Deformations of some infinite-dimensional Lie algebras

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Abstract

The concept of a versal deformation of a Lie algebra is investigated and obstructions to extending an infinitesimal deformation to a higher-order one are described. The rigidity of the Witt algebra and the Virasoro algebra is deduced from cohomology computations for certain Lie algebras of vector fields on the real line. The Lie algebra of vector fields on the line that vanish at the origin also turns out to be rigid. All the affine Lie algebras are rigid; this is derived from the cohomology of their maximal nilpotent subalgebra. On the other hand, the maximal nilpotent subalgebras in both the Virasoro and affine cases are not rigid and have interesting nontrivial deformations (in fact, most vector field Lie algebras are not rigid).

I Introduction

It is known that in characteristic zero a semisimple Lie algebra has no nontrivial deformations ([1]). The same is true for infinite-dimensional classical Lie algebras from the "Cartan series". We say that those Lie algebras are rigid. However, in nonzero characteristics, both semisimple and Cartan type Lie algebras have nontrivial deformations, as was shown by Rudakov (1971) in [2] and Dzumadil'daev (1980) in [3]. For some other modular Lie algebras there are also interesting results ([4]). It is known that solvable and nilpotent Lie algebras have very many nontrivial deformations. All infinitesimal deformations (defined later) of the maximal nilpotent subalgebra of simple finite-dimensional Lie algebras are known (Leger–Luks [5]), but the infinitesimal deformations for some other of their subalgebras are impossible to classify (Piper, 1971 [6]).

In infinite dimension, we consider Lie algebras having a triangular decomposition ([7]): affine algebras, Virasoro algebra, and their nilpotent subalgebras.

In this paper I am going to present my results on the deformations of these infinite-dimensional Lie algebras.

II Preliminaries

For computing deformations, we need cohomology theory. We should mention that although a highly developed general theory existed, there were very few computations. The situation changed in the late 1960s, after the important Russian works of Gelfand and Fuks on the cohomology with trivial coefficients of Lie algebras of vector fields on a smooth manifold ([8]). In 1976 the cohomology of the maximal nilpotent subalgebras of affine algebras with trivial coefficients was also computed (Garland–Lepowsky [9]).

(i) For computing deformations, we have to compute cohomology with oefficients in the adjoint representation.

Let L be a Lie algebra (finite or infinite dimensional). Let us define a Lie superalgebra structure on the cochain complex $C^{\cdot}(L;L)$. For $a \in C^p(L;L)$, $b \in C^q(L;L)$, define $ab \in C^{p+q-1}(L;L)$ by

$$ab(g_1, \ldots, g_{p+q-1}) := \sum_{\sigma} \operatorname{sgn}(\sigma) a\big(b(g_{i_1}, \ldots, g_{i_q}), g_{j_1}, \ldots, g_{j_{p-1}}\big),$$

where σ runs over all the shuffle permutations with $i_1 < \cdots < i_q$ and $j_1 < \cdots < j_{p-1}$. Put

$$[a, b] := ab - (-1)^{(p-1)(q-1)} ba.$$

The differential of degree 1 acts on brackets by the rule:

$$d([a, b]) = [da, b] - (-1)^{p-1}[a, db].$$

It is easy to verify that the cochain complex $C^{\cdot}(L;L)$ is a differential Lie superalgebra. The superbracket multiplication can be lifted to the cohomology space:

$$H^p(L;L) \otimes H^q(L;L) \to H^{p+q-1}(L;L),$$

(see [10] for details).

(ii) Recall the intuitive definition of a deformation of a Lie algebra L_0. It is a family of Lie algebras L_t with the same underlying vector space, and with the bracket

$$\begin{aligned} \mu_t(x, y) &= \mu_0(x, y) + \varphi(t)(x, y) \\ &= [x, y] + \varphi_1 t + \varphi_2 t^2 + \cdots, \end{aligned}$$

where $x, y \in L_0$, $\varphi(t) = \sum_{i=1}^{\infty} t^i \varphi_i$, and $\mu_0(x, y)$ is the original bracket in L_0. Obviously $\varphi_i \in C^2(L_0; L_0)$ and the Jacobi identity means

$$-d\varphi = \frac{1}{2}[\varphi, \varphi],$$

or for each k,

$$-2 \sum_k d\varphi_k = \sum_k \sum_{i+j=k} [\varphi_i, \varphi_j] \quad \operatorname{mod}\ (t^{k+1}),$$

where d and $[\ ,\ ]$ were defined in (i). (This is the so-called "deformation equation".)

