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AN EXAMPLE OF FORMAL DEFORMATIONS OF LIE ALGEBRAS

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Introduction

In this work we are going to investigate the formal deformations of an infinite dimensional Lie algebra of vector fields on the line with polynomial coefficients. This Lie algebra L_1 consists of the fields which vanish with their first derivative at the origin. For finding the deformations, we have to consider the cohomology with coefficients in the adjoint representation.

In Section 1 we recall – following Nijenhuis and Richardson [6] – the construction of the differential Lie superalgebra structure in the cochain complex of an arbitrary Lie algebra with coefficients in the adjoint representation. In Section 2 we apply the general theory of Schlessinger [8] to the formal deformations of a Lie algebra. In Section 3 we compute the cohomology $H^\bullet(L_1;L_1)$ with the help of the Feigin–Fuks spectral sequences [1]. In Section 4 we deal with the obstruction theory of Lie algebras and give concrete computations in the case of L_1. In Section 5 we give examples of deformations of this infinite dimensional Lie algebra.

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1. The differential Lie superalgebra $C^\bullet(L;L)$

Let L be a Lie algebra. For a positive integer q denote by $C^q(L;L)$ the space of q-linear, antisymmetric, L-valued functions on L. This is the space of q-dimensional cochains of L with coefficients in the adjoint representation. For $q<0$ put $C^q(L;L)=0$. Let $d_q=d$ denote the differential or coboundary operator $d_q=d:C^q(L;L)\to C^{q+1}(L;L)$ which acts as follows.

For $q \geqq 0$, $\varphi \in C^q(L;L)$

$$d\varphi(g_1,\dots,g_{q+1}) := \sum_{1\leqq s<t\leqq q+1} (-1)^{s+t-1}\varphi([g_s,g_t],g_1,\overset{\hat{s}\,\hat{t}}{\dots},g_{q+1}) + \\ + \sum_{1\leqq s\leqq q+1}(-1)^s\big[g_s,\varphi(g_1,\overset{\hat{s}}{\dots},g_{q+1})\big]$$

where $\hat{}$ means that the element with the indicated index is missing. For $q<0$ let $d_q = 0$. From the definition it follows that $d_{q+1}\circ d_q = 0$, so we get a complex $C^\bullet(L;L)$.

By a differential Lie superalgebra we mean a complex $C = (X_n,d)_{n=0}^\infty$ with an operation [,] such that for $x \in X_p$, $y \in X_q$

$$[x,y] = -(-1)^{p\cdot q}[y,x] \tag{1}$$

where p is the degree of x; the super-Jacobi identity is satisfied for $x\in X_p$, $y\in X_q$, $z\in X_r$:

$$(-1)^{p\cdot q}[[x,y],z] + (-1)^{q\cdot r}[[y,z],x] + (-1)^{r\cdot p}[[z,x],y] = 0 \tag{2}$$

and the differential d of degree $+1$ is such that

$$d([x,y]) = [dx,y] - (-1)^p[x,dy]. \tag{3}$$

Proposition. *The complex $C^\bullet(L;L)$ is a differential Lie superalgebra, the degree of $a\in C^p(L;L)$ being $p-1$.*

Proof. For $a\in C^p(L;L)$ and $b\in C^q(L;L)$ define the cochain $ab\in C^{p+q+1}(L;L)$ by

$$ab(g_1,\dots,g_{p+q-1}) := \sum_\sigma \operatorname{sgn}(\sigma)\, a\big(b\big(g_{i_1},\dots,g_{i_q}\big),g_{j_1},\dots,g_{j_{p-1}}\big)$$

where the sum runs over the shuffles

$$\{1,\dots,p+q+1\} = \{i_1,\dots,i_q\}\cup\{j_1,\dots,j_{p-1}\}$$

$(i_1<\dots<i_q, j_1<\dots<j_{p-1})$.

Put $[a,b] = ab-(-1)^{(p-1)(q-1)}ba$.

It is easy to verify that for this superbracket operation the identities (1)–(3) are satisfied with $c\in C^r(L;L)$.

From (3) it follows that if a,b are cocycles then the superbracket $[a,b]$ is also a cocycle, and the cohomology class of $[a,b]$ depends only on the class of a and b. That means that a multiplication can be defined in the cohomology space

$$H^p(L;L)\otimes H^q(L;L) \longrightarrow H^{p+q-1}(L;L)$$

which satisfies (1) and (2) with $a\in H^p(L;L)$, $b\in H^q(L;L)$, $c\in H^r(L;L)$. □

Corollary. *The Lie superalgebra structure on $C^\bullet(L;L)$ induces a structure of Lie superalgebra on the cohomology space, in which the usual grading is reduced by one. In this way we get an analogy with the Kodaira–Spencer theory* (see [6]).

2. Formal deformations of Lie algebras. General theory

In this section we explain, how the general theory of Schessinger applies to formal deformations of Lie algebras.

