli spaces of algebras.

5. Stratification of the nilpotent algebras

The nilpotent algebras are divided into 15 different strata. There are 4 projective 1-parameter families of algebras: $d_{75}(p:q)$, $d_{78}(p:q)$, $d_{83}(p:q)$ and $d_{86}(p:q)$, where $(p:q)$ is a projective coordinate. For some of these families, there is also an action of the symmetric group Σ_2, given by permutation of the coordinates, so that the algebra associated to the parameter $(p:q)$ is isomorphic to the algebra given by the parameter $(q:p)$. In this case, the family is parameterized by $\mathbb{P}^1/\Sigma_2$. Otherwise the family is parameterized by $\mathbb{P}^1$. The 11 algebras d_{73}, d_{74}, d_{76}, d_{77}, d_{80}, d_{81}, d_{82}, d_{79}, d_{84}, d_{85} and d_{87} each determine a stratum consisting of a single algebra. We will give a description of each of the strata below (see Table 1).

6. The fifteen families of codifferentials

6.1. Type 73

The algebra $d_{73} = \psi_2^{31} + \psi_2^{13} + \psi_1^{33} + \psi_2^{34} + \psi_2^{44}$ has a 1-dimensional kernel $M = \langle e_2 \rangle$. Its core is the 3-dimensional nilpotent algebra $d_{20}(0:0) = \psi_2^{33}$. The algebra d_{73} is isomorphic to its opposite algebra. We compute

$$H^2(d_{73}) = \langle \delta^1, \delta^2, \delta^3 \rangle,$$

where

$$\begin{aligned}
\delta^1 &= -\psi_2^{1,1} - \psi_2^{2,3} - \psi_2^{3,2} + \psi_1^{3,4} + \psi_1^{4,4} \\
\delta^2 &= -\psi_2^{1,1} + \psi_2^{2,3} + \psi_2^{3,2} + 2\psi_3^{3,3} + \psi_4^{3,4} + \psi_4^{4,3} \\
\delta^3 &= -2\psi_4^{1,3} - 2\psi_4^{3,1} + 2\psi_2^{3,2} + \psi_1^{1,4} + 3\psi_2^{2,4} + \psi_3^{3,4} - 2\psi_4^{3,4} + \psi_1^{4,1} + 3\psi_2^{4,2} + \psi_3^{4,3}
\end{aligned}$$

Table 1
The cohomology of the algebras $d_{73}\dots d_{87}$.

Codifferential	H^0	H^1	H^2	H^3
$d_{73}=\psi^{31}_2+\psi^{13}_2+\psi^{33}_1+\psi^{34}_2+\psi^{44}_2$	0|2	2|0	0|3	4|0
$d_{74}=\psi^{33}_2+\psi^{41}_2+\psi^{43}_1+\psi^{14}_2+\psi^{34}_1+\psi^{44}_3$	0|4	4|0	0|4	4|0
$d_{75}(p:q)=\psi^{33}_2+p\psi^{43}_1+q\psi^{34}_1+\psi^{44}_2$	0|2	3|0	0|5	8|0
$d_{75}(1:1)=\psi^{33}_2+\psi^{34}_2+\psi^{44}_1$	0|2	3|0	0|6	10|0
$d_{75}(1:-1)=\psi^{42}_1+\psi^{24}_3+\psi^{44}_1$	0|2	3|0	0|6	11|0
$d_{75}(1:0)=\psi^{33}_2+\psi^{43}_1+\psi^{44}_2$	0|2	3|0	0|5	8|0
$d_{75}(0:0)=\psi^{33}_2+\psi^{44}_2$	0|4	8|0	0|17	41|0
$d_{76}=\psi^{31}_2+\psi^{13}_2+\psi^{33}_1+\psi^{44}_2$	0|4	5|0	0|6	8|0
$d_{77}=\psi^{31}_2+\psi^{13}_2+\psi^{33}_1+\psi^{34}_2$	0|2	3|0	0|6	10|0
$d_{78}(p:q)=\psi^{33}_2+p\psi^{41}_2+q\psi^{14}_2+\psi^{34}_2+\psi^{44}_2$	0|2	3|0	0|6	12|0
$d_{78}(1:1)=\psi^{33}_2+\psi^{41}_2+\psi^{14}_2+\psi^{34}_2+\psi^{44}_2$	0|2	3|0	0|6	12|0
$d_{78}(1:-1)=\psi^{33}_2+\psi^{41}_2-\psi^{14}_2+\psi^{34}_2+\psi^{44}_2$	0|2	3|0	0|6	12|0
$d_{78}(1:0)=\psi^{33}_2+\psi^{41}_2+\psi^{34}_2+\psi^{44}_2$	0|2	3|0	0|8	23|0
$d_{78}(0:0)=\psi^{33}_2+\psi^{34}_2+\psi^{44}_2$	0|2	6|0	0|15	37|0
$d_{79}=\psi^{31}_2+\psi^{13}_2+\psi^{33}_1$	0|4	6|0	0|10	18|0
$d_{80}=\psi^{33}_2+\psi^{43}_1-\psi^{34}_1+\psi^{44}_2$	0|2	4|0	0|9	17|0
$d_{81}=-\psi^{31}_2+\psi^{13}_2+\psi^{44}_2$	0|2	5|0	0|10	18|0
$d_{82}=\psi^{33}_2+\psi^{43}_1+\psi^{34}_1+\psi^{44}_2$	0|4	6|0	0|10	18|0
$d_{83}(p:q)=p\psi^{42}_1+q\psi^{24}_1+\psi^{44}_1+\psi^{44}_3$	0|2	5|0	0|9	16|0
$d_{83}(1:1)=\psi^{42}_1+\psi^{24}_1+\psi^{44}_1+\psi^{44}_3$	0|4	7|0	0|12	21|0
$d_{83}(1:-1)=\psi^{42}_1-\psi^{24}_1+\psi^{44}_1+\psi^{44}_3$	0|2	5|0	0|10	18|0
$d_{83}(1:0)=\psi^{42}_1+\psi^{44}_1+\psi^{44}_3$	0|2	5|0	0|11	21|0
$d_{83}(0:0)=\psi^{44}_1+\psi^{44}_3$	0|4	10|0	0|28	82|0
$d_{84}=-\psi^{31}_2+\psi^{13}_2+\psi^{14}_2$	0|2	4|0	0|10	26|0
$d_{85}=\psi^{31}_2+\psi^{13}_2+\psi^{33}_2+\psi^{44}_2$	0|4	7|0	0|14	28|0
$d_{86}(p:q)=p\psi^{42}_1+q\psi^{24}_1+\psi^{44}_1$	0|2	6|0	0|15	37|0
$d_{86}(1:1)=\psi^{42}_1+\psi^{24}_1+\psi^{44}_1$	0|4	8|0	0|17	41|0
$d_{86}(1:-1)=\psi^{42}_1-\psi^{24}_1+\psi^{44}_1$	0|2	6|0	0|15	38|0
$d_{86}(1:0)=\psi^{42}_1+\psi^{44}_1$	0|2	6|0	0|19	52|0
$d_{86}(0:0)=\psi^{44}_1$	0|4	10|0	0|28	82|0
$d_{87}=-\psi^{31}_2+\psi^{13}_2$	0|2	8|0	0|17	42|0

