MODULI SPACES OF LOW DIMENSIONAL REAL LIE ALGEBRAS

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ABSTRACT. Moduli spaces of low dimensional complex Lie algebras have a unique stratification, determined by deformation theory, in which each stratum has the structure of a projective orbifold, in fact, it is given by an action of a finite group on complex projective space. For low dimensional real Lie algebras, the picture is similar, except that the strata are orbifolds given by the action of finite groups on spheres. We give a complete decomposition of the moduli spaces of real three and four dimensional Lie algebras, and compare the structure of these moduli spaces to that of the corresponding complex Lie algebras.

1. INTRODUCTION

In this paper, we study the moduli spaces of three and four dimensional real Lie algebras. The classification of real and complex Lie algebras of low dimension has been going on for a long time. Sophus Lie himself classified all complex Lie algebras of dimension three or less as early as 1897 [11], and the real Lie algebras of dimension three were

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classified by Bianchi in 1918 [3]. The complex and real four dimensional Lie algebras were classified by Mubarakzanov in 1963 [12, 13]. Since then, many authors have studied the four dimensional Lie algebras, and several different notations for the algebras, as well as divisions into families have appeared in the literature (see for example [4, 1, 18]).

In [7], the moduli space of complex three dimensional Lie algebras was studied from the point of view of deformation theory. Versal deformations of each of the algebras were constructed, and these were used to divide the algebras into families. It emerged that the classical division into families was not completely consistent with the picture that arises from deformation theory. Moreover, the moduli space was seen to have a completely unique, natural stratification, with each stratum consisting of either a singleton point, or a projective orbifold given by the action of a finite group on $\mathbb{P}^1$.

In [8], the authors studied the complex four dimensional Lie algebras using the same approach, and again discovered that the moduli space has a unique stratification by projective orbifolds, given by the action of the symmetric group on $\mathbb{CP}^1$ and $\mathbb{CP}^2$.

The authors have also studied moduli spaces of low dimensional L_∞ and associative algebras, and have observed the same pattern of stratification by projective orbifolds.

In [10], the deformations of real three dimensional Lie algebras were calculated. For a study of contractions of low dimensional Lie algebras, see [14]. In [9], the versal deformation of real Lie algebras of dimension three was studied, for the purpose of studying the relationship between contractions of Lie algebras and jump deformations. It was noticed that the stratification seemed to be based not on projective geometry, as in the complex case, because the stratum is constructed from a quotient of $\mathbb{R}^2$ by $\mathbb{R}_+$ instead of $\mathbb{R}^*$. We speculated at the time that the pattern for real moduli spaces might be described by orbifolds given by an action of a finite group on S^n, rather than $\mathbb{P}^n$, as in the complex case.

In this paper, we study the moduli spaces of real three and four dimensional real Lie algebras. It turns out that in both cases, the moduli space has a unique stratification by orbifolds, each of which is given by the action of a finite group on either S^2 or S^1. The manner in which the spheres S^2 and S^1 are assembled from the elements in the moduli space is a bit surprising.

Our results in this paper lead us to speculate that moduli spaces of real Lie algebras of a fixed dimension may have a decomposition into orbifolds based on actions of finite groups on spheres.

2. Preliminaries

In general, an orbifold is given by gluing together sets which are obtained by the action of a group on either $\mathbb{R}^n$ or $\mathbb{C}^n$. The simplest case, where the orbifold is given by the quotient of a manifold by a finite group is the only one we need for this paper. An orbifold point is a point of an orbifold whose stabilizer is nontrivial, in other words, it is fixed by some non identity element.

A typical example of the type of orbifold we encounter in this paper is $\mathbb{CP}^n$ quotiented out the action of the symmetric group Σ_{n+1} given by permuting the projective coordinates $(z_0 : \cdots : z_n)$. In the case of Σ_2 acting on $\mathbb{P}^1$, the points $(1 : 1)$ and $(1 : -1)$ are the only orbifold points.

A 1-parameter deformation of a Lie algebra d_0 is a family d_t of Lie algebras, depending smoothly on the parameter t. We call the deformation a jump deformation if for every nonzero value of the parameter t, the deformed Lie algebras d_t are isomorphic, but not isomorphic to d_0. (Note that in the real case, the deformation parameter may be defined only on a half-interval with 0 as an endpoint.) A deformation for which the algebras d_t are nonisomorphic for distinct values of the parameter t is called a smooth deformation.

A Lie algebra structure on a vector space W is a map $d : \bigwedge^2 W \to W$, which is a codifferential in the space of all coderivations of the exterior algebra $\bigwedge W$, in other words, it satisfies the Jacobi identity. In general, a map $\varphi : \bigwedge^n W \to W$ is called an n-cochain, so a Lie algebra is given by a 2-cochain d which satisfies the codifferential condition $[d, d] = 0$ in the graded Lie algebra of coderivations of $\bigwedge W$.

A formal 1-parameter deformation of a Lie algebra structure d is a 2-cochain of the form

$$d_t = d + t\delta_1 + t^2\delta_2 \cdots ,$$

where the δ_i are 2-cochains, and d_t is required to formally satisfy the Jacobi condition. This means that

$$D(\delta_{n+1}) = -\tfrac{1}{2} \sum_{k=1}^{n} [\delta_k, \delta_{n+1-k}] = 0,$$

for $n = 1\ldots$, where $D(\varphi) = [d, \varphi]$ is the coboundary operator which determines the cohomology $H(d)$ of the Lie algebra. We will call this formal deformation an actual deformation if the series converges in some neighborhood of 0.

One can also consider deformations in which more than one parameter appears, or more generally, with a complete local algebra base. We refer the reader to [5]. Following that article, we call a deformation over a complete local algebra base versal (miniversal) if it induces all nonequivalent deformations and its infinitesimal part is universal. A construction of versal deformations was given in [6].

If we let $H^2 = \langle\delta^1, \cdots, \delta^n\rangle$, where we identify H^2 with a subspace of the 2-cocycles, then the universal infinitesimal deformation d^{inf} is given by

$$d^{\mathrm{inf}} = d + t_i\delta^i,$$

where $t_1, \cdots, t_n$ are parameters. The universality of the deformation has to do with the fact that it uniquely determines all infinitesimal deformations. Up to first order, $[d^{\mathrm{inf}}, d^{\mathrm{inf}}] = 0$, but there are higher order corrections and relations on the parameters that are necessary in order to construct a versal deformation

$$d^\infty = d + t_i\delta^i + t_it_j\varphi^{ij} + \cdots,$$

which satisfies $[d^\infty, d^\infty] = 0$ if certain relations on the parameters are satisfied. These relations are called the relations on the base of the versal deformation, and all formal deformations can be derived from the versal deformation. In particular, if we substitute values for the parameters $t_1, \cdots, t_n$ which satisfy the relations, we may obtain an actual deformation; that is, the series may converge to yield a Lie algebra.

For simple cases, the method described above can be applied in practice. For example, for 3-dimensional Lie algebras, the infinitesimal deformation is always versal, at least for a good choice of basis of H^2. However, there is another method for computing the versal deformation which is often more effective.

If the space B^3 of 3-coboundaries has basis β^i, then there is a subspace P^2 of 2-cochains, with basis γ^i, such that $D(\gamma^i) = -\frac{1}{2}\beta^i$. The two spaces P^2 and B^3 are isomorphic, and so have the same dimension. Let H^3 have basis α^i, and let P^3, with basis τ^i, be a subspace of 3-cochains mapping isomorphically to the space B^4 of 4-coboundaries. Then the versal deformation can be expressed in the form

$$d^\infty = d + \delta^i t_i + \gamma^j x_j,$$

where x_j is a formal power series of order at least 2 in the parameters t_i. Formally, one has

$$[d^\infty, d^\infty] = \alpha^i r_i + \beta^i s_i + \tau^i u_i,$$

where s_i and u_i are formal power series in the parameters t_i.