Let L be a Lie algebra over a field K. Let $\mathcal{C}$ be the category of local finite dimensional algebras A over K. For such an A there exists a unique maximal ideal m_A such that $A/m_A = K$ and $\dim_K A$ is finite. Let us denote by ε the canonical map $A \longrightarrow A/m_A = K$. If $t_1, \dots, t_n$ are elements of m_A such that their images in m_A/m_A^2 form a basis, then $A = K[[t_1, \dots, t_n]]/I$ where I contains a power of the maximal ideal of $K[[t_1, \dots, t_n]]$. The morphisms in $\mathcal{C}$ are the homomorphisms of local algebras (so commuting with ε).

A deformation L_A of L parametrized by $A \in \mathcal{C}$ is a Lie algebra structure over A on $L \otimes_K A$ such that the Lie algebra structure on

$$L = (L_A) \otimes_A K = (L \otimes A) \otimes_A K$$

is the given one on L (obtained from L_A by the extension of the scalars given by ε). If $f : A \to B$ is a morphism in $\mathcal{C}$ then the Lie algebra $L_B = (L_A) \otimes_A B$ is the deformation of L parametrized by B induced by f from L_A.

Two deformations L_A and L'_A of L parametrized by A are equivalent, if there exists a Lie algebra isomorphism over A of L_A on L'_A inducing the identity of $L_A \otimes_A K = L$ on $L'_A \otimes_A K = L$.

The functor $F : \mathcal{C} \to \text{Sets}$ associates to $A \in \mathcal{C}$ the set $F(A)$ of equivalence classes of deformations of L parametrized by A.

The algebra A is of order less or equal to k if $m^{k+1} = 0$. (For instance if $k = 1$, $K = \mathbb{C}$, $t_1, \dots, t_n$ form a basis of m/m^2, then $A = \mathbb{C} \cdot 1 \oplus \mathbb{C}t_1 \oplus \cdots \oplus \mathbb{C}t_n$ and $t_i t_j = 0$ for all $1 \leqq i, j \leqq n$.)

A deformation L_A is of order k if A is of order $\leqq k$. An infinitesimal deformation is a deformation of order 1, parametrized by $K[t]/(t^2)$.

More generally, let $\hat{\mathcal{C}}$ be the category of complete local algebras A over K such that $A/m_A^n \in \mathcal{C}$ for all n (complete means that $A = \varprojlim A/m_A^n$).

A deformation of L parametrized by A is the projective limit $\varprojlim L_A/m_A^n$ of deformations of L parametrized by A/m_A^n. In other words, a deformation of L parametrized by $A \in \hat{\mathcal{C}}$ is a Lie algebra structure over A on $L \hat{\otimes} A = \varprojlim L \otimes A/m_A^n$, inducing the given structure on L.

There is an analogous definition for isomorphisms of deformations parametrized by $A \in \hat{\mathcal{C}}$. The functor F can be extended as a functor $\hat{F} : \hat{\mathcal{C}} \to \text{Sets}$.

A deformation L_R of L parametrized by $R \in \hat{\mathcal{C}}$ is called *formally versal* if for any deformation L_A of L parametrized by $A \in \hat{\mathcal{C}}$ there is a morphism $f : R \to A$ such that $L_R \otimes_R A$ is equivalent to L_A and if the map $m_R/m_R^2 \to m_A/m_A^2$ induced by f is unique. (In particular, all the other deformations of L can be induced from the versal one.)

A versal deformation up to order k is defined similarly, where $\hat{\mathcal{C}}$ is replaced by the subcategory of algebras of order $\leqq k$.

Theorem (Schlessinger). *If the space $H^2(L;L)$ is finite, then there exists a formal versal deformation of L.*

Proof. This follows from the general Theorem of Schlessinger ([8, Theorem 2.11]), provided we check the following properties of the functor F.

Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathcal{C}$, and consider the map

$$\tau : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

associating to the equivalence class of a deformation of L parametrized by $A' \times_A A''$ the equivalence classes of the deformations of L parametrized by A' and A'' respectively, induced by the morphism $A' \times_A A'' \to A'$ and $A' \times_A A'' \to A''$. Then

a) τ is surjective whenever $A'' \to A$ is surjective;

b) τ is bijective when $A = K$.

To check a) consider two deformations $L_{A'}$ and $L_{A''}$ of L parametrized by A' and A'' respectively, and such that there exists an equivalence ξ of $L'_A = L_{A'} \otimes_{A'} A$ on $L''_A = L_{A''} \otimes_{A''} A$ of the associated deformations of L parametrized by A. The equivalence classes of $L_{A'}$ and $L_{A''}$ give an element of $F_{A'} \times_{F_A} F_{A''}$. As the morphism $A'' \to A$ is surjective and as $L_{A''}$ is a free A''-module, after changing $L_{A'}$ in its equivalence class, we can assume that the equivalence ξ is the identity of $L_A = L \otimes A$, using the canonical A-module isomorphism of L'_A and L''_A with $L \otimes A$. Then the image by τ of the equivalence class of the deformation $L_{A'} \times_{L_A} L_{A''}$ of L parametrized by $A' \times_A A''$ will be the given element of $F_{A'} \times_{F_A} F_{A''}$.