More precisely, the cohomology classes of these coderivations give a basis of H^2, but it is convenient to identify the cohomology classes with *representative cocycles*, which give a pre-basis of the cohomology.

The third order deformation is versal, with four relations

$$t_3^2=0,\qquad t_2t_3=0,\qquad t_2t_3=0,\qquad t_2^2(t_1+t_3)=0.$$

Note that the number of relations is precisely the dimension of H^3, and that the third relation is redundant. From the first relation, we see that in any deformation, $t_3=0$. Since t_3 vanishes by the first relation, the second and third relations give no additional constraints. Finally, the fourth relation gives that either $t_1=0$ or $t_2=0$. When $t_1=0$, we obtain a 1-parameter deformation

$$\begin{aligned} d_{t_2} &= \psi_2^{31} + \psi_2^{13} + \psi_1^{33} + \psi_2^{34} + \psi_2^{44} - t_2\psi_2^{11} + t_2\psi_2^{23} + t_2\psi_2^{32} + 2t_2\psi_3^{33} + t_2\psi_4^{34} \\ &\quad + t_2\psi_4^{43} - t_2^2\psi_1^{11} - t_2^2\psi_2^{12} - t_2^2\psi_2^{21} - t_2^2\psi_3^{13} - t_2^2\psi_3^{31} - t_2^2\psi_4^{14} - t_2^2\psi_4^{41} \end{aligned}$$

which is a deformation to a neighborhood of the codifferential $d_{37}(1:1)$. In fact, one can show that $d_{t_2} \sim d_{37}(x:y)$ where $t_2 = -\frac{(x-y)^2}{xy}$, whenever $t_2 \neq 0$. From this expression, we see that as $t_2 \to 0$, the algebra $d_{t_2}(x:y) \to d_{37}(1:1)$. However, when $t_2 = 0$, we obtain the original algebra d_{73}. What happens is that the matrix which expresses the isomorphism between the algebras d_{t_2} and $d_{37}(x:y)$ becomes singular when $x = y$.

The second solution of the relations, given by setting $t_2 = 0$, gives a 1-parameter deformation in a neighborhood of $d_{38}(1:1)$. As in the case of the first solution, we do not obtain a deformation to the algebra $d_{38}(1:1)$.

There are no jump deformations to either $d_{37}(1:1)$ or $d_{38}(1:1)$, which we found surprising, since in our previous constructions, we have always observed that when a member of one stratum deforms smoothly to a neighborhood of a point in a different stratum, that the smooth deformation factored through a jump deformation to that point, which just means that there is a jump deformation to the point.

6.2. *Type 74*

The algebra $d_{74} = \psi_2^{33} + \psi_2^{41} + \psi_1^{43} + \psi_1^{34} + \psi_2^{14} + \psi_3^{44}$ has a 1-dimensional kernel $M = \langle e_2\rangle$. Its core is the 3-dimensional nilpotent algebra $d_{19} = \psi_1^{43} + \psi_1^{34} + \psi_3^{44}$. This algebra is commutative.