Because of the method of construction of the versal deformation, the coboundary terms in this bracket must vanish; *i.e.*, $s_i = 0$ for all i. Since there are exactly as many terms s_i as x_i, and $s_i = x_i$ plus quadratic terms of the form $t_i x_j$ and $x_i x_j$, one can, in principle, solve these equations to express the x_j as functions of the t_i. Plugging the solutions for the x_i into the expressions r_i, one obtains the relations on the base of the versal deformation. By the general construction in [6], the terms u_i automatically will lie in the ideal generated by the r_i, in other words, they vanish in the base of the versal deformation.

What makes the idea above attractive, even though it is not guaranteed to have a computable solution, is that in practice, the solution can be obtained in a closed form, not as a formal power series, but as a rational function of the parameters, so that the relations on the base are given by a set of rational expressions. This means that the relations can be solved, and the values of the parameters which give these solutions can be plugged into the expression for d^∞ to obtain an actual Lie algebra. It is precisely the solutions for small values of the parameters t_i that we use to determine a local neighborhood of the Lie algebra .

Using the versal deformations, we determined neighborhoods of a Lie algebra as follows. If the Lie algebra d has a jump to the Lie algebra d', then d' is not in its neighborhood. Moreover, if d' has a smooth family of deformations d'_t, it will happen that there is a corresponding smooth family d_t of deformations of d. These points are also not in the neighborhood of d. We say in this case that the smooth deformation d_t factors through a jump deformation. A neighborhood of a point d is determined by the Lie algebras which can be obtained by smooth deformations that do not factor through jumps. If the versal deformation of d is given in terms of the parameters $t_1, \cdots, t_n$, then the points

$(t_1, \cdots, t_n)$ in some small neighborhood of 0 which arise from values satisfying the relations on the base determine a local neighborhood of d, if we exclude those values which correspond to jump deformations and those which factor through jump deformations.

It may happen that there are no smooth deformations of an algebra which do not factor through jumps. In this case, we say that the algebra is isolated. In particular, rigid Lie algebras are always isolated, because they have no deformations. In the case where an algebra has smooth deformations, these smooth deformations determine a family, which, in our investigations, seems to always have a nice structure of an orbifold. Every isolated Lie algebra and family of Lie algebras determines a stratum.

For finite dimensional vector spaces, there are only a finite number of strata, which can be organized into levels. A family or isolated point is said to be at level k if its only deformations are to elements of the same family or to elements of lower level, with at least one member of the family having a deformation to an element of level $k-1$. For level 0, we require simply that the elements deform only along their own family. Actually, in the cases we study, and probably in general, level 0 consists of the rigid Lie algebras.

For simplicity of presentation, we don't provide many details about the computations of the versal deformations, limiting ourselves to pointing out some interesting cases.

A Lie algebra of dimension n can be represented by an $n \times \binom{n}{2}$ matrix in the following manner. Given a basis $\{e_1, \cdots, e_n\}$ of a vector space

W, take the basis of $\wedge^2 W$ given by $e_1e_2, e_1e_3, e_2e_3, e_1e_4, e_2e_4, e_3e_4, \cdots$. Then the column of A corresponding to the basis element e_ie_j of $\bigwedge^2 W$ is just the column vector representing $[e_i, e_j]$. We can also encode this information in codifferential form, $d = \psi^{ij}_k c^k_{ij}$, where c^k_{ij} are the structure constants of the Lie algebra, so that $[e_i, e_j] = c^k_{ij} e_k$, and $\psi^{ij}_k : \bigwedge^2 W \to W$ is given by $\psi^{ij}_k(e_me_n) = \delta^{ij}_{mn} e_k$.

In classifying real and complex Lie algebras, the extensions of the trivial n-dimensional Lie algebra by a 1-dimensional Lie algebra are always classified by the equivalence classes of an $n \times n$ matrix under similarity, up to multiplication of the set of eigenvalues by a nonzero constant. As a consequence, similarity classes of such matrices play an important role. Over the complex field, one can use the Jordan decomposition, but over the reals, one has to use the rational canonical form for the matrix, or some variant of this form.

The original classification of real Lie algebras of low dimension in [13], essentially used the Jordan form of the matrix, which leads to a somewhat different decomposition of the moduli space. In [17], the Frobenius normal form was used to classify solvable Lie algebras of dimension less than or equal to 4 over perfect fields. For the complex case, the Jordan normal form was used in [2] to study Lie algebras of dimension up to 4, and produces a similar decomposition of the moduli space as we give, except for issues, which we will describe later, related to whether to include the defective or nondefective matrices in the main family.

3. Complex Three Dimensional Lie Algebras

In [7, 16], the moduli space of three dimensional complex Lie algebras was studied. Let us note that three dimensional complex lie algebras have a very simple decomposition into three special algebras, given by codifferentials which we denoted in [7] by d_1, d_2 and d_3, and one family $d_2(r:s)$, which is determined by projective parameters $(r:s)\in\mathbb{P}^1$, up to an action of the symmetric group Σ_2 given by interchanging r and s. In other words, the family $d_2(r:s)$ is parameterized by the *complex projective orbifold* $\mathbb{P}^1/\Sigma_2$.

The codifferential d_1, given by the matrix $\left[\begin{smallmatrix}0&0&1\\0&0&0\\0&0&0\end{smallmatrix}\right]$ represents the nilpotent algebra $\mathfrak{n}_3(\mathbb{C})$, d_2, given by the matrix $\left[\begin{smallmatrix}0&1&0\\0&0&1\\0&0&0\end{smallmatrix}\right]$, represents the solvable algebra $\mathfrak{r}_{3,1}(\mathbb{C})$, d_3, given by the matrix $\left[\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\right]$, represents the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, while $d_2(r:s)$, given by the matrix $\left[\begin{smallmatrix}0&r&1\\0&0&s\\0&0&0\end{smallmatrix}\right]$, represents the algebra $\mathfrak{r}_{3,r/s}$ unless $r=s$, when it represents the algebra $\mathfrak{r}_3(\mathbb{C})$.

In the classical decomposition of the moduli space, the Lie algebra $\mathfrak{r}_{3,r/s}$ is given by the matrix $\left[\begin{smallmatrix}0&r&0\\0&0&s\\0&0&0\end{smallmatrix}\right]$, which seems more natural than our decomposition, until one considers the deformation theory. Already, in the complex case, we found that it was necessary to rearrange the families from the classical description in order to give the correct decomposition of the moduli space dictated by deformation theory. The change is quite modest, we simply interchange the roles of d_2 and $d_2(1:1)$ so that $d_2(1:1)$ belongs to the 1-parameter family of Lie algebras instead of d_2.

The algebras $d_2(r:s)$, d_1 and d_2 arise when considering extensions of the 1-dimensional complex Lie algebra by the trivial 2-dimensional Lie algebra, The matrices of these codifferentials vanish except for a 2×2 matrix in the upper right hand part of the 3×3 matrix representing the codifferential. The equivalence classes of the codifferentials represented by these matrices are given by the equivalence classes of 2×2 matrices under conjugation, up to multiplication by a nonzero constant. As is well known, the equivalence classes of complex matrices under conjugation are completely determined by the Jordan canonical form.

The codifferentials $d_2(r:s)$ are given by matrices with eigenvalues r and s, except that when $r=s$ the matrix is the Jordan 2-block with eigenvalue 1. The codifferential d_1 comes from the Jordan 2-block with eigenvalue 0, while d_2 comes from the identity matrix. Although it may seem more natural to group the identity matrix with the family coming from matrices with 2 distinct eigenvalues, as in the classical decomposition, the uniqueness of decomposition given by deformations, as described in [8, 16], dictates otherwise.