The condition b) can be easily verified because in that case $B = A' \times_A A''$ is equivalent to $K \cdot 1 \oplus m_{A'} \oplus m_{A''}$ with $m_{A'} \cdot m_{A''} = 0$ and a deformation of L parametrized by $A' \times_A A''$ is characterized by the induced deformation parametrized by A' and A''. □

To link this section to the preceding one we can give a more concrete description of the Lie algebra L_A.

Suppose $A \in \hat{\mathcal{C}}$, then $A = K[[t_1, \ldots, t_n]]/I$. We can express $L\hat{\otimes}A$ in the form $L\hat{\otimes}K[[t_1, \ldots, t_n]]/I$. The Lie algebra structure on L_A will be described by a bilinear alternate map

$$\mu_t : L \times L \to L\hat{\otimes}K[[t_1, \ldots, t_n]] \text{ such that}$$
$$\mu_t(x,y) = [x, yl \otimes 1 + \sum_{|\alpha| \geqq 1} \varphi_\alpha(x,y) \otimes t^\alpha$$

where $\varphi_\alpha \in C^2(L;L)$ and $\mu_t(x,y)$ composed with the projection on $L\hat{\otimes}A$ is equal to $[x \otimes 1, y \otimes 1]_{L_A}$. This lifting is unique mod I.

The map μ_t will define a bracket in $L\hat{\otimes}A$ verifying the Jacobi identity iff

$$2d\varphi + [\varphi, \varphi] \equiv 0 \bmod I$$

where d and $[\ ,\ ]$ were defined in Section 2. This means that the coefficients of the formal power series obtained from

$$2 \sum_{|\alpha| \geqq 1} (d\varphi_\alpha) t^\alpha + \sum_{|\alpha| \geqq 1} \sum_{\beta+\gamma=\alpha} [\varphi_\beta, \varphi_\gamma] t^\beta t^\gamma$$

by applying to each coefficient an arbitrary linear form on $C^2(L;L)$ belongs to I.

In case $|\alpha| = 1$ we get $d\varphi_\alpha = 0$ for each α which means that φ_α is a cocycle.

The elements of $H^2(L;L)$ correspond bijectively to the equivalent classes of infinitesimal deformations. Suppose that $\dim H^2(L;L)$ is finite. Let φ_1, φ_2, …, φ_n be cocycles whose cohomology classes form a basis of $H^2(L;L)$. Then a versal deformation of order 1 of L is parametrized by $K[t_1,\dots,t_n]/(m^2)$ and is given by the bilinear alternate map

$$\mu_t = \mu_0 + \varphi_1 t_1 + \dots + \varphi_n t_n$$

where μ_0 is the original bracket in L.

Let us try to extend this deformation to a versal deformation of order 2 parametrized by $K[t_1,\dots,t_n]/I$ where I contains m^3. The bracket should be of the form

$$\mu_t = \mu_0 + \sum_{i=1}^{n} \varphi_i t_i + \sum \varphi_{ij} t_i t_j$$

with the conditions that

$$-2\Sigma d\varphi_{ij} t_i t_j \equiv \Sigma[\varphi_i, \varphi_j] t_i t_j \bmod I.$$

This means that the right-hand side (which is always a three-cocycle) must be coboundary. So the ideal I is generated by the polynomials, obtained by composing the cohomology class of the right-hand side with linear forms on $H^3(L;L)$, and m^3. For φ_{ij} one can choose any 2-cochain satisfying the above condition.

3. Computation of $H^\bullet(L_1;L_1)$

Let $W^{\mathrm{pol}} = W_1$ be the Lie algebra of vector fields on the line with polynomial coefficients $f(x)\frac{d}{dx}$. This Lie algebra has an additive algebraic basis

$$e_i = x^{i+1}\frac{d}{dx}, \qquad i \geqq -1.$$

In this basis the bracket operation is

$$[e_i, e_j] = (j-i)e_{i+j}.$$

Let us introduce the subalgebra L_i, $i \geqq 0$ of W_1 which is generated by the basis elements $\{e_i, e_{i+1}, \dots\}$.

We shall investigate the subalgebra L_1. The Lie algebra L_1 is naturally graded, the weight of e_i equals i. With this grading L_1^{pol} is a graded Lie algebra: $L_1^{\mathrm{pol}} = \bigoplus_{m=1}^{\infty} L_1^{(m)}$. We consider the cohomology of L_1 with coefficients in the adjoint representation. The cochain complex is defined in the graded sense:

$$C^q(L_1;L_1) = \oplus C^q_{(m)}(L_1;L_1),$$

where for the cochain $\varphi \in C^q_{(m)}(L_1;L_1)$ the weight of $\varphi(e_{i_1},\dots,e_{i_q})$ is $m + i_1 + \dots + i_q$. The grading is inherited by the cohomology spaces $H^q(L_1;L_1)$.