When we compute a miniversal deformation, we obtain that the fourth order deformation is versal, and all the relations vanish. The miniversal deformation gives jump deformations to d_2, d_6, d_7, d_8, d_9, d_{31}, d_{32}, d_{35}, d_{36}, d_{49}, and d_{50}.

It is interesting to note that all the algebras to which d_{74} deforms are also commutative. It is well-known that in order for an algebra to deform to a commutative algebra, it must be commutative. However, it is possible, and even common, for commutative algebras to deform to noncommutative algebras.

6.3. *Type 75(p : q)*

The family of algebras $d_{75}(p:q)$ are generically given by the codifferentials $\psi_2^{33} + p\psi_1^{43} + q\psi_1^{34} + \psi_2^{44}$. However, for certain special values of the parameters, the codifferential is given by a different codifferential. We have $d_{75}(1:-1) = \psi_1^{43} + \psi_2^{34} + \psi_1^{44}$ and $d_{75}(1:1) = \psi_2^{33} + \psi_2^{34} + \psi_1^{44}$. The family is parameterized by $\mathbb{P}^1/\Sigma_2$, so that $d_{75}(p:q) \sim d_{75}(q:p)$. It might be possible to find a family $d(p:q)$ of codifferentials which give a parametrization of the entire family, but we did not discover such a representation. In all cases, the algebra $d_{75}(p:q)$ has a 2-dimensional kernel $M = \langle e_1, e_2\rangle$ and its core is the trivial nilpotent 2-dimensional algebra.

For projective families of algebras, generically, the dimension of H^n does not vary, and there is a generic pattern for the deformations. There are a few values of $(p:q)$

for which the pattern varies from the generic case, in that the dimension of H^n may be larger, and there may be extra deformations. Usually, these special values are $(1:1)$, $(1:-1)$, $(1:0)$ and $(0:0)$. In fact, $(0:0)$ is always special, and there are always jump deformations from the $(0:0)$ codifferential to every codifferential in the family. When there is no action of Σ_2 on the family, then the codifferential corresponding to $(p:q)$ is not generically isomorphic to that for $(q:p)$, the value $(0:1)$ is also special. Occasionally, there are additional special values.

6.3.1. *Generic case*

In the case of $d_{75}(p:q)$, generically, there are jump deformations to d_3, d_4 and d_5, as well as smooth deformations along the family in a neighborhood of $d_{75}(p:q)$.

6.3.2. $(p:q)=(1:0)$

The algebra $d_{75}(1:0)$ has additional jump deformations to the algebras $d_{22},\cdots,d_{30}$ and the algebras $d_{45},\cdots,d_{48}$.

6.3.3. $(p:q)=(1:-1)$

The algebra $d_{75}(1:-1)$ has additional jump deformations to d_{14}, d_{15}, d_{16}, d_{22}, d_{23}, d_{27}, d_{28}, d_{29}, d_{30}, d_{58}, and d_{59}.

6.3.4. $(p:q)=(1:1)$

The algebra $d_{75}(1:1)$ has additional jump deformations to d_{22}, d_{23}, d_{27}, d_{28}, d_{29}, d_{30}, $d_{37}(x:y)$ for all $(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$ for all $(x:y)$ except $(1:1)$ and $(0:0)$, and d_{73}.

6.4. $(p:q)=(0:0)$

The algebra $d_{75}(0:0)$ representing the "generic element" in $\mathbb{P}^1$, will automatically have jump deformations to every element in the family $d_{75}(p:q)$ except itself. In fact, it is not uncommon for the generic element to coincide with an element in some other family, and this happens in this case, as $d_{75}(p:q)\sim d_{86}(1:1)$. As a consequence, we also know that $d_{75}(0:0)$ will deform in a neighborhood of $d_{86}(1:1)$. Our general method of listing the codifferentials is organized around the principle that the order should be preserved by deformations in the sense that elements only deform to elements whose numbers are lower. However, the generic element in a family cannot be expected to preserve that principle, as in some sense, it is on a higher level than members of its family.

In addition, we also expect the generic element to have jump deformations to all algebras to which any other member of its family jumps. In fact, the algebra $d_{75}(0:0)$ has jump deformations to $d_1,\cdots,d_9,d_{14},d_{15},d_{16},d_{20},\cdots,d_{36},d_{37}(x:y),d_{38}(x:y),d_{39}$, $d_{40},d_{45},\cdots,d_{50},d_{57},\cdots,d_{60},d_{65},d_{68},d_{73},d_{74},d_{75}(x:y),d_{76},d_{77},d_{78}(x:y)$ except $(0:0)$, $d_{83}(1:1),d_{80},d_{82},\cdots,d_{85}$, as well as deforming in a neighborhood of $d_{83}(1:1)$.