Figure 1 below illustrates the moduli space of complex 3-dimensional Lie algebras.

In the figure, the plane represents the stratum $d_2(r:s)$, which is parameterized by $\mathbb{CP}^1/\Sigma_2$. The jump deformations from d_1 to this stratum are represented by a tetrahedron. The jump deformations from d_2 to $d_2(1:1)$ and from $d_2(1:-1)$ to d_3 are represented by arrows.

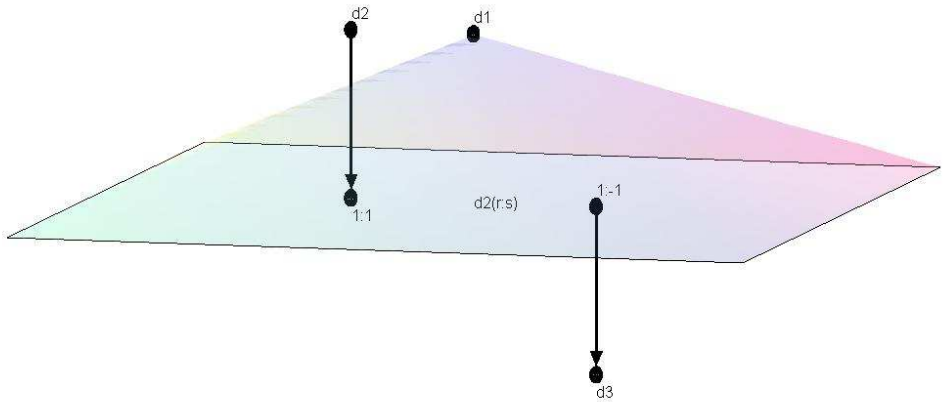

Figure 1. The Moduli Space of complex 3-dimensional Lie Algebras.

4. Real Three Dimensional Lie Algebras

In [9], the moduli space of three dimensional real Lie algebras was analyzed. We summarize the results as follows. As in the complex case, the central extensions of the trivial 1-dimensional Lie algebra by a 2-dimensional vector space are determined by equivalence classes of 2×2 matrices, representing the upper right hand side of the 3×3 matrix of the codifferential, under similarity. Because we are considering real matrices, these similarity classes are characterized by the rational canonical form of the matrix, rather than the Jordan decomposition, as in the complex case. After a careful analysis of the deformations we arrived at the following decomposition of the moduli space into strata.

There is a family $d_2(r:s)$ which is represented by the matrix $\left[\begin{smallmatrix} 0 & r \\ |r| & s \end{smallmatrix}\right]$. Moreover, the points $(r:s)$ are not projective, since the equivalence relation is given by an action of $\mathbb{R}_+$, rather than $\mathbb{R}_*$, on $\mathbb{R}^2$. In other words, $(r,s) \sim q(r,s)$ when $q \in \mathbb{R}_+$. There is also an action of $\mathbb{Z}_2$

on this family given by negating the s parameter. Thus the family is parameterized by $S^1/\mathbb{Z}_2$. We denote this real family by $d_2(r:s)$, since it captures all the real codifferentials coming from the complex family $d_2(r:s)$.

We also obtain the codifferentials d_1 and d_2, which are the same as in the complex case. However, the codifferential d_3 corresponds now to the real Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, and there is a new codifferential $\tilde{d}_3 = \psi_3^{12} - \psi_2^{13} + \psi_1^{23}$, corresponding to the simple real Lie algebra $\mathfrak{su}_2$. In the complex case, the codifferential $d_2(1:-1)$ had a jump deformation to d_3. The two real codifferentials $d_2(\pm 1 : 0)$ both correspond to the complex codifferential $d_2(1:-1)$. The codifferential $d_2(-1:0)$ deforms to both d_3 and $\tilde{d}_3$, while $d_2(1:0)$ deforms only to d_3. Note that $d_2(\pm 1:0)$ are the only real codifferentials in the family $d_2(r:s)$ which come from the same complex codifferential. It is interesting to note that these points are also the only orbifold points, that is, points whose stabilizer is not simply the identity.

The codifferential $d_2(-1:2)$ corresponds to the complex codifferential $d(1:1)$, which is interesting because d_2 has a jump deformation to $d(1:1)$, so it deforms to the real codifferential $d_2(-1:2)$ as well.

The table below gives the cohomology of the elements of the moduli space of real three dimensional Lie algebras. We also give a comparison to the decomposition of the moduli space given by Levy-Nahas in [10], in terms of the structures A_1–A_9, as well as to the standard notation, which we give in Fraktur script.

Type	Codiff	H^0	H^1	H^2	H^3
$d_1=\mathfrak{n}_3=A_3$	ψ_1^{23}	1	4	5	2
$d_2=\mathfrak{r}_{3,1}(\mathbb{R})=A_4$	$\psi_1^{13}+\psi_2^{23}$	0	3	3	0
$d_2(1:0)=A_7$	$\psi_2^{13}+\psi_1^{23}$	0	1	2	1
$d_2(-1:0)=A_6$	$\psi_2^{13}-\psi_1^{23}$	0	1	2	1
$d_2(0:1)=\mathfrak{r}_2(\mathbb{R})\oplus\mathbb{R}=A_3$	ψ_2^{23}	1	2	1	0
$d_2(r:s)=A_5(\frac{4r^2}{s\lvert s\rvert}-1)$	$\psi_2^{13}\lvert r\rvert+\psi_1^{23}r+\psi_2^{23}s$	0	1	1	0
$d_3=\mathfrak{sl}_2(\mathbb{R})=A_9$	$\psi_3^{12}+\psi_2^{13}+\psi_1^{23}$	0	0	0	0
$\tilde{d}_3=\mathfrak{su}_2=A_8$	$\psi_3^{12}-\psi_2^{13}+\psi_1^{23}$	0	0	0	0

Table 1. Cohomology of Real Three Dimensional Algebras

Morally speaking, the dimension of H^2 should correspond to a the number of different directions in which the algebra can deform. For example, generically, $H^2(d_2(r:s))$ has dimension 1, so deformations occur in only one direction. On the other hand, $H^2(d_2(1,0))$ has dimension 2, so there should be 2 independent directions of deformation, one corresponding to the deformation along the family, and one corresponding to a jump deformation to d_3.

In reality, the situation is more complex. For example, $H^2(d_1)$ has dimension 5, too large to account for by the rule of thumb given above. In fact, there are 2 relations on the base of the versal deformation for d_1, which means that the parameter space of the deformations of

d_1 is essentially 3-dimensional, a feature which is not captured in the dimension of H^2.

It is more accurate to think of the dimension of H^2 as being the dimension of the tangent space at the point. However, because the moduli space is not a manifold, it can happen that the not all tangent directions correspond to actual deformations. Thus H^2 comes more close to measuring the space spanned by all tangent directions. However, even this analogy is not completely precise in general. For example, there are finite dimensional Lie algebras which have no deformations, but for which H^2 does not vanish.

5. Construction of the moduli space

In this section, we compute the versal deformations of the real 3 dimensional Lie algebras, and show how they glue the moduli space together. In [10], a complete description of the deformations of the real 3-dimensional Lie algebras was given. The language of versal deformations for Lie algebras had not yet been invented, so the description there is a bit more complicated than we give, but coincides in the essential details.

To construct the versal deformation, one first chooses a pre-basis for H^2, also called a set of representative cocycles for the cohomology, and a parameter for each representative cocycle is given. One also needs to choose a basis of the 3-coboundaries, as well as a corresponding pre-basis, and a pre-basis of the 4-coboundaries. After making these choices, the computation of the versal deformation is completely mechanical. In good cases, the versal deformation will be given as a codifferential with coefficients which are rational functions of the parameters.