A *deformation* is said to be *of order* k if the Jacobi identities are satisfied mod (t^{k+1}). A deformation of order 1 is called an *infinitesimal deformation.*

III Versal deformations

First we give a general definition of Lie algebra deformations. There are four steps in the generalization (for details on this see [10], [11]).

(i) Consider L_t as a Lie algebra over $K((t))$.

(ii) Generalize over $K[[t_1, \ldots, t_r]]$.

(iii) Let the parameter space A be a local finite-dimensional algebra. We say that L_A is a deformation of the Lie algebra L, parametrized by a local finite dimensional algebra A if L_A is a Lie algebra structure over A on $L \otimes_K A$ such that the Lie algebra structure on $L = L_A \otimes_A K = (L \otimes A) \otimes_A K$ is the given one on L.

(iv) Let the parameter space be a complete local algebra A ($A = \lim A/m_A^n$, where A/m_A^n are local finite dimensional for each n). A *deformation* of L parametrized by a complete local algebra A is a projective limit of deformations of L, parametrized by ${}^A/m_A^n$. Two deformations, L_A and L'_A, parametrized by A are called *equivalent* if there exists a Lie algebra isomorphism over A of L_A on L'_A, inducing the identity of $L_A \otimes_A K = L$ on $L'_A \otimes_A K = L$.

Define a functor from the category $\widehat{C}$ of complete local algebras into the equivalence classes of deformations of L, parametrized by A.

The generalizations (i)–(iv) are necessary in order to define the so-called versal deformations, which induce all the other deformations of a given Lie algebra.

Definition. A deformation L_R of L parametrized by $R \in \widehat{C}$ is a *versal deformation* if for any L_A, parametrized by $A \in \widehat{C}$, there exists a morphism $f : R \to A$ such that (i) $L_R \otimes_R A$ is equivalent to L_A; (ii) if the map $m_R/m_R^2 \to m_A/m_A^2$ induced by f is unique.

Theorem 1. *If $H^2(L; L)$ is finite dimensional then there exists a versal deformation.*

Proof. The statement follows from a general theorem of Schlessinger [12]. See the details in [10].

Remark. Suppose $A = \mathbb{C}[t_1, \ldots, t_n]/I$. Then the deformation equation is the following (see [11]):

$$2 \sum_{|\alpha| \geq 1} (d\varphi_\alpha) t^\alpha + \sum_{|\alpha| \geq 1} \sum_{\beta + \gamma = \alpha} [\varphi_\beta, \varphi_\gamma] t^\beta t^\gamma \equiv 0 \pmod{I}.$$

For $|\alpha| = 1$ we get $d\varphi_\alpha = 0$ for each α, which means that φ_α has to be a cocycle.

Proposition. *The elements of $H^2(L; L)$ correspond bijectively to the nonequivalent infinitesimal deformations.*

Proof. This well-known fact can be proved by direct computation.

Corollary. *The condition $H^2(L; L) = 0$ is sufficient for L to be rigid (but not necessary).*

IV Obstructions

After defining the nontrivial infinitesimal deformations, the next natural question arises: Is it possible to extend an infinitesimal deformation to a deformation of higher order? The answer is "no" in general. To extend it (represented by a cocycle φ_1) to second order, parametrized by $\mathbb{C}[t]/(t^3)$, it is necessary and sufficient that $[\varphi_1, \varphi_1]$ is cohomologous to 0, which means that it must be a coboundary. The Jacobi identity of order 2 is

$$-2d\varphi_2 = [\varphi_1, \varphi_1].$$

If φ_2 is a cochain such that this identity is satisfied, then we can define a deformation of order 2 with the bracket

$$\mu_t = \mu_0 + \varphi_1 t + \varphi_2 t^2,$$

where φ_2 is well defined up to a two-cocycle. The cohomology class of $[\varphi_1, \varphi_1]$ is the *first obstruction* to forming a one-parameter family of deformations whose first term is cohomologous to φ_1. If it vanishes, another obstruction may show up at the next level. To extend it to a third-order deformation parametrized by $\mathbb{C}[t]/(t^4)$, it is necessary and sufficient that $[\varphi_1, \varphi_2]$ is also cohomologous to zero. If φ_3 is a cochain such that

$$-2d\varphi_3 = [\varphi_1, \varphi_2] + [\varphi_2, \varphi_1],$$

(the class $[\varphi_1, \varphi_1]$ is zero), then we can define a deformation of order three with the bracket

$$\mu_t = \mu_0 + \varphi_1 t + \varphi_2 t^2 + \varphi_3 t^3.$$

Here, φ_3 is also defined up to a two-cocycle. The cohomology class of $[\varphi_1, \varphi_2]$ is the *second obstruction* to forming a one-parameter family of deformations, whose first term is cohomologous to φ_1.