6.5. *Type 76*

The algebra $d_{76} = \psi_2^{31} + \psi_2^{13} + \psi_1^{33} + \psi_2^{44}$ has a 1-dimensional kernel $M = \langle e_2 \rangle$. Its core is the 3-dimensional nilpotent algebra $d_{20}(0:0) = \psi_1^{33}$. It is commutative. It has jump deformations to d_2, d_6, d_7, d_8, d_9, d_{31}, d_{32}, d_{35}, d_{36}, $d_{37}(1:1)$, $d_{38}(1:1)$, d_{49}, d_{50}, d_{73}, and d_{74}, as well as deforming smoothly in a neighborhood of $d_{37}(1:1)$ and $d_{38}(1:1)$. With the exception of d_{73}, the algebras it jumps to are all commutative, but the algebras $d_{37}(x:y)$ and $d_{38}(x:y)$ to which it deforms smoothly are not in general commutative.

6.6. *Type 77*

The algebra $d_{77} = \psi_2^{31} + \psi_2^{13} + \psi_1^{33} + \psi_2^{34}$ has a 1-dimensional kernel $M = \langle e_2 \rangle$. Its core is the 3-dimensional nilpotent algebra $d_{20}(0:0) = \psi_1^{33}$. It is isomorphic to its opposite algebra. It has jump deformations to $d_3, d_4, d_5, d_{22}, \cdots, d_{30}, d_{37}(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$ except $(1:1)$ and $(0:0), d_{45}, \cdots, d_{48}, d_{58}, d_{59}$, and d_{73}.

6.7. *Type 78(p : q)*

The family of algebras $d_{78}(p:q)$ are given by the codifferentials $\psi_2^{33} + p\psi_2^{41} + q\psi_2^{14} + \psi_2^{34} + \psi_2^{44}$. The family is parameterized by $\mathbb{P}^1/2$, but the linear automorphism which transforms $d_{78}(p:q)$ to $d_{78}(q:p)$ cannot be given in a completely generic form. When $p=1$ and $q=0$ or $p=1$ and $q=2$, the matrix which gives the equivalence does not fall into the same pattern as will work generically. None of the algebras in this family are commutative, owing to the ψ_2^{34} term in the codifferential representing the algebra.

Generically, the kernel of this algebra is the 1-dimensional ideal $M = \langle e_2 \rangle$, and its core is the trivial 3-dimensional algebra. For the element $d_{78}(0:0)$, the dimension of the kernel is 2, and we also have that $d_{78}(0:0) \sim d_{86}(1:\gamma)$, where γ is a primitive 6-th root of unity.

The special cases of the parameters $(p:q)$ include not only the standard values $(1:1)$, $(1:-1)$, $(1:0)$ and $(0:0)$, but also the values $(1:2)$ and $(1:\gamma)$, where γ is a primitive 6th root of unity.

6.7.1. *Generic case*

Generically, an algebra in the family $d_{78}(p:q)$ has jump deformations to d_1, $d_{37}(p:q)$ and $d_{38}(p:q)$, as well as smooth deformations in a neighborhood of $d_{37}(p:q)$, $d_{38}(p:q)$ and $d_{78}(p:q)$.

6.7.2. $(p:q) = (1:2)$

We mentioned something unusual in this special case already, namely that the matrices which give rise to the isomorphisms between $d_{78}(1:2)$ and $d_{78}(2:1)$ do not fit the generic pattern. Other than this, the deformation pattern is entirely generic.

6.7.3. $(p : q) = (1 : 0)$

The algebra $d_{78}(1 : 0)$, in addition to the generic deformations, has jump deformations to $d_1, d_3, d_4, d_5, d_{22}, \cdots, d_{30}, d_{39}, d_{40}$, and $d_{45}, \cdots, d_{48}$.

6.7.4. $(p : q) = (1 : -1)$

The algebra $d_{78}(1 : -1)$ has jump deformations to d_{20} and d_{21} in addition to the generic deformations.

6.7.5. $(p : q) = (1 : 1)$

The algebra $d_{78}(1 : 1)$ does not quite follow the generic pattern in that while it does deform in a neighborhood of $d_{37}(1 : 1)$ and $d_{38}(1 : 1)$, it does not jump to either one of them. In fact, since both $d_{37}(1 : 1)$ and $d_{38}(1 : 1)$ are commutative, while $d_{78}(1 : 1)$ is not commutative, it has no possibility of having a jump deformation to either of them. It does have a jump deformation to d_1, which fits the generic pattern.

6.8. Type 79

The algebra $d_{79} = \psi_2^{31} + \psi_2^{13} + \psi_1^{33}$ has a 2-dimensional kernel $M = \langle e_2, e_4\rangle$. Its core is the nontrivial 2-dimensional nilpotent algebra d_6. It is commutative. It has jump deformations to d_2–d_9, d_{22}–d_{36}, $d_{37}(x : y)$, $d_{38}(x : y)$, both for all $(x : y)$, d_{45}–d_{50}, d_{58}, d_{59}, d_{65}, d_{68}, d_{73}, d_{74}, d_{76}, and d_{77}.