Theorem (see [2]). *For $q>0$, $H^q_{(m)}(L_1;L_1)\cong H^{q-1}_{(m)}(L_2;\mathbb{C})$. The cohomology space $H^q(L_1;L_1)$ has dimension $2q-1$ and is generated by elements of weight $-\frac{3q^2-q}{2}+i$ where $i=1,2,\dots,2q-1$.*

In particular, $H^1(L_1;L_1)$ is of dimension 1 and has weight 0; the space $H^2(L_1;L_1)$ is three-dimensional with generators of weight $-2,-3$ and -4, while $\dim H^3(L_1;L_1)=5$ with generators of weight $-7,-8,-9,-10$ and -11.

Proof. Define the module F_λ over W_1, where $\lambda\in\mathbb{C}$ is arbitrary, as the space of expressions $f(x)dx^{-\lambda}$ where $f(x)$ is a formal power series of x (see [1]). Then the formula

$$\left(g\frac{d}{dx}\right)fdx^{-\lambda}=(gf'-\lambda fg')dx^{-\lambda}$$

gives the action of W_1 in F_λ. (For λ an integer they are modules of formal tensor fields; formal power series for $\lambda=0$, formal differential 1-forms for $\lambda=-1$ and formal vector fields for $\lambda=1$.) The module F_λ has an additive basis $\{f_j\mid j=0,1,\dots\}$ where $f_j=x^jdx^{-\lambda}$ and the action on the basis elements is

$$e_if_j=(j-(i+1)\lambda)f_{i+j}.$$

Denote by $\mathcal{F}_\lambda$ the W_1-module which is defined in the same way, only the index j runs over all integers. The adjoint modules F'_λ, $\mathcal{F}'_\lambda$ are defined as modules of linear functionals $F_\lambda\to\mathbb{C}$, $\mathcal{F}_\lambda\to\mathbb{C}$ which are finite in the sense that they take nonzero value only on a finite number of f_j-s. That means F'_λ and $\mathcal{F}'_\lambda$ are generated by elements f'_j and W_1 acts on them by the formula

$$e_if'_j=\begin{cases}(-(j-i)+(i+1)\lambda)f'_{j-i} & \text{if } f'_j\in\mathcal{F}'_\lambda \text{ or } j\geqq i,\\ 0 & \text{if } f'_j\in F'_\lambda \text{ and } j<0.\end{cases}$$

The correspondence $f'_j\leftrightarrow f_{-1-j}$ defines for any λ an isomorphism $\mathcal{F}'_\lambda=\mathcal{F}_{-1-\lambda}$ and as $F_{-1-\lambda}=\operatorname{ann}F_\lambda$, it follows that $F'_\lambda=\mathcal{F}_{-1-\lambda}/F_{-1-\lambda}$.

For $\lambda\neq 0$ the W_1-module F_λ is irreducible. But if we consider it as an L_0-module, it is reducible. For obtaining the L_0-submodules of the module F_λ it is sufficient to take the subspace generated by those f_j-s with $j\geqq\mu$ where μ is some positive integer. Denote the L_0-module we get by $F_{\lambda,\mu}$. We can define it directly as the subspace, generated – like F_λ – by the elements f_j, $j=0,1,\dots$, on which L_0 acts by

$$e_if_j=(j+\mu-(i+1)\lambda)f_{i+j}.$$

In this definition μ can be an arbitrary complex number. (For positive integer μ the embedding $F_{\lambda,\mu}\to F_\lambda$ is defined by the formula $f_j\mapsto f_{j+\mu}$.) Let $F'_{\lambda,\mu}$ denote the module, conjugate to $F_{\lambda,\mu}$. At last define the modules $\mathcal{F}_{\lambda,\mu}$ over W_1 as $F_{\lambda,\mu}$ above, without requiring the positivity of j. Obviously $\mathcal{F}'_{\lambda,\mu}=\mathcal{F}_{-1-\lambda,-\mu}$ and $F'_{\lambda,\mu}=\mathcal{F}_{-1-\lambda,-\mu}/F_{-1-\lambda,-\mu}$.

All these modules are graded. Their basis elements f_j are homogeneous and the grading is defined by $\deg f_j=j$, $\deg f'_j=-j$. (Mention that in $F_{\lambda,\mu}$ and $\mathcal{F}_{\lambda,\mu}$ the grading is independent of λ and μ.)

Our aim is to calculate the homology of the Lie algebra L_1 with coefficients in $\mathcal{F}_{\lambda,\mu}$ and $F_{\lambda,\mu}$. (The calculations for the modules $\mathcal{F}_\lambda$ and F_λ see in [1] or [3].)