In general, let us define in $H^{\cdot}(L; L)$ higher operations, called *Massey operations*. These operations of order n are partially defined and they are well-defined modulo those of order $(n-1)$. It is enough to define them on the homogeneous elements. For $y_1 \in H^2(L; L)$ and $y_2 \in H^2(L; L)$ the Massey operation of order 2 is the superbracket. Suppose that for $y_1 \in H^2(L; L)$, $y_2 \in H^2(L; L)$, and $y_3 \in H^2(L; L)$, $[y_i, y_j] = 0$. Then for the cocycles x_i representing y_i, $[x_i x_j] = dx_{ij}$, where x_{ij} are one-cochains. Then the Massey operation of order 3, $[y_1, y_2, y_3]$ takes value in the factor space

$$H^3(L; L) \Big/ \left([y_1,\, H^2(L; L)] + [y_2,\, H^2(L; L)] + [y_3,\, H^2(L; L)]\right),$$

and is equal to the cohomology class of the cocycle

$$[x_{12}, x_3] - [x_{23}, x_1] + [x_{13}, x_2].$$

This cocycle is not well defined and depends on the choice of x_{ij}, but its image in the factor space is well defined.

In general, Massey operations are defined on n classes of any cohomology space $H^{k_1}\ (L;L)$, $H^{k_2}\ (L;L),\dots$ such that the operations of smaller order are all cohomologous to zero ([10], [13]). More generally, Massey operations can be defined in $H^{\cdot}(L;A)$, where A is any L module. In the case $A=L$, the Massey operations take value in $H^3\ (L;L)$, and they are closely connected with the obstructions to "expanding" an infinitesimal deformation of the Lie algebra.

Theorem 2. *If all the Massey products from $H^2(L;L)$ of an infinitesimal deformation are cohomologous to zero then there exists a formal deformation of the Lie algebra L, continuing the given infinitesimal deformation, see [13]. (The converse is obviously true.)*

Remark. Whether this series converges or not remains a question.

After finding the obstructions, we can compute a versal deformation for the given Lie algebra step by step. A versal deformation of order one is given with the help of infinitesimal deformations and is parametrized by $K[t_1,\dots,t_n]/(m^2)$, where m is the maximal ideal in $K[t_1,\dots,t_n]$:

$$\mu_1=\mu_0+\varphi_1 t_1+\cdots+\varphi_n t_n.$$

Let us try to extend this deformation to a versal deformation of order two parametrized by $K[t_1,\dots,t_n]/I$ where I contains m^3. The bracket should be of the form

$$\mu_t=\mu_0+\sum_{i=1}^{n}\varphi_i t_i+\sum\varphi_{ij}t_i t_j,$$

with the conditions that

$$-2\sum d\varphi_{ij}t_i t_j=\sum[\varphi_i,\varphi_j]t_i t_j \pmod I$$

(see [10] and [11]). The conditions for the coefficients of a versal deformation can be obtained from the deformation equation step by step.

V Virasoro algebra and its subalgebras

Consider the complexification $\mathfrak{L}$ of the Lie algebra of polynomial vector fields on the circle:

$$e_k\to(\exp ik\varphi)\frac{d}{d\varphi},$$

where φ is the angular parameter. The Lie algebra $\mathfrak{L}$ is called the *Witt algebra*. It is well known (see [14]) that $\mathfrak{L}$ has a unique nontrivial one-dimensional central extension. The extended Lie algebra $\widehat{\mathfrak{L}}$ is called the *Virasoro algebra*.

Theorem 3. *For the maximal nilpotent subalgebra L_1 of the Virasoro algebra,*

$$\dim H^q\ (L_1;L_1)=2q-1,$$

and the space $H^q\ (L_1;L_1)$ is generated by elements of weight $-(3q^2-q)/2+i$, where $q>0$, $i=1,2,\dots,2q-1$.

Proof. There are two alternate proofs. The first proof ([11]) uses Feigin–Fuks spectral sequences ([15]) and Goncharowa's result ([16]) on the cohomology with trivial coefficients (see also [17]). The second proof is similar to the procedure to determine the cohomology of maximal nilpotent subalgebra of a complex semisimple Lie algebra with coefficients in an irreducible representation, see [18] for details.