6.9. Type 80

The algebra $d_{80} = \psi_2^{33} + \psi_1^{43} - \psi_1^{34} + \psi_2^{44}$ has a 2-dimensional kernel $M = \langle e_1, e_2\rangle$ and its core is the trivial nilpotent 2-dimensional algebra. In fact, this element is the $(1 : -1)$ case of the family of codifferentials $\psi_2^{33} + p\psi_1^{43} + q\psi_1^{34} + \psi_2^{44}$, which correspond to $d_{75}(p : q)$ in most cases.

The reason this element is relegated to a different position than one would expect is that there are two codifferentials which deform in a neighborhood of $d_{75}(1 : -1)$. To decide which one actually belongs in the family, we consider the following data. First, the one in the family has the smaller cohomology H^2. Secondly, this element deforms to the other one, and we would expect that the one which does not deform to the other actually belongs in the family.

This algebra has jump deformations to $d_1, d_3, d_4, d_5, d_{14}, d_{15}, d_{16}, d_{20}, \cdots, d_{23}, d_{27}, \cdots, d_{30}, d_{39}, d_{40}, d_{57}, \cdots, d_{60}$, and $d_{75}(1 : -1)$.

6.10. Type 81

The algebra $d_{81} = \psi_2^{13} - \psi_2^{31} + \psi_2^{44}$ has a 1-dimensional kernel $M = \langle e_2\rangle$. Its core is the 3-dimensional trivial nilpotent algebra. It is isomorphic to its opposite algebra. It has jump deformations to d_1, d_{20}, d_{21}, $d_{37}(1 : -1)$, $d_{38}(1 : -1)$, d_{40}, d_{51}, d_{52}, $d_{78}(1 : -1)$ and deforms in neighborhoods of the algebras $d_{37}(1 : -1)$, $d_{38}(1 : -1)$ and $d_{78}(1 : -1)$.

6.11. *Type 82*

The algebra $d_{81} = \psi_2^{33} + \psi_1^{43} + \psi_1^{34} + \psi_2^{44}$ has a 2-dimensional kernel $M = \langle e_1, e_2 \rangle$. Its core is the 2-dimensional trivial nilpotent algebra. It is isomorphic to its opposite algebra. It has jump deformations to $d_3, d_4, d_5, d_{22}, d_{23}, d_{27}, \cdots, d_{30}, d_{37}(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$ except $(1:1)$ and $(0:0)$, d_{73}, $d_{75}(1:1)$ and deforms in a neighborhood of $d_{75}(1:1)$.

6.12. *Type 83(p : q)*

The family of algebras $d_{83}(p:q)$ are given by the codifferentials $p\psi_1^{42}+q\psi_1^{24}+\psi_1^{44}+\psi_3^{44}$. The family is parameterized by $\mathbb{P}^1$, and does not have an action of $\mathbb{Z}_2$, so that $d_{83}(p:q)$ is not, in general, isomorphic to $d_{83}(q:p)$. Generically, the kernel of this algebra is the 2-dimensional ideal $M = \langle e_1, e_3 \rangle$, and its core is the trivial 2-dimensional algebra. For the element $d_{83}(0:0)$, the dimension of the kernel is 3, and we also have that $d_{83}(0:0) \sim d_{86}(0:0)$.

The special cases of the parameters $(p:q)$ include not only the standard values $(1:1)$, $(1:-1)$, $(1:0)$ and $(0:0)$, but also $(0:1)$ as $d_{83}(1:0)$ is not isomorphic to $d_{83}(1:0)$, so must be treated separately.

6.12.1. *Generic case*

Generically, an algebra in the family $d_{83}(p:q)$ has jump deformations to $d_3, d_4, d_5, d_{14}, d_{15}, d_{16}, d_{22}, \cdots, d_{30}, d_{37}(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$, except $(1:1)$ and $(0:0)$, $d_{45}, \cdots, d_{48}, d_{58}, d_{59}, d_{73}, d_{75}(x:y)$ except $(0:0)$, and d_{77} as well as smooth deformations in a neighborhood of $d_{83}(p:q)$.

6.12.2. $(p:q) = (1:0)$

The algebra $d_{83}(1:0)$, in addition to the generic deformations has jump deformations to d_{11}, d_{12}, d_{42}, d_{44}, d_{67}, and d_{69}.

6.13. $(p:q) = (0:1)$

The algebra $d_{83}(0:1)$, in addition to the generic deformations has jump deformations to d_{10}, d_{13}, d_{41}, d_{43}, d_{66}, and d_{70}. Note that these are precisely the opposite algebras to the extra deformations of the algebra $d_{83}(1:0)$.

6.14. $(p:q) = (1:-1)$

The algebra $d_{83}(1:-1)$ has jump deformations to d_1, d_{20}, d_{21}, d_{39}, d_{40}, d_{51}, d_{52}, d_{60} and d_{80}, in addition to the generic deformations.

6.15. $(p:q)=(1:-1)$

In addition to the generic deformations, the algebra $d_{83}(1:1)$ has jump deformations to $d_2, d_6, d_7, d_8, d_9, d_{31}, \cdots, d_{36}, d_{37}(1:1), d_{38}(1:1), d_{49}, d_{50}, d_{65}, d_{68}, d_{74}$, and d_{82}. Note that except for d_{65} all of these algebras are commutative.