The space of chains $C_q^{(m)}(L_1;\mathcal{F}_{\lambda,\mu})$ is generated by "monomials", i.e. by the chains

$$f_j \otimes e_{i_1} \wedge \cdots \wedge e_{i_q} \quad \text{with} \quad j + i_1 + \cdots + i_q = m.$$

Denote by $G_pC_q^{(m)}(L_1;\mathcal{F}_{\lambda,\mu})$ the subspace of the space $C_q^{(m)}(L_1;\mathcal{F}_{\lambda,\mu})$, generated by monomials with $i_1 + \cdots + i_q \leqq p$. Evidently, $\{G_pC_q^{(m)}(L_1;F_{\lambda,\mu})\}_p$ is a decreasing filtration in $C_\bullet^{(m)}(L_1;\mathcal{F}_{\lambda,\mu})$.

Denote the spectral sequence corresponding to this filtration by $E(\lambda, m-\mu)$. In this spectral sequence $E^0_{p,q} = C^{(p)}_{p+q}\big(L_1;(\mathcal{F}_{\lambda,\mu})_{m-p}\big)$ where $(\mathcal{F}_{\lambda,\mu})_{m-p}$ is considered as a trivial L_1-module, and $d^0_{p,q}$ is the differential

$$d_{p+q} : C^{(p)}_{p+q}(L_1;\mathbb{C}) \to C^{(p)}_{p+q-1}(L_1;\mathbb{C}).$$

Hence

$$E^1_{p,q} = H^{(p)}_{p+q}(L_1;\mathbb{C}).$$

From [5] it follows that $E^1_{p,q} = \mathbb{C}$ for $p = \frac{3r^2 \pm r}{2}$, $p+q = r$ and $E^1_{p,q} = 0$ for other p and q. We set $E^r_p = \oplus_q E^r_{p,q}$; $d^r_p = \oplus_q d^r_{p,q}$, then obviously

$$H^{(m)}_q(L_1;\mathbb{C}) \cong E^\infty_{\frac{3q^2+q}{2}} \oplus E^\infty_{\frac{3q^2-q}{2}}.$$

If the coefficients are taken not in $\mathcal{F}_{\lambda,\mu}$ but in $F_{\lambda,\mu}$ then the filtration is the same. The new spectral sequence can be mapped into the old one. On E^1_p with $p \leqq m$ this map is an isomorphism, and for $p > m$ we have in the new spectral sequence $E^1_p = 0$. If we "truncate" the spectral sequence $E(\lambda, m-\mu)$ from the other side, leaving in it the part corresponding to $p \geqq m$, then obviously we get a spectral sequence converging to $H^{(m)}_\bullet(L_1;F'_{-1-\lambda,-\mu})$. (We recall that $F'_{-1-\lambda,-\mu} = \mathcal{F}_{\lambda,\mu}/F_{\lambda,\mu}$.)

Let us set $e(t) = (3t^2 + t)/2$ (Euler polynomial) and define the k-th parabola ($k = 0, 1, 2, \dots$) as a curve on the complex plane with the parametric equations

$$\lambda = e(t) - 1$$
$$m - k = e(t) + e(t+k) - 1.$$

If in the second equation we take a negative integer k then we get another parametric equation for the $|k|$-th parabola.

For $k_1, k_2 \in \mathbb{Z}$ we set

$$P(k_1,k_2) = (e(k_1) - 1, e(k_1) + e(k_2) - 1)$$

(the points $P(k_1,k_2)$ are pairwise distinct) and let

$$\mathbb{P} = \{P(k_1,k_2) \mid k_1,k_2 \in \mathbb{Z}\}.$$

Lemma. (i) *If the point* $(\lambda, m-\mu)$ *does not lie on any parabola then in the spectral sequence* $E(\lambda, m-\mu)$ *all the first differentials are different from zero.*

(ii) *If the point* $(\lambda, m-\mu)$ *lies on the k-th parabola but does not lie on parabolas with smaller indices, and is not contained in* $\mathbb{P}$*, then the differentials* $d_1(r)$ *with* $r \leqq k$ *and* $d_2(k+2s,1)$*,* $d_2(k+2s-1,2)$ *with* $s>0$ *are nontrivial in the spectral sequence* $E(\lambda, m-\mu)$.

For the proof of this Lemma see [1, Lemma 3.1.(A) and (B)].

From the Lemma the generated version of Theorem 4.1(A) and 4.2(A) in [1] follows easily. □

Theorem a). *If* $(\lambda, m-\mu) \notin \mathbb{P}$ *then*

$$H_{\bullet}^{(m)}(L_1;F_{\lambda,\mu}) = 0;$$

if the point $(\lambda, m-\mu)$ *does not lie on any of the parabolas of the Lemma then*

$$H_q^{(m)}(L_1;F_{\lambda,\mu}) = H_q^{(m)}(L_2;\mathbb{C}).$$

Another theorem we need is the generalization of Theorem 3.1 in [1] for $F'_{\lambda,\mu}$-modules.