Corollary. *The Lie algebra L_1 has three nonequivalent infinitesimal deformations. Denote the cocycles representing the different cohomology classes by α, β, and γ, where α is of weight $-2, \beta$ of weight -3, and γ of weight -4. Such cocycles are given explicitly in [10].*

Theorem 4. *The Witt algebra and the Virasoro algebra are rigid.*

Proof. This follows from [16] and Theorem 3.

Theorem 5. *For the Lie algebra L_1, the infinitesimal deformation of weight -2 can be extended to a real deformation. The one with weight -3 can be extended to a deformation of order 2, but not of higher order, and the infinitesimal deformation of weight -4 cannot be extended at all.*

Proof. It follows from computing the possible Massey operations ([10]). A nice realization of the extended deformation of weight -2 is the following. Denote by $L_1(t)$ the Lie algebra of vector fields $(x^2+t)\varphi(t)(d/dx)$. Define a linear isomorphism $\epsilon_t : L_1 \to L_1(t)$ by the formula

$$\epsilon_t(e_i) = (x^2+t)x^{i-1}\left(\frac{d}{dx}\right) = e_i + te_{i-2}.$$

Then

$$[e_i, e_j] = \epsilon_t^{-1}[\epsilon_t(e_i), \epsilon_t(e_j)]$$

defines a deformation of L_1 of weight -2.

Theorem 6. *Let L_k denote the subalgebra of the Witt algebra with the basis $e_j = x^{j+1}(d/dx)$, $j = k, k+1, \dots$. The cohomology spaces $H^q\ (L_k; L_s)$ are finite dimensional for each positive integer k and s:*

$$\dim H^q\ (L_k; L_s) \leqslant k \dim H^q\ (L_{k+1}; \mathbb{C}) + (k+1-s) \dim H^q\ (L_k; \mathbb{C}).$$

Proof. The cohomology $H^{\cdot}(L_k; L_s)$ can be computed with the help of the spectral sequence, associated to the filtration

$$L_s \supset L_{s+1} \supset L_{s+2} \supset \cdots$$

in the coefficient module ([19]).

Corollary. *Each of the Lie algebras L_k, $k \geqslant 1$ has a finite number of nonequivalent deformations.*

Remark. The upper bound seems to be rather crude because for $q = 1$ and $k = s$ it gives

$$\dim H^1\ (L_k; L_k) \leqslant k^2 + 3k + 1,$$

while a direct computation shows that the precise dimension of this cohomology space is k.

Theorem 7. *For the Lie algebra L_0 of vector fields on the line, vanishing at the origin,*

$$H^q\ (L_0; L_0) = 0, \quad \text{for each} \ \ q \geqslant 1.$$

Particularly, the Lie algebra L_0 is rigid.

Proof. It follows from constructing the corresponding spectral sequence and the results for trivial coefficient cohomology:

$$H^q\ (L_0) = \begin{cases} \mathbb{C}, & \text{for} \ \ q = 0, 1, \\ 0, & \text{for} \ \ q > 1. \end{cases}$$

VI Affine algebras and their subalgebras

Let $\mathfrak{g}$ denote an affine Kac–Moody Lie algebra (twisted or untwisted) and $\mathfrak{g}_+$ denote its maximal nilpotent subalgebra ($\mathfrak{g} = \mathfrak{g}_- \oplus h \oplus g_+$, see e.g., [20]). Recall that the one-dimensional cohomology space with coefficients in the adjoint representation corresponds to the exterior derivations of the given Lie algebra.

Theorem 8. *A basis in the space of exterior derivations of the Lie algebra $\mathfrak{g}_+$ is the following:*

$$\mathbf{h}_i : \mathfrak{g} \to [h_i, \mathfrak{g}], \quad i = 1, \ldots, n-1,$$
$$\boldsymbol{\tau}_i : t^{is+1} \left(\frac{d}{dt}\right), \ i = 0, 1, 2, \ldots,$$

where s is the order of the exterior automorphism of the corresponding finite-dimensional finite-dimensional simple Lie algebra.