6.16. Type 84

The algebra $d_{84} = \psi_2^{13} - \psi_2^{31} + \psi_2^{14}$ has a 1-dimensional kernel $M = \langle e_2 \rangle$. Its core is the trivial 3-dimensional nilpotent algebra. It has jump deformations to d_1, d_3, d_4, d_5, d_{14}, d_{15}, d_{16}, d_{20}–d_{30}, $d_{37}(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$ except $(1:1)$ and $(0:0)$, d_{39}, d_{40}, d_{45}–d_{48}, d_{57}–d_{60}, d_{73}, d_{77}, $d_{78}(x:y)$ except $(0:0)$.

6.17. Type 85

The algebra $d_{85} = \psi_2^{31} + \psi_2^{13} + \psi_2^{33} + \psi_2^{44}$ has a 1-dimensional kernel $M = \langle e_2 \rangle$. Its core is the trivial 3-dimensional nilpotent algebra. It has jump deformations to $d_3, d_4, d_5, d_{22}, d_{23}, d_{27}, \cdots, d_{30}, d_{37}(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$ except $(1:1)$ and $(0:0)$, and d_{73}.

6.18. Type 86$(p:q)$

The family of algebras $d_{86}(p:q) = \psi_1^{42}p + \psi_1^{24}q + \psi_1^{44}$ is parameterized by $\mathbb{P}^1/\Sigma_2$. The algebras are not commutative, except for $d_{86}(1:1)$ and $d_{86}(0:0)$. Generically, the elements in this family have a 2-dimensional kernel $M = \langle e_1, e_3 \rangle$, and their core is the trivial 2-dimensional algebra, except that $d_{86}(0:0)$ has 3-dimensional kernel and 1-dimensional core.

6.18.1. Generic case

Generically, $d_{86}(p:q)$ has jump deformations to d_1, d_3, d_4, d_5, d_{14}, d_{15}, d_{16}, d_{20}–d_{30}, $d_{37}(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$ except $(1:1)$ and $(0:0)$, d_{39}, d_{40} d_{45}–d_{48}, d_{57}–d_{60}, d_{73}, $d_{75}(x:y)$ except $(0:0)$, d_{77}, $d_{78}(x:y)$ except $(0:0)$, $d_{83}(p:q)$, d_{80}, and d_{84}, as well as deforming smoothly in a neighborhood of $d_{83}(p:q)$ and $d_{86}(p:q)$.

6.18.2. $(p:q)=(1:0)$

The algebra $d_{86}(1:0)$ has additional jump deformations to $d_{10}, \cdots, d_{14}, d_{41}, d_{42}, d_{43}$, $d_{61}, d_{62}, d_{63}, d_{77}$ and d_{84}.

6.18.3. $(p:q)=(1:1)$

The algebra $d_{86}(1:1)$ has additional jump deformations to $d_2, d_6, d_7, d_8, d_9, d_{31}, \cdots$, $d_{36}, d_{49}, d_{50}, d_{65}, d_{68}, d_{74}, d_{76}, d_{82}$ and d_{79}. With the exception of d_{65}, all of these additional jump deformations are to commutative algebras, to which it is not possible for generic elements of the family to deform.

6.18.4. $(p:q)=(1:-1)$

The algebra $d_{86}(1:1)$ has additional jump deformations to d_{51}, d_{52} and d_{81}.

6.19. Type 87

The algebra $d_{87}=-\psi_2^{31}+\psi_2^{13}$ has a 2-dimensional kernel $M=\langle e_2,e_4\rangle$. Its core is the trivial 2-dimensional nilpotent algebra. It has jump deformations to d_1,d_3,d_4,d_5, $d_{14},d_{15},d_{16},d_{20},\cdots,d_{30},d_{37}(x:y)$ except $(1:1)$ and $(0:0)$, $d_{38}(x:y)$ except $(1:1)$ and $(0:0),d_{39},d_{40},d_{45},\cdots,d_{48},d_{51},\cdots,d_{54},d_{57},\cdots,d_{60},d_{73},d_{75}(x:y)$ except $(0:0)$, d_{77}, $d_{78}(x:y)$ except $(0:0)$, $d_{83}(1:-1)$, d_{80}, d_{81}, d_{84} and $d_{86}(1:-1)$ and deforms in a neighborhood of $d_{83}(1:-1)$ and $d_{86}(1:-1)$.

6.20. How we constructed the algebras

In Section 1, we discussed the fact that any nilpotent algebra has a unique kernel, consisting of all elements whose product with any element is zero, and core, consisting of the quotient of the algebra by its kernel, so that, in principle, it should be easier to construct all algebras of a given dimension by considering extensions of algebras by ideals in this manner.

However, our motivation for the constructions is not simply to give a list of all algebras, but to give a decomposition of the moduli space into strata which are dictated by deformation theory. A single stratum may include algebras whose kernels do not all have the same dimension, so this fact alone dictates the need for another method of identifying the strata. Moreover, our constructions have revealed that the strata seem to consist of projective orbifolds, and it is necessary to identify these orbifolds in some natural way. This has led us to the following method of construction.