Theorem b). *For those* $(\lambda, m-\mu)$ *considered in Theorem a),*

$$H_q^{(m)}(L_1;\mathcal{F}_{\lambda,\mu}/F_{\lambda,\mu}) = H_{q-1}^{(m)}(L_1;F_{\lambda,\mu}) \oplus H_q^{(m)}(L_1;\mathcal{F}_{\lambda,\mu}).$$

For the proof let us consider the short exact sequence

$$0 \to F_{\lambda,\mu} \to \mathcal{F}_{\lambda,\mu} \to \mathcal{F}_{\lambda,\mu}/F_{\lambda,\mu} \to 0.$$

One should check that the homomorphism $H_q(L_1;F_{\lambda,\mu}) \to H_q(L_1;\mathcal{F}_{\lambda,\mu})$ is always trivial, which is evident from Theorem a) for dimensional reasons.

The adjoint representation is $F_{1,1}$. Remark that $H^q(L_1;F_{1,1})$ is dual to $H_q(L_1;F'_{1,1})$ in the graded sense, i.e.

$$H^q_{(-m)}(L_1;L_1) = H_q^{(m)}(L_1;L'_1) = H_q^{(m)}(L_1;F'_{1,1}) = H_q^{(m)}\big(L_1;\mathcal{F}_{-2,-1}/F_{-2,-1}\big).$$

In our case $\lambda = -2$, $\mu = -1$. The line $\lambda = -2$ does not intersect any of the parabolas of the Lemma. From Theorem a) and b) we obtain that

$$\dim H^q_{(-m)}(L_1;L_1) = \dim H^q_{(-m)}(L_2;\mathbb{C}).$$

The spaces $H^\bullet(L_q;\mathbb{C})$ are calculated in [5]. By comparing the results of that calculations with the above one we get the required Theorem.

Remark. A cocycle φ, representing a generator of $H^1(L_1;L_1)$ has the form $\varphi(e_i) = ie_i$. We know that each element of $H^1(L;L)$ defines a Lie algebra, containing L_1 as an ideal of codimension 1. In the present case we get L_0.

Let us denote by α,β and λ the three homogeneous nonzero elements of weights $-2,-3$ and -4 in $H^2(L_1;L_1)$. It is not difficult to find cocycles in those cohomology classes.

Proposition. *Such cocycles are for instance* $\overline{\alpha} \in C^2_{(-2)}(L_1;L_1)$, $\overline{\beta} \in C^2_{(-3)}(L_1;L_1)$, $\overline{\gamma} \in C^2_{(-4)}(L_1;L_1)$, *defined as follows.*

$$\overline{\alpha}(e_2,e_3) = 4e_3,$$
$$\overline{\alpha}(e_2,e_j) = je_j, \quad \overline{\alpha}(e_3,e_j) = -(j-1)e_{j+1} \text{ for } j \geqq 4,$$
$$\overline{\alpha}(e_i,e_j) = 0 \text{ for other } i,j;$$
$$\overline{\beta}(e_2,e_3) = 8e_2, \quad \overline{\beta}(e_2,e_4) = 4e_3, \quad \overline{\beta}(e_3,e_4) = -10e_4,$$
$$\overline{\beta}(e_2,e_j) = (j+1)e_{j-1},\ \overline{\beta}(e_3,e_j) = -2je_j,\ \overline{\beta}(e_4,e_j) = (j-1)e_{j+1} \text{ for } j \geqq 5,$$
$$\overline{\beta}(e_i,e_j) = 0 \text{ for other } i,j;$$
$$\overline{\gamma}(e_2,e_3) = 14e_1, \quad \overline{\gamma}(e_2,e_5) = 8e_3, \quad \overline{\gamma}(e_3,e_4) = -24e_3,$$
$$\overline{\gamma}(e_3,e_5) = -16e_4, \quad \overline{\gamma}(e_4,e_5) = 18e_5$$
$$\left.\begin{array}{ll} \overline{\gamma}(e_2,e_j) = (j+2)e_{j-2}, & \overline{\gamma}(e_3,e_j) = -3(j+1)e_{j-1} \\ \overline{\gamma}(e_4,e_j) = 3je_j, & \overline{\gamma}(e_5,e_j) = -(j-1)e_{j+1} \end{array}\right\} \text{ for } j \geqq 6,$$
$$\overline{\gamma}(e_i,e_j) = 0 \text{ for other } i,j.$$

Proof. The fact that $\overline{\alpha},\overline{\beta}$ and $\overline{\gamma}$ are cocycles, follows from direct computations, and the fact that they are not cohomological to zero follows from the next

Lemma. *Each class of* $H^2_{(-m)}(L_1;L_1)$ *with* $m \geqq 2$ *is represented by a unique cocycle* ω, *which vanishes on* $e_1 : \omega(e_1,e_j) = 0$ *for all* j. □

Remark. Using this lemma we can give an elementary proof of the fact that α,β and γ generate $H^2(L_1;L_1)$.

Remark. By constructing the cocycles $\overline{\alpha},\overline{\beta}$ and $\overline{\lambda}$ we can give all the nonequivalent infinitesimal deformations of the Lie algebra L_1.