Proof. Let $\mathfrak{g} = \oplus_{i>0} \mathfrak{g}_i$ be a nilpotent graded Lie algebra and $B = \oplus B_i$ a graded $\mathfrak{g}$ module. The space of k chains $C_k^{(m)}(\mathfrak{g}; B)$ is spanned by monomials of the form

$$\mathfrak{g}_1 \wedge \cdots \wedge \mathfrak{g}_k \otimes b,$$

where $\mathfrak{g}_s \in g_{i_s}$, $b \in B_j$, $i_1 + \cdots + i_k + j = m$. Denote by $F_p C_k^{(m)}(\mathfrak{g}; B)$ the subspace of $C_k^{(m)}(\mathfrak{g}; B)$ generated by monomials with $i_1 + \cdots + i_k \leqslant p$. Obviously, $\{F_p\}$ is decreasing filtration. Let us apply the spectral sequence corresponding to this filtration to the computation of the homology of $\mathfrak{g}_+$ with coefficients in the coadjoint representation $\mathfrak{g}_+^*$ (which is equivalent to the computation of the cohomology of $\mathfrak{g}_+$ with coefficients in the adjoint representation). For each of the affine Lie algebras the terms and differentials of this spectral sequence can be explicitly determined, see [21].

Examples of infinitesimal deformations of the Lie algebra $\mathfrak{g}_+$:

(i) If $\alpha \in H^1(\mathfrak{g}_+;\mathfrak{g}_+)$ and $\beta \in H^1(\mathfrak{g}_+;\mathbb{C})$ then $\alpha\beta \in H^2(\mathfrak{g}_+;\mathfrak{g}_+)$. The number of such deformations is $\dim H^1(\mathfrak{g}_+;\mathfrak{g}_+)\cdot \dim H^1(\mathfrak{g}_+;\mathbb{C})$. (In Theorem 8 we saw that the first factor, and hence the product, is infinite.)

(ii) Let $1 \leqslant i \leqslant n$, where n is the rank of $\mathfrak{g}$. Define a deformation of $\mathfrak{g}_+$ inside $\mathfrak{g}$: e_i deforms into $e_i + tf_i$, the other additive generators of $\mathfrak{g}_+$ do not change. The number of such deformations is n.

(iii) Let $1 \leqslant i, j \leqslant n$, such that in the Cartan matrix $a_{ij} = -1$. If $a_{ij} = a_{ji}$, choose $i < j$. The Lie algebra $\mathfrak{g}_+$ again deforms inside $\mathfrak{g}$, with e_i deforming into $e_i + tf_j$, and $[e_i, e_j]$ into $[e_i, e_j] - th_j$, while the other additive generators do not change. The number of such deformations is the number of nonzero pairs (a_{ij}, a_{ji}) with $i \neq j$.

Theorem 9. *For all affine Lie algebras* $\mathfrak{g}$ *except* $\widetilde{A}_1$*:*

(i) *All the homogeneous infinitesimal deformations of* $\mathfrak{g}_+$ *may be extended to real deformations.*

(ii) *The space of infinitesimal deformations,* $H^2(\mathfrak{g}_+;\mathfrak{g}_+)$*, is spanned by deformations described in* (i), (ii)*, and* (iii). *Thus all deformations arise from these infinitesimal ones.*

Proof There are two methods. The first proof uses filtration in the cochain complex and the corresponding spectral sequence ([21]). The other method (see [22]) uses results for the maximal nilpotent algebra n_+ of finite-dimensional semisimple Lie algebras. We know $H^{\cdot}(n_+; s)$ where s is the adjoint representation of the finite-dimensional semisimple algebra s. Consider the exact sequences of n_+ modules:

$$0 \to n_+ \to s \to s/n_+ \to 0, \quad 0 \to h \to s/n_+ \to n_+^* \to 0$$

(h is the Cartan subalgebra of s). These sequences allow us to reduce the computation of $H^2(n_+; n_+)$ to that of $H^1(n_+; n_+^*)$ which can be computed directly. Generalizing this method for the infinite dimensional affine algebras, we get the statement of Theorem 9.

Theorem 10. *The case of* $\widetilde{A}_1$ *is an exceptional case because there are two additional infinitesimal deformations not listed in* (i)*,* (ii) [*type* (iii) *does not exist*]. *These two infinitesimal deformations cannot be extended to real deformations of* $\mathfrak{g}_+$*, even not order two (because their Massey square is nonzero).*

Proof By direct computation with the help of the described spectral sequence ([21]). The generalization of the finite dimensional method ([22]) does not work for this particular case.

Theorem 11. *All the affine Lie algebras are rigid.*

Proof. It follows from Theorem 9 for $\mathfrak{g} \neq \widetilde{A}_1$ and by direct computation for $\mathfrak{g} = \widetilde{A}_1$. For an independent proof see [23].

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