We first need to know all $(n-1)$-dimensional nilpotent algebras, and then we use the fact that there is always a codimension 1 ideal in any nilpotent algebra to realize all algebras as extensions of the trivial 1-dimensional algebra by an $(n-1)$-dimensional nilpotent algebra. In the case of dimension 4 complex nilpotent algebras, there are 3 nontrivial nilpotent 3-dimensional algebras, as well as the trivial nilpotent algebra, which come into play. Here we will consider how our algebras arise by this method of construction.

6.21. Extensions by the nilpotent algebra $d_{19}=\psi_2^{31}+\psi_2^{13}+\psi_1^{33}$

The algebras d_{73}, d_{76}, d_{77}, and d_{79} all arise as extensions of the trivial algebra by the 3-dimensional algebra d_{19}. From the compatibility condition $[\mu,\lambda]=0$ and the Maurer Cartan equation, we can reduce λ to the form $\lambda=\psi_2^{34}x$, where x can be chosen to be either 1 or 0, ψ can be taken to be 0, and τ is of the form $\tau = \psi_2^{44}c$, where again, c is either 0 or 1. This gives 4 distinct possibilities, which are the four algebras listed above.

6.22. Extensions by the nilpotent algebra $d_{20}(p:q) = \psi_2^{13}p + \psi_2^{31}q + \psi_2^{33}$

The algebras d_{73}, d_{74}, $d_{75}(x:y)$, d_{76}, $d_{78}(x:y)$, $d_{83}(1:1)$, $d_{83}(0:0)$, d_{80}, d_{81}, d_{82}, d_{84}, d_{85}, and $d_{86}(x:y)$ arise as extensions of the trivial algebra by the nilpotent algebra $d_{20}(p:q)$. In this case, the compatibility condition gives rise to more than one solution, and these solutions depend on the variable $(p:q)$. For the special values $(1:0)$ and $(0:0)$, there are nongeneric solutions in addition to the generic case.

Let us examine the generic case first. In this case, we can reduce λ to the form $\lambda = \psi_2^{43}r + \psi_2^{14}s$, where $(r:s)$ is a projective coordinate, assume $\psi = 0$, and $\tau = \psi_2^{44}c$. However, after further analysis, we obtain that the algebras which arise in this fashion are d_{81}, d_{84}, d_{85} and $d_{86}(p:q)$. Thus the projective coordinates $(r:s)$ don't arise in the final description of the algebra.

When $(p:q) = (1:0)$, we obtain the same format for λ and τ, and the algebras we obtain are $d_{78}(x:y)$ except $(0:0)$, d_{84} and $d_{86}(1:0)$.

Finally, when $(p:q) = (0:0)$, we obtain 4 distinct solutions for the format of λ. For example, in one of the patterns, we have $\lambda = \psi_2^{41}p + \psi_2^{14}q + \psi_2^{34}u$, where $(p:q)$ is a projective coordinate, and the choice of u can be reduced to $u = 1$ or $u = 0$. Thus we obtain a new projective coordinate, and this coordinate is retained in some of the solutions, so we obtain new projective families of algebras which don't arise from the old projective families. After some analysis, we obtain that algebras arising in this case are d_{73}, d_{74}, $d_{75}(x:y)$, d_{76}, $d_{78}(x:y)$, $d_{83}(1:1)$, $d_{83}(0:0)$, d_{80}, d_{82}, and d_{85}.

6.23. Extensions by the nilpotent algebra $d_{21} = \psi_2^{13} - \psi_2^{31}$

The algebras $d_{78}(1:-1)$, d_{81}, d_{84}, and d_{87} all arise as extensions of the nilpotent 3-dimensional algebra d_{21}.

6.24. Extensions by the trivial 3-dimensional nilpotent algebra

The algebras d_{77}, $d_{83}(x:y)$, d_{80}, d_{79}, d_{84}, $d_{86}(x:y)$, and d_{87} arise as extensions of the trivial 1-dimensional algebra by the trivial 3-dimensional nilpotent algebra.

From the description above, it is clear that many of these algebras arise as extensions of the trivial algebra by different 3-dimensional nilpotent algebras.

7. Commutative algebras

There are 20 distinct nonnilpotent commutative algebras, of which 9 are unital (see Table 2). Every commutative algebra is a direct sum of algebras which are ideals in quotients of polynomial algebras. Every finite dimensional unital commutative algebra

Table 2
Nilpotent 4-dimensional commutative algebras.

Codifferential	Structure
d_{74}	$x\mathbb{C}[x]/(x^5)$
$d_{75}(1:1)$	$(x,y) \le \mathbb{C}[x,y]/(x^2-y^2, yx^2, xy^2)$
$d_{75}(0:0) = d_{86}(1:1)$	$\mathbb{C}_0 \oplus (x,y) \le \mathbb{C}_0 \oplus \mathbb{C}[x,y]/(x^2-y^2, xy)$
d_{76}	$(x,y) \le \mathbb{C}[x,y]/(y^3-x^2, xy)$
d_{79}	$\mathbb{C}_0 \oplus x\mathbb{C}[x]/(x^4)$
$d_{83}(1:1)$	$(x,y) \le \mathbb{C}[x,y]/(y^2, x^2y, x^3)$
$d_{83}(0:0) = d_{86}(0:0)$	$\mathbb{C}_0^2 \oplus x\mathbb{C}[x]/(x^3)$
d_{85}	$(x,y,z) \le \mathbb{C}[x,y,z]/(x^2-y^2, y^2-yz, xy, xz, z^2)$
d_0	$\mathbb{C}_0^4$

Table 3
The levels of the algebras.