4. Obstructions

A natural question is when is it possible to extend an infinitesimal deformation to a deformation of higher order. To extend an infinitesimal deformation represented by a cocycle φ_1 to a second order deformation, parametrized by $\mathbb{C}[t]/(t^3)$ it is necessary and sufficient that $[\varphi_1,\varphi_1]$ is cohomological to zero. If φ_2 is a cochain such that

$$-2d\varphi_2 = [\varphi_1,\varphi_1]$$

then we can define a 2-order deformation with the bracket

$$\mu_t(x,y) = [x,y] + \varphi_1(x,y)t + \varphi_2(x,y)t^2.$$

Here φ_2 is well-defined up to a 2-cocycle. The cohomology class of $[\varphi_1,\varphi_1]$ is the first obstruction to forming a one-parameter family of deformations whose first term is cohomological to φ_1.

To extend now a second order deformation to a third-order one, parametrized by $\mathbb{C}[t]/(t^4)$, it is necessary and sufficient that $[\varphi_1,\varphi_2]$ is also cohomological to zero. If φ_3 is a cochain such that

$$-2d\varphi_3 = [\varphi_1,\varphi_2]$$

then we can define a 3-order deformation with the bracket

$$\mu_t = \mu_0 + \varphi_1 t + \varphi_2 t^2 + \varphi_3 t^3.$$

Here φ_3 is also well-defined up to a 2-cocycle. The cohomology class of $[\varphi_1,\varphi_2]$ is the second obstruction to forming a one-parameter family of deformations whose first term is cohomological to φ_1. If we take another cocycle $\varphi_2' = \varphi_2 + c$ then the obstruction is the class of $[\varphi_1,\varphi_2]+[\varphi_2,c]$ which is in the factorspace $H^3(L_1;L_1)/H^2(L_1;L_1)$.

We can extend a given deformation step by step, until the first obstruction appears. As we see, each further step depends on the choice of the earlier ones.

In general, let us define in $H^\bullet(L;L)$ higher order operations, called Massey operations. These n-order operations are partially defined and they are well-defined mod the $(n-1)$-order ones. The second-order operation is the superbracket. Suppose that $y_1 \in H^p(L;L)$, $y_2 \in H^q(L;L)$ and $y_3 \in H^r(L;L)$ are such that $[y_i,y_j]=0$, i.e. for cocycles x_i representing y_i, $[x_i,x_j]=dx_{ij}$. Then the third-order $\langle y_1,y_2,y_3\rangle$ Massey operation (cube) takes value in the factorspace

$$H^{p+q+r-3}(L;L)/[y_1,H^{q+r-2}(L;L)+[y_2,H^{p+r-2}(L;L)]+[y_3,H^{p+q-2}(L;L)]$$

and is equal to the image of the cohomology class of the cocycle

$$[x_{12},x_3]+[x_1,x_{23}]+(-1)^{q\cdot r}[x_{13},x_2].$$

The cohomology class of this cocycle depends on the choice of x_{ij}, but its image in the factorspace is well-defined. The next Massey operation $\langle y_1,y_2,y_3,y_4\rangle$ is defined on four elements $y_1 \in H^p(L;L)$, $y_2 \in H^q(L;L)$, $y_3 \in H^r(L;L)$, $y_4 \in H^s(L;L)$ when all the operations of lower order are defined and are cohomological to zero. It takes value in the factorspace

$$\begin{aligned} H^{p+q+r+s-5}(L;L)/[y_1,H^{q+r+s-3}]&+[y_2,H^{p+r+s-3}] \\ &+[y_3,H^{p+q+s-3}]+[y_4,H^{p+q+r-3}] \end{aligned}$$

etc. For the general definition see Retakh [7].

If $y_i \in H^2(L;L)$ then these operations take value in a factorspace of $H^3(L;L)$. The Massey operations are closely connected with the "obstructions" to "extending" an infinitesimal deformation of the Lie algebra.

Theorem (Retakh, [7]). *Given an element α belonging to $H^2(L;L)$ there exists a formal deformation of the Lie algebra L parametrized by $K[[t]]$ with infinitesimal deformation α if and only if all the Massey products $\langle\alpha,\dots,\alpha,\alpha\rangle$ are zero.*

Remark. The question of convergence of this formal power series remains open in general.

Theorem 1. *In the case of L_1 the Massey products $\underbrace{\langle\alpha,\alpha,\dots,\alpha\rangle}_{i}$ are zero for all i, the brackets $[\beta,\beta],[\alpha,\beta]$ and $[\alpha,\gamma]$ are trivial, while $[\gamma,\gamma]$ and $[\beta,\gamma]$ are not. The only nontrivial 3-products are $\langle\beta,\beta,\beta\rangle$ and $\langle\alpha,\beta,\beta\rangle$. The higher operations are either not defined or they are trivial.*

Proof. The superbrackets $[\alpha,\alpha],[\beta,\beta]$ are trivial, because the weight of $[\alpha,\alpha]$ and $[\beta,\beta]$ equals -4 and -6 and any such three-dimensional cohomology class is zero. Similarly, by dimensional considerations we have $[\alpha,\beta]=[\alpha,\gamma]=0$.