There are also some relations which must be satisfied by the parameters, called the relations on the base of the versal deformation. In order for a deformation given by substituting values of the parameters in the formula for the versal deformation to correspond to an actual codifferential, the relations on the base must be satisfied.

Generically, the versal deformation of $d_2(r : s)$ has matrix $\left[\begin{smallmatrix} 0 & t & r \\ 0 & |r| & s \\ 0 & 0 & 0 \end{smallmatrix}\right]$. and there are no relations on the parameter t. This matrix corresponds to a codifferential which is equivalent to $d_2(\sqrt{r^2 - st} : s + t)$ if $r > 0$ and $d_2(-\sqrt{r^2 + st} : s + t)$ when $r < 0$. These deformations are just along the family.

The case $d_2(0 : 1)$ is more interesting. The deformation fits the generic pattern, which means that it gives $d_2(-\sqrt{t}, 1 + t)$ when $t > 0$, and $d_2(\sqrt{-t}, 1 + t)$, when $t < 0$. In other words, it deforms along the set of matrices which we have given for the family $d_2(r : s)$. This fact shows that the gluing of the two sets of matrices $\left[\begin{smallmatrix} 0 & r \\ -r & s \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & r \\ r & s \end{smallmatrix}\right]$ together as single family is not an artificial construction, but is precisely the manner in which deformation theory dictates.

Now let us consider $d_2(1 : 0)$. Its versal deformation has matrix $\left[\begin{smallmatrix} 0 & t_2 & 1 \\ 0 & 1 & 0 \\ t_1 & 0 & 0 \end{smallmatrix}\right]$. This time, there is a relation on the base of the versal deformation: $t_1 t_2 = 0$. This means that we get two curves $t_1 = 0$ and $t_2 = 0$, so that the deformations of $d_2(1 : 0)$ do not really give a 2-dimensional space, but two transversal 1-dimensional spaces, leading to an H^2 of dimension 2. The first curve, $t_1 = 0$, gives a codifferential equivalent to $d_2(1 : t_2)$, which is just a deformation along the family. The second deformation, given by $t_2 = 0$ is equivalent to d_3, for all nonzero values of the parameter t_1. Thus there is a jump deformation from $d_2(1 : 0)$ to d_3.

For $d_2(-1 : 0)$, the versal deformation has matrix $\left[\begin{smallmatrix} 0 & t_2 & -1 \\ 0 & 1 & 0 \\ t_1 & 0 & 0 \end{smallmatrix}\right]$, with relation $t_1t_2 = 0$. When $t_1 = 0$, it deforms to $d_2(-1 : t_1)$. When $t_2 = 0$, it deforms to d_3 when $t_1 > 0$ and to $\tilde{d}_3$ when $t_1 < 0$. Thus, we jump to two completely different codifferentials depending on whether one goes backwards or forwards along the deformation curve. This type of phenomenon does not occur for jump deformations over $\mathbb{C}$.

The codifferential d_2 has versal deformation given by the matrix $\left[\begin{smallmatrix} 0 & 1+t_3 & t_2 \\ 0 & t_1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. The codifferential corresponding to this matrix is equivalent to $d_2\left(-\sqrt{1 + t_3 - t_2t_1} : 2 + t_3\right)$. Note that when $t_1 = t_3 = 0$, this is always equivalent to $d_2(-1 : 2)$, which means that we obtain a jump deformation from d_2 to $d_2(-1 : 2)$. Because the smooth deformations along the family $d_2(r : s)$ factor through this jump deformation, they are not part of the stratum to which d_2 belongs. In fact, d_2 is an isolated point.

The matrix of the versal deformation of d_1 is $\left[\begin{smallmatrix} t_1 & t_3 & 1 \\ t_2 & t_4 & 0 \\ t_5 & -t_2 & 0 \end{smallmatrix}\right]$. There are 2 relations on the base of the versal deformation:

$$t_4t_1 - t_3t_2 = 0$$

$$t_2t_1 + t_3t_5 = 0.$$

The solutions to these relations give three separate conditions. Either $t_1 = t_3 = 0$, $t_2 = t_3 = t_4 = 0$ or $t_5 = -\frac{t_4 {t_1}^2}{{t_3}^2}$ and $t_2 = \frac{t_4 t_1}{t_3}$. Thus the deformations live on subvarieties of codimension at least 2 in the parameter space.

Without going into all the details, we note that d_1 has jump deformations to every codifferential except d_2. For example, from the first solution to the relations, we obtain that on the plane $t_5 = 0$, the deformation is equivalent to $d_3(0:1)$ except when $t_3 = t_1 = 0$,and this means that there is a jump deformation from d_1 to $d_2(0:1)$.

One could give an explicit curve which yields $d(0:1)$ as a jump deformation, for example, we can take d_t to be the coderivation given by substituting $t_3 = 0$, $t_5 = 0$ and $t_3 = t$ in the first solution to the relations. In practice, to determine which codifferentials d' arise as jump deformations d, one checks to see if there are solutions to $d^\infty \sim d'$ for arbitrarily small values of the parameters, rather than finding an explicit form of a jump deformation.

The deformation theory of three dimensional real Lie algebras is essentially the same as in the complex case. The main consideration, when comparing the complex with the real case is that in determining the equivalence classes of complex algebras, one is allowed to use transformation matrices with complex coefficients, so it has to be checked carefully that the equivalence classes of real deformations agree with the complex results.

There are two main differences between the real and complex cases. First, there are two extra real equivalence classes, corresponding to the two real forms for $\mathfrak{sl}2$. Secondly, there are two solvable algebras $d_2(\pm 1, 0)$, both of which are equivalent as complex Lie algebras to the algebra $d(1 : -1)$.

The real subtlety arises in the interpretation in terms of families. In the complex case, we obtain one family $d(r : s)$, parameterized by the eigenvalues r and s of a 2×2 matrix, up to multiplication by a complex number, so that $(r : s)$ is a projective coordinate. There is an action of Σ_2 given by interchanging coordinates, so $d(r : s) \sim d(s : r)$.

In the real case, we do not simply take those matrices in $d(r : s)$ which have real eigenvalues. In fact, at first glance, it would appear that the real case gives two circles. For example, in [15], the corresponding real matrices are decomposed into two families, $\mathfrak{r}_{3,\lambda}$, represented by matrices $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix}\right]$, and $\mathfrak{r}'_{3,\lambda}$, given by matrices $\left[\begin{smallmatrix} \lambda & -1 \\ 1 & \lambda \end{smallmatrix}\right]$. This description is completely acceptable if the goal is to classify the Lie algebras up to equivalence, but deformation theory adds some new features to the description of the moduli space.

First, one has to classify into families which agree with deformation theory. Secondly, it turns out to be natural to see the stratum as an orbifold. In the real case, the two families which appear in [15] are really parts of a single family which is given by the action of $\mathbb{Z}_2$ on a circle. This description is not simply an artifact of the particular choice of parameterization which we gave for the family $d_2(r : s)$.