Level	Codifferentials
1	1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 39, 53, 54, 55, 56
2	6, 7, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 40, 41, 42, 43, 44, 57
3	8, 9, 31, 32, 45, 46, 47, 48, 58, 59, 60, 61, 62, 63, 64
4	33, 34, 35, 36, 49, 50, 66, 67, 69, 70
5	$37(p:q)$, $38(p:q)$, 65, 68, 74
6	$37(0:0)$, $38(0:0)$, 51, 52, 73, $75(p:q)$
7	71, 72, 76, 77, $78(p:q)$, 80, 82, 85
8	79, 81, $83(p:q)$, 84
9	$75(0:0)$, $78(0:0)$, $86(p:q)$
10	$83(0:0)$, $86(0:0)$, 87

is a quotient of a polynomial algebra, while every finite dimensional nonunital algebra is an ideal in such an algebra. In [5] we gave a table showing the nonnilpotent commutative algebras, which we will not reproduce here.

These algebras were classified by Hazlett [7], and also given in [9]. There are 8 nontrivial nilpotent commutative algebras.

We note that commutative algebras may deform into noncommutative algebras, but noncommutative algebras never deform into commutative algebras. The fact that commutative algebras have noncommutative deformations plays an important role in physics, and deformation quantization describes a certain type of deformation of a commutative algebra into a noncommutative one.

8. Levels of algebras

We give a table showing the levels of each algebra (see Table 3). For completeness, we include the levels of the nonnilpotent algebras, so that the reader can see how the nilpotent algebras fit into the general picture. To define the level, we say that a rigid algebra has level 1, an algebra which has only jump deformations to an algebra on level one has level two, and so on. To be on level $k+1$, an algebra must have a jump deformation to an algebra on level k, but no jump deformations to algebras on a level higher than k. For families, if one algebra in the family has a jump to an element on level k, then we place the entire family on at least level $k+1$. Thus, even though generically, elements of the family $d_{37}(p:q)$ deform only to members of the same family, there is an element in the family which has a jump to an element on level 4. For the generic element in a family, we consider it to be on a higher level than the other elements because it has jump deformations to the other elements in its family.

9. Analysis

The reader may note that every element in the family $d_{86}(p:q)$ has a jump deformation to every element in the families $d_{75}(x:y)$ and $d_{78}(x:y)$. As a consequence, it is difficult to say which element in the family $d_{86}(p:q)$ really should be the $(0:0)$ element in those families. Of course, for the choice of codifferentials representing $d_{75}(p:q)$, the corresponding member of the family is precisely the element $d_{86}(1:1)$, but this can be an artifact of the form of the codifferential.

To illustrate this point, we consider the following family of algebras, given by the codifferentials $d(p:q)=\psi_2^{13}+\psi_2^{33}+p\psi_2^{41}+q\psi_2^{34}$. It is easily checked that $d(q:p)\sim d(p:q)$, and that the family is given by projective coordinates. In fact, $d(p:q)\sim d_{78}(x:y)$ where $\frac{q}{p}=\frac{xy}{x^2+xy+y^2}$, so in general, the family $d(p:q)$ and $d_{78}(x:y)$ determine the same collection of algebras. However, $d(0:0)$ is isomorphic to $d_{86}(1:0)$, not $d_{86}(1:\alpha)$ (where α is a primitive 6th root of unity), so we see that there may be no natural way to identify the generic element in a projective family as a specific codifferential.

What is important is that there always is an algebra corresponding to the ‘so called’ generic element of the $\mathbb{P}^1$ which parameterizes a family.

There are two additional features of this moduli space that did not arise in the lower dimensional moduli spaces of ordinary complex associative algebras. The first is that we were unsuccessful in describing the family $d_{75}(p:q)$ with a single family of codifferentials with parameters p and q.

The second feature we saw was that for the algebras d_{73} and $d_{75}(1:1)$, there were deformations in a neighborhood of elements of another family, without there being a jump to the point in whose neighborhood the smooth deformation occurred.

The authors and collaborators have written a series of articles describing moduli spaces of Lie, associative, L_∞, and A_∞ algebras. It first became clear when studying the 3-dimensional complex Lie algebras that there was a stratification of the moduli space by projective orbifolds, each of the form $\mathbb{P}^n$, possibly with an action of Σ_{n+1}, given by interchanging coordinates. Our example of 4-dimensional complex associative algebras fits this pattern nicely. We have been conjecturing that this type of stratification by projective orbifolds ought to hold in general, but as of yet, do not have the tools to prove this.

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