In fact, let us introduce a different parameterization. One can consider the matrices $\left[\begin{smallmatrix} r & 1 \\ 0 & s \end{smallmatrix}\right]$, in other words, the matrices corresponding to $d(r : s)$ in the complex case, along with the matrices $\left[\begin{smallmatrix} r & 1 \\ -s^2 & 1 \end{smallmatrix}\right]$. The first set captures the matrices with real, and the second with complex, eigenvalues, and we can visualize each of these families as parameterized by a circle, with an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ producing the identifications. In the first case, the action of one of the factors of $\mathbb{Z}_2$ is by $(r : s) \mapsto (-r : -s)$, and the second, by $(r : s) \mapsto (r : -s)$. In the second circle, we have a similar action of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Evidently, the two families of matrices have an overlap when we choose $r = s$ in the first family and $s = 0$ in the second. What is necessary to get the right picture is to cut the first circle along the line $r = s$, taking say the semicircle given by $r \leq s$, and cut the second circle along the line $s = 0$, taking say the semicircle $s \geq 0$, and joining these two semicircles. Amazingly, deformation theory agrees with this seemingly strange gluing. Note that in each of the semicircles, the group acting reduces to $\mathbb{Z}_2$, and the action agrees where they are joined together.

If the reader wishes, one could make the joining of the semicircles seem more natural if we replace the left family by $d(r : r + s)$ instead. Then the group action of $\mathbb{Z}_2$ changes to $(r : s) \mapsto (-r - s : s)$, and we could join the elements with $s \leq 0$ to the elements in the second family with $s \geq 0$, for a more natural seeming picture. The price one pays is that the group action is more complicated, although note that on the boundary $s = 0$, the two group actions agree.

The point is, however you cut the moduli space up, the pieces reassemble to give the same structure as an orbifold. Moreover, the pieces each have an action of $\mathbb{R}^*$, rather than just $\mathbb{R}_+$, but in order to assemble them according to deformation theory, one has to glue two half circles together, thereby killing off part of the group action. We shall see that this same phenomenon occurs in the four dimensional moduli space, this time with 2-spheres instead of circles.

The moduli space of three dimensional lie algebras is represented pictorially in figure 2 below. Another way to represent the elements of the moduli space is in terms of levels. The lowest level, level 0, consists of those elements which have no deformations. They may or may not be cohomologically rigid, although in three dimensional case, level 0 elements are d_3 and d_4, which are simple algebras, and thus rigid. The next level, level 1, consists of those elements whose deformations either lie on the same family or to level 0 objects. The family $d_2(r : s)$ is on level 1. Finally, to complete the three dimensional picture, there are two level 2 elements, d_2 and d_1. In general, any element on level k may deform to elements on any level below it.

6. Four Dimensional Complex Lie algebras

In [8], the moduli space of four dimensional complex Lie algebras was studied, and the versal deformations of all the elements were computed. The four dimensional Lie algebras had previously been classified, see for example [2, 4, 15], but the decompositions given did not completely align with the picture given by deformation theory. There are several conventions for naming the algebras, and we also introduced our own notation in [8], which we will use here, because it is consistent with the decomposition of the moduli space according to deformation theory.

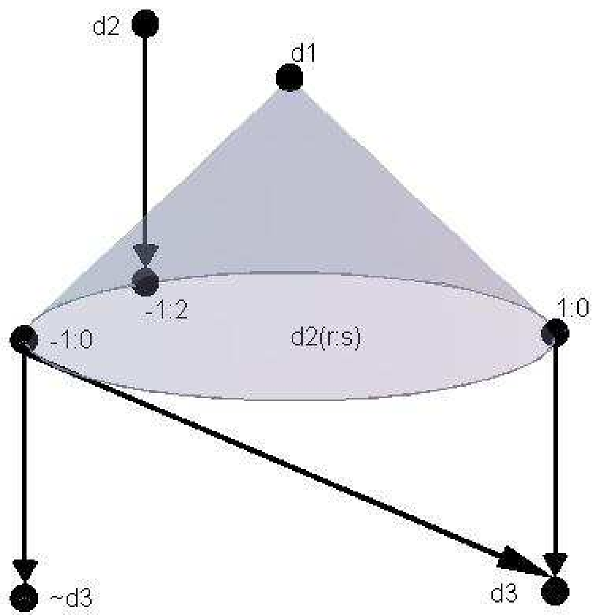

FIGURE 2. The Moduli Space of real 3-dimensional Lie Algebras.

There are two 1-parameter families $d_1(p:q)$ and $d_3(p:q)$, one 2-parameter family $d_3(p:q:r)$, and six isolated elements, d_1, $d_1^\sharp$, $d_2^\sharp$, d_2^*, d_3 and d_3^*.

Five of these codifferentials arise from extensions of the one dimensional complex Lie algebra by a trivial three dimensional algebra. Such extensions are determined by the equivalence class of a 3×3 matrix under similarity and multiplication by a nonzero number. The matrices are listed below. Note that this 3×3 matrix is just the upper right hand part of the 4×6 matrix of the codifferential, if e_1, e_2, e_3 is taken as the basis of the extension algebra, and e_4 is the basis of the one dimensional algebra. The 3×3 matrix represents the module structure of the extension, with input basis e_1e_4, e_2e_4, e_3e_4, and output basis e_1, e_2, e_3. The five Lie algebras are given as follows:

$$\begin{array}{ccccc} d_3(p:q:r) & d_3(p:q) & d_3^* & d_2* & d_1 \\ \begin{bmatrix} p & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & r \end{bmatrix} & \begin{bmatrix} p & 0 & 0 \\ 0 & p & 1 \\ 0 & 0 & q \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

There are two codifferentials arising from the nontrivial extensions of the three dimensional Heisenberg algebra (d_1 in our notation). Note that the three dimensional Heisenberg algebra itself is listed above. These extensions are classified by the similarity classes of a 2×2 matrix, up to multiplication by a nonzero constant. The matrices of these extensions, denoted by $d_1(p:q)$ and $d_1^\sharp$ are given below.

$$\begin{array}{cc} d_1(p:q) & d_1^\sharp \\ \begin{bmatrix} 0 & 0 & 1 & p+q & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 1 \\ 0 & 0 & 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Finally, we have the matrices of the Lie algebra d_3, which is the direct product of the simple 3-dimensional Lie algebra and the trivial 1-dimensional Lie algebra, and the rigid Lie algebra $d_2^\sharp$, which is the direct product of the nontrivial 2-dimensional Lie algebra with itself.

$$d_3 \qquad\qquad d_2^\sharp$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As far as levels go, we have d_3 and $d_2^\sharp$ on level 0, $d_1(p:q)$ on level 1, $d_3(p:q:r)$ and $d_1^\sharp$ on level 2, $d_3(p:q)$ and d_2^* on level 3, and finally, d_1 and d_3^* on level 4. Let us describe the deformations on each level. The family $d_1(p:q)$ generically deforms only along its own family, but the element $d_1(1:-1)$ has an extra deformation, a jump deformation to d_3. This pattern replicates the pattern of the 3-dimensional complex algebras, even to the existence of a jump deformation from the codifferential $d_1^\sharp$ to $d_1(1:1)$.

The family $d_3(p:q:r)$ generically deforms only along its own family. There are three special subfamilies $d_3(p:p:q)$, $d_3(p:q:0)$ and $d_3(p:q:p+q)$, which have extra deformations. An element of the form $d_3(p:q:p+q)$ has a jump deformation to $d_1(p:q)$. An element of the form $d_3(p:q:0)$ has a jump deformation to $d_2^\sharp$. Elements of the form $d_3(p:p:q)$ are special because there is a jump deformation from $d_3(p:q)$ to $d_3(p:p:q)$. The element d_2^* jumps to every element in the family $d_3(p:q:r)$. On the highest level, d_3^* jumps to $d_3(1:1)$, while d_1 has jump deformations to every codifferential except d_3^*. Complete details

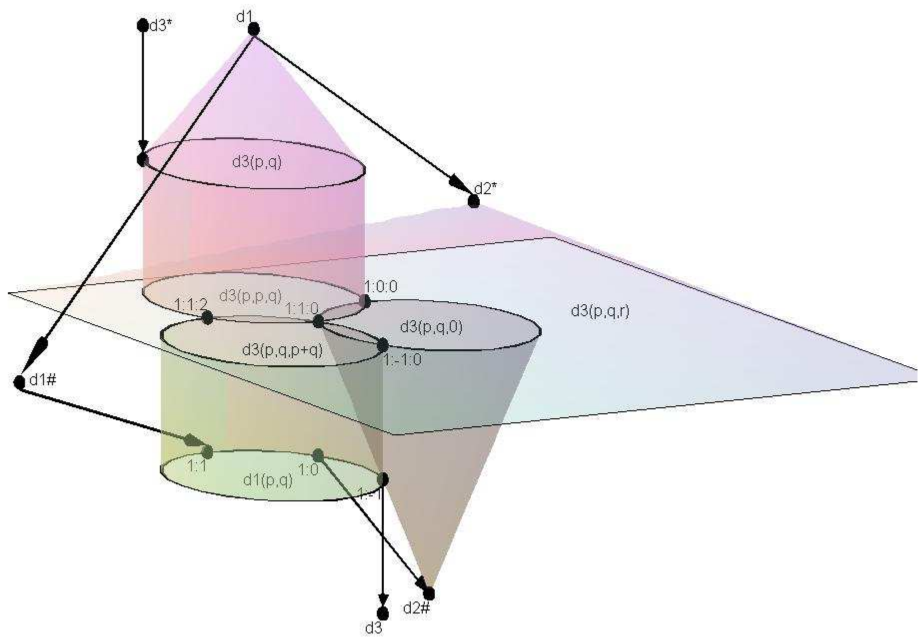

Figure 3. The Moduli Space of complex 4 dimensional Lie Algebras

on the deformation theory of four dimensional complex Lie algebras are available in [8]. The moduli space of complex 4-dimensional Lie algebras is illustrated in Figure 3.

In the figure, we represent the strata $d_3(p, q, r)$ by a plane, a standard way of presenting $\mathbb{CP}^2$, although in this case, it really represents a $\mathbb{CP}^2/\Sigma_3$. The strata $d_3(p, q)$ and $d_1(p, q)$, as well as the elements of the form $d_3(p, q, 0)$, $d_3(p, q, p + q)$ and $d_3(p, p, q)$ are represented by circles, although they really represent either a $\mathbb{CP}^1$ or a $\mathbb{CP}^1/\Sigma_2$. Although it is more standard to represent $\mathbb{CP}^1$ as a line, in our case, the curves $d_3(p, q, 0)$, $d_3(p, q, p+q)$ and $d_3(p, p, q)$ intersect in more than one point, because they are really equivalence classes of $\mathbb{CP}^1$'s under the action of

Σ_3 on $\mathbb{CP}^2$. For example, $d_3(p,q,0)$ and $d_3(p,q,p+q)$ are parameterized by $\mathbb{CP}^1/\Sigma_2$, while $d_3(p,p,q)$ is parameterized by $\mathbb{CP}^1$. The jump deformations from d_1 to $d_3(p,q)$ and $d_3(p,q,0)$ to $d_2^\sharp$ are represented by cones, the jump deformations from d_2^* to $d_3(p,q,r)$ are represented by a tetrahedron, the jump deformations from $d_3(p,q)$ to corresponding elements in $d_3(p,p,q)$ and from elements in $d_3(p,q,p{+}q)$ to corresponding elements in $d_1(p,q)$ are represented by cylinders, while the arrows in the picture represent the jump deformations from single elements to other single elements. The levels are represented in ascending order.

7. Real four dimensional Lie algebras

In the three dimensional case, we saw that two types of phenomena occur. First, some complex codifferentials split into two separate real codifferentials. In fact, d_3 became d_3 and $\tilde{d}_3$, and $d(1:-1)$ split into two real representatives, $d_2(1:0)$ and $d_2(-1:0)$. Secondly, the complex family $d(p,q)$, which was parameterized by complex eigenvalues, has to be modified so as to only include the elements with either two real, or two conjugate complex eigenvalues, giving a new family $d_2(p,q)$. These two types of codifferentials glue together to give an $S^1/\mathbb{Z}_2$.

There are three complex families of four dimensional Lie algebras. We study the real version $\tilde{d}_3(p,q,r)$ of the complex 2-parameter family $d_3(p,q,r)$ first.

7.1. **The family $\tilde{d}_3(p:q:r)$.** The complex family $d_3(p,q,r)$ is given by certain 3×3 matrices, with three complex eigenvalues, where repeated eigenvalues are given by a single Jordan block. If one considers the eigenvalues of a real 3×3 matrix, then one of them must be real. This means that the matrices can either have three real eigenvalues, or one real and two conjugate eigenvalues, in which case the eigenvalues are distinct. The matrices $\begin{bmatrix} p & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & r \end{bmatrix}$,which we will call Type 1, and $\begin{bmatrix} p & 1 & 0 \\ 0 & r & 1 \\ 0 & -q^2 & r \end{bmatrix}$, which we will call Type 2, represent codifferentials with three real eigenvalues and those with one real eigenvalue, respectively. Notice that essentially, we have used one of the decompositions of the real 2×2 equivalence classes of matrices discussed in the section on three dimensional real Lie algebras.

We wish to glue the Type 1 and Type 2 codifferentials together as a single family, dictated by deformation theory. Let us study the groups acting on the Type 1 and Type 2 matrices which give the equivalence classes of codifferentials. For Type 1 matrices, in addition to the action of $\mathbb{R}_+$ on triples $(p : q : r)$ by multiplying the triple by a positive real number, we also have an action of $\mathbb{Z}_2 \times \Sigma_3$, where the $\mathbb{Z}_2$ acts by negating the triple, and Σ_3 acts by permuting the elements in the triple. For Type 2 codifferentials, in addition to the action of $\mathbb{R}_+$, we have an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on triples $(p : q : r)$, where the first factor acts by negating the q coordinate, and the second by negating both the p and r coordinates.

The gluing is accomplished by cutting the sphere for the Type 1 matrices along the lines $p = q$ and $q = r$, and taking the quadrant given by $p \leq q \leq r$. By taking this piece of a sphere, effectively we have killed off most of the $\mathbb{Z}_2 \times \Sigma_3$ action, but there remains a $\mathbb{Z}_2$ action given by $(p : q : r) \mapsto (-r : -q : -p)$. Next, consider the hemisphere of

Type 2 matrices with $q \geq 0$. The action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on this hemisphere reduces to an action of $\mathbb{Z}_2$ given by $(p : q : r) \mapsto (-p : q : -r)$. Thus the actions in both cases are reduced to $\mathbb{Z}_2$ actions.

The semicircle $p = q$ on the quadrant $p \leq q \leq r$ of type 1 matrices is glued to the semicircle $(r, 0, p)$ of type 2 matrices with $p \leq r$. The semicircle $q = r$ of type 1 matrices is glued to the semicircle of type 2 matrices of the form $(p, 0, r)$ with $p \leq r$. It is easily checked that the action of $\mathbb{Z}_2$ is consistent with this gluing. What is more important is that the gluing is consistent with deformation theory. Deformations from points on the boundaries of these two regions move along both types in a natural fashion.

This gluing operation produces a sphere with a global action of $\mathbb{Z}_2$. There are exactly two fixed points, one coming from each type. In type 1, the point $(-1 : 0 : 1)$ is evidently fixed, and is the only fixed point. In type 2, the only fixed point is $(0 : 1 : 0)$. These two points both correspond to the same complex codifferential, since the eigenvalues of the second one are 0, i and $-i$. These are the only two points on the sphere corresponding to the same complex codifferential.

Let us denote by $\tilde{d}_3(p : q : r)$ the real codifferential of type 1 or type 2 whose 3×3 submatrix has eigenvalues p, q and r. Either all three eigenvalues are real, in which case, the codifferential is given by a type 1 matrix or two of the eigenvalues are complex conjugates, so the codifferential is given by a type 2 matrix. With this notation, every element in $\tilde{d}_3(p : q : r)$ corresponds to the same element $d_3(p : q : r)$ as a complex codifferential. We do have to keep in mind that

$\tilde{d}_3(-1:0:1)$ is not equivalent as a real codifferential to $\tilde{d}_3(0:i:-i)$, which distinguishes the family $\tilde{d}_3(p:q:r)$ from its progenitor family $d_3(p:q:r)$. In studying the deformation theory, it is still convenient at times to study the codifferentials of Type 1 and Type 2 separately. We can parameterize the codifferentials in the family $d_3(p:q:r)$ by $S^2/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ has been described above.

7.2. **The family $d_3(p:q)$.** The complex family $d_3(p:q)$ is given by matrices with a repeated geometric eigenvalue p. As a consequence, the real versions of such codifferentials must have three real eigenvalues, so that essentially, the real family $d_3(p:q)$ coincides with a subfamily of the complex family, where $(p:q)$ represents an element of S^1, given by the action of $\mathbb{R}_+$ on $\mathbb{R}^2$. Moreover, $d_3(p:q)\sim d_3(-p:-q)$, so that the family $d_3(p:q)$ is parameterized by S^1/Z_2. In this case, the action of $\mathbb{Z}_2$ on S^1 determines the quotient space $\mathbb{RP}^1$, so there are no orbifold points.

7.3. **The family $\tilde{d}_1(p:q)$.** In the complex case, the codifferentials in $d_1(p:q)$ arise from extensions of the 1-dimensional complex Lie algebra by the Heisenberg algebra d_1, which is also denoted by $\mathfrak{n}_3(\mathbb{C})$. These extensions are parameterized by $\mathbb{CP}^1/\Sigma_2$, and arise from classifying 2×2 complex matrices under similarity. Thus the real matrices have the same classifying scheme as the 1-parameter real 3-dimensional algebras. Like that case, we need to glue together two semicircles represented by matrices of the form $\begin{bmatrix} 0 & 0 & 1 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & |p| & q \end{bmatrix}$. We denote the codifferentials in this family by $\tilde{d}_1(p:q)$, to distinguish them from their complex analogues $d_1(p:q)$.

As in the three dimensional case, we obtain that $\tilde{d}_1(\pm 1:0)$ both correspond to the same complex codifferential $d_1(1:-1)$, but otherwise the elements $\tilde{d}_1(p:q)$ correspond to distinct elements in the complex family $d_1(x:y)$.

The picture of this family is similar to the 1-parameter family from the real three dimensional case. It is parameterized by S^1/Σ_2, but is not $\mathbb{RP}^1$. The action of $\mathbb{Z}_2$ is given by $(p:q) \mapsto (p:-q)$, so the two orbifold points coincide with the two points which determine the same complex codifferential.

7.4. **Special Cases.** In the complex case, there were 6 isolated codifferentials. Since each of them has a representative with real coefficients, they all have real counterparts. The matrices for the complex algebras d_1, $d_1^\sharp$, d_2^*, $d_2^\sharp$, d_3 and d_3^* have been given above. There are two additional special points $\tilde{d}_3$ and $\tilde{d}_2^\sharp$.

As the notation suggests, the complex codifferential d_3, which represents $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$, splits into two real codifferentials d_3, representing $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}$, and $\tilde{d}_3$, representing $\mathfrak{su}_2 \oplus \mathbb{R}$. In addition, the complex codifferential $d_2^\sharp$, representing $\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$, splits into two real codifferentials $d_2^\sharp$, representing $\mathfrak{r}_2(\mathbb{R}) \oplus \mathfrak{r}_2(\mathbb{R})$, and $\tilde{d}_2^\sharp$. Both of these codifferentials arise by extensions of the 1-dimensional real Lie algebra by the Lie algebra given by d_2. The matrix of $\tilde{d}_2^\sharp$ is $\begin{bmatrix} 0&0&1&1&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{bmatrix}$. Like $d_2^\sharp$, $\tilde{d}_2^\sharp$ is completely rigid, but it is not the direct sum of two real Lie algebras.

In the table below, we give the cohomology for the real codifferentials.

8. Gluing the moduli space

We omit the details about the computation of the versal deformations of the real 4-dimensional algebras, but just give a description of the moduli space.

The level 0 algebras are d_3, $\tilde{d}_3$, $d_2^\sharp$ and $\tilde{d}_2^\sharp$. The last two are cohomologically rigid, and none of them have any deformations.

Generically, $H^2(\tilde{d}_1(p:q))$ is of dimension 1, and deformations of elements in this family lie along the family itself. The exception to this case are the two elements $\tilde{d}_1(\pm 1:0)$ which are the two orbifold points of the $S^1/\mathbb{Z}_2$ which parameterizes this family, and the element $\tilde{d}_1(0:1)$. The element $\tilde{d}_1(0:1)$ deforms to both $d_2^\sharp$ and $\tilde{d}_2^\sharp$. This is similar to the complex picture, where $d_1(0:1)$ deforms to $d_2^\sharp$. The element $\tilde{d}_1(1:0)$ deforms to d_3, while the element $\tilde{d}_1(-1:0)$ deforms to both d_3 and $\tilde{d}_3$. Because of these exceptional points, the family $d_1(p:q)$ is at level 1, and it is the only level one object.

The element $d_1^\sharp$ has a jump deformation to $\tilde{d}_1(-1:2)$, corresponding to the complex jump deformation to $d_1(1:1)$. Thus $d_1^\sharp$ lives on level 2.

The family $\tilde{d}_3(p:q:r)$ also lives on level 2, because it has a subfamily $\tilde{d}_3(p:q:p+q)$ which deforms to the $\tilde{d}_1(p:q)$ family. To see which element it jumps to, keep in mind that as a complex codifferential, there is a jump deformation from $d_3(p,q,p+q)$ to $d_1(p,q)$. But $d_1(p,q)\sim\tilde{d}_1(\sqrt{-pq},p+q)$ when $pq\le 0$ and $d_1(p,q)\sim\tilde{d}_1(-\sqrt{pq},p+q)$ when $pq>0$, which gives the point which $\tilde{d}_3(p,q,p+q)$ jumps to when p and q are real. When p and q are complex conjugates, then $pq>0$, so we get a jump deformation to the point $\tilde{d}_1(-\sqrt{pq},p+q)$, whose coordinates are real. The circle $(p:q:p+q)$ gives the only points on the sphere $(p:q:r)$ which deform to the family $\tilde{d}_1(p:q)$.

There is another circle, given by the elements $\tilde{d}_3(0:q:r)$, whose elements all deform to either $d_2^\sharp$, when the eigenvalues are real or $\tilde{d}_2^\sharp$ otherwise. The special cases $\tilde{d}_3(0:0:1)$ and $\tilde{d}_3(0:1:1)$ are boundary points between the elements on the circle $\tilde{d}_3(0:q:r)$ with real and nonreal roots. The point $\tilde{d}_3(0:1:1)$ jumps to both $d_2^\sharp$ and $\tilde{d}_2^\sharp$, but the point $\tilde{d}_3(0:0:1)$ only jumps to $d_2^\sharp$. The two points $\tilde{d}_3(0:1:-1)$ and $\tilde{d}_3(0:i:-i)$, both of which lie on the circle, have additional jump deformations, the first to d_3, and the second to both d_3 and $\tilde{d}_3$.

The subfamily $\tilde{d}_3(p:q:-p-q)$ was listed in Table 2, because its cohomology is not generic. However, H^2 is of the usual dimension for the family $d_3(p:q:r)$, and the deformations of this subfamily lie along $\tilde{d}_3(p:q:r)$ as in the generic case. In addition, the subfamily $\tilde{d}_3(p:p:q)$, which is the boundary between the elements with all real eigenvalues and the elements with two nonreal eigenvalues, while not special in terms of its deformations, plays a special role as it is the image of jump deformations from the family $d_3(p:q)$. Note that elements in $\tilde{d}_3(p:p:q)$ have real eigenvalues, so correspond precisely to their complex counterparts $d_3(p:p:q)$.

On level 3, the element d_2^* has jump deformations to every element of the family $\tilde{d}_3(p:q:r)$, and thus deforms to everything to which any element of this family deforms as well. Every element of the family $d_3(p:q)$ has a jump deformation to $\tilde{d}_3(p:p:q)$. The points $d_3(0:1)$, $d_3(1:0)$, $d_3(1:2)$ and $d_3(1:-2)$ all lie over special points or subfamilies of the $\tilde{d}_3(p:q:r)$ family, and thus have extra deformations, arising from the extra deformations of their image points.

On level 4, d_3^* has a jump deformation to $d_3(1:1)$ and thus to everything to which $d_3(1:1)$ deforms. This accounts for all the deformations of d_3^*. Finally, d_1 has jump deformations to every element except d_3^*.

This gives the complete deformation theory of real four dimensional Lie algebras. The picture is almost identical with the deformation theory for complex Lie algebras, and the variations are all accounted for by the splitting of some of the complex Lie algebras into two real forms.

Figure 4 below gives a representation of the moduli space of four dimensional real Lie algebras.

In the figure, the strata $\tilde{d}_3(p:q:r)$, which is parameterized by $S^2/\mathbb{Z}_2$, is represented as a plane. The stratum $d_3(p:q)$, and the substratum $\tilde{d}_3(p:p:q)$ both of which are parameterized by $\mathbb{RP}^2$ are represented by circles. The stratum $\tilde{d}_1(p:q)$, and the substrata $\tilde{d}_3(p:q:p+q)$ and $\tilde{d}_3(p:q:0)$ all of which are parameterized by

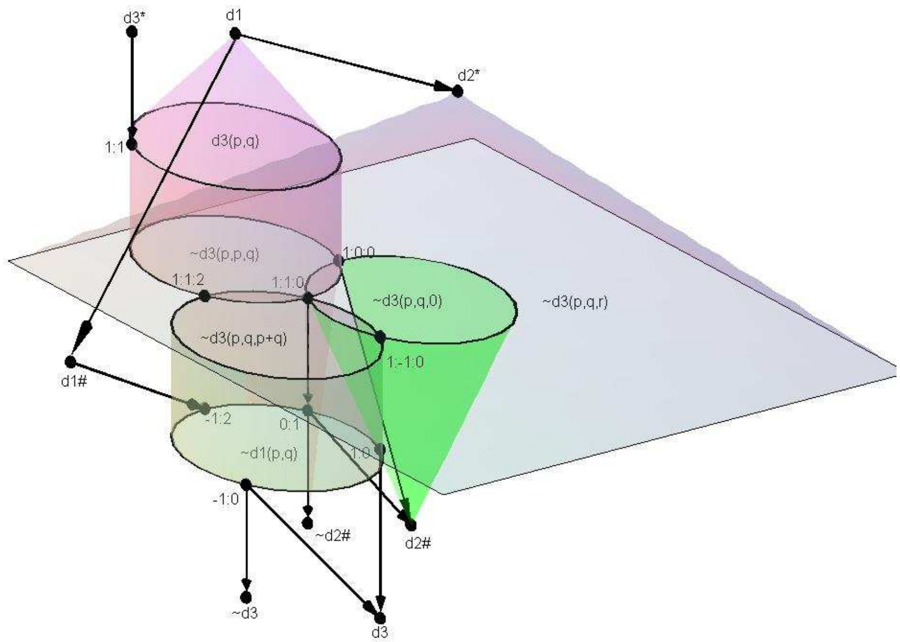

FIGURE 4. The Moduli Space of 4-dimensional real Lie Algebras.

$S^1/\mathbb{Z}_2$, are also represented by circles. The jump deformations from d_1 to $d_3(p:q)$ are represented by a cone, while the jump deformations from the arc from $\tilde{d}_3(1:1:0)$ to $\tilde{d}_3(1:0:0)$ to $\tilde{d}_2^\sharp$, representing the codifferentials in $\tilde{d}_3(p:q:0)$ with nonreal eigenvalues, and the jump deformations from the other arc from $\tilde{d}_3(1:1:0)$ to $\tilde{d}_3(1:0:0)$ to $d_2^\sharp$, representing the codifferentials in $\tilde{d}_3(p:q:0)$ with real eigenvalues, are represented by cones as well. The jump deformations from d_3^* to $\tilde{d}_3(p:q:r)$ are represented by a tetrahedron. Finally, the arrows represent jump deformations from single elements to other single elements in the moduli space.

9. Conclusions

From our examples, it appears that the moduli spaces of real Lie algebras may be parameterized by spheres modulo the action of a finite group. Actually, in our examples, more seems to be true. In fact, the finite group was always $\mathbb{Z}_2$, and in most cases the $\mathbb{Z}_2$ action was given by a rotation of the sphere, so that it had exactly 2 fixed points, both of which represented the same complex codifferential. In the case where the action of $\mathbb{Z}_2$ produced a projective space, there were no fixed points, and every element had a distinct complex image. Note that if the $\mathbb{Z}_2$ action on a sphere is given by a rotation, then great circles on the sphere which go through the axis of rotation will also inherit an action of $\mathbb{Z}_2$, consisting of a reflection through the axis points, while the great circle perpendicular to the axis will inherit a $\mathbb{Z}_2$ action consisting of a rotation. This is reflected in the fact that some of the substrata of $\tilde{d}_3(p:q:r)$, for example $\tilde{d}_3(p:q:0)$, are given by $S^2/\mathbb{Z}_2$, while the substratum $\tilde{d}_3(p:p:q)$ is parameterized by $\mathbb{RP}^1$.

It would be interesting to see if a similar pattern emerges for higher dimensional moduli spaces of real Lie algebras.

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Type	H^0	H^1	H^2	H^3	H^4
$\tilde{d}_3(-1:0:1)$	1	3	5	5	2
$\tilde{d}_3(0:i:-i)$	1	3	5	5	2
$\tilde{d}_3(0:q:r)$	1	3	3	1	0
$\tilde{d}_3(p:q:p+q)$	0	2	3	1	0
$\tilde{d}_3(p:q:-p-q)$	0	2	2	1	1
$\tilde{d}_3((p:q:r)$	0	2	2	0	0
$d_3(0:1)$	2	6	6	2	0
$d_3(1:0)$	1	5	7	3	0
$d_3(1:2)$	0	4	5	1	0
$d_3(1:-2)$	0	4	4	1	1
$d_3(p:q)$	0	4	4	0	0
$\tilde{d}_1(0:1)$	0	1	2	1	0
$\tilde{d}_1(\pm1:0)$	1	2	2	2	1
$\tilde{d}_1(r:s)$	0	1	1	0	0
d_1	2	8	13	10	3
$d_1^\sharp$	0	3	3	0	0
$d_2^\sharp$	0	0	0	0	0
$\tilde{d}_2^\sharp$	0	0	0	0	0
d_2^*	1	4	6	5	2
d_3	1	1	0	1	1
$\tilde{d}_3$	1	1	0	1	1
d_3^*	0	8	8	0	0

Table 2. Cohomology of Four Dimensional Real Lie Algebras