The triviality of the class $\underbrace{\langle\alpha,\alpha,\dots,\alpha\rangle}_{i}$ for any i follows from the fact that there exists a deformation with infinitesimal deformation, equal to $-\frac{1}{3}\alpha$. Namely, the deformation is

$$[e_i,e_j]_t=(j-i)e_{i+j}+(j-i)te_{i+j-2}.$$

A geometric realization of this deformation is the following. Denote by $L_1(t)\subset W_1$ the algebra of vector fields $(x^2+t)\varphi(x)\frac{d}{dx}$. Define a linear isomorphism $\varepsilon_t: L_1\to L_1(t)$ by the formula

$$\varepsilon_t(e_i)=(x^2+t)x^{i-1}\frac{d}{dx}=e_i+te_{i-2}.$$

Then

$$[e_i,e_j]_t=\varepsilon_t^{-1}[\varepsilon_t(e_i),\varepsilon_t(e_j)]$$

gives the above deformation of the Lie algebra L_1.

To prove $[\gamma,\gamma]\neq 0$ substitute $\overline{\gamma}$ into the superbracket formula. The three-dimensional cocycle we obtain is not cohomological to zero, as its value on a homology class of weight -8 is different from zero. Direct computation shows that $[\overline{\gamma},\overline{\gamma}]$ has nonzero value on the class of weight -8.

Let us verify that $[\beta,\gamma]\neq 0$. Here the triviality would imply that L_1 has a deformation over $K[t_1,t_2]/(t_1^2,t_2^2)$ such that in the deformed algebra

$$[e_i,e_j]_{t_1,t_2}=(j-i)e_{i+j}+\overline{\beta}(e_i,e_j)t_1e_{i+j-3}+\overline{\gamma}(e_i,e_j)t_2e_{i+j-4}+\kappa(e_i,e_j)t_1t_2e_{i+j-7}.$$

Straight calculation shows that such an algebra cannot exist. Namely, the numbers $\kappa_{i,j}=\kappa(e_i,e_j)$ can be defined step by step. At the 12-th step we get a system of equations for κ_{ij} from the Jacobi identities, which has no solution.

The nontriviality of the class $\langle\beta,\beta,\beta\rangle$ is equivalent to the fact that for any Lie algebra over $K[t]/(t^4)$ with the basis $\{e_i, i=1,2,\dots\}$ the bracket cannot be of the following form:

$$[e_i,e_j]_t=(j-i)e_{i+j}+t\overline{\beta}(e_i,e_j)e_{i+j-3}+t^2\kappa_1(e_i,e_j)e_{i+j-6}+t^3\kappa_2(e_i,e_j)e_{i+j-9}.$$

Here $\kappa_1(e_i,e_j)$ and $\kappa_2(e_i,e_j)$ can be defined step by step $(i+j=1,2,\dots)$ from the system of equations, following from the Jacobi identity. For $i+j=12$ we get a contradiction.

The three-bracket $\langle\alpha,\alpha,\beta\rangle$ is also defined, because the two-brackets are zero. It is defined mod $[\beta,H^2(L_1;L_1)]+[\alpha,H^2(L_1;L_1)]$. As the weight of $[\beta,\gamma]$ equals to -7, the three-bracket $\langle\alpha,\alpha,\beta\rangle$ is trivial. The last three-bracket which can be defined is $\langle\alpha,\beta,\beta\rangle$. For computing it we have to choose a coboundary for $[\alpha,\beta],[\alpha,\alpha]$ and $[\beta,\beta]$. It turns out that $\langle\alpha,\beta,\beta\rangle$ is not containing zero.

The only four-bracket which can be defined is $\langle\alpha,\alpha,\alpha,\beta\rangle$ which occurs to be the trivial cohomology class. □

A versal deformation of the Lie algebra L_1 of order 1 is given by the bilinear map

$$\mu_{t_1,t_2,t_3}=\mu_0+\overline{\alpha}t_1+\overline{\beta}t_2+\overline{\gamma}t_3$$

and is parametrized by $\mathbb{C}[t_1,t_2,t_3]/(t^2)$.

The nontrivial superbrackets give the equation for the parameter space of the versal deformation of order 2:

$$[\beta,\gamma]t_2t_3+[\gamma,\gamma]t_3^2=0.$$

As $[\beta,\gamma]$ and $[\gamma,\gamma]$ have different grading, we conclude that the parameter space is $\mathbb{C}[t_1,t_2,t_3]/I$ where I is generated by t_2t_3, t_3^2 and m^3.

Theorem 2. *A versal deformation of order two of the Lie algebra L_1 is parametrized by $\mathbb{C}[t_1,t_2,t_3]/I$ and is of the form*

$$\mu_{t_1,t_2,t_3}=\mu_0+\overline{\alpha}t_1+\overline{\beta}t_2+\overline{\gamma}t_3+\sum_{i,j=1}^{3}\varphi_{ij}t_it_j$$

where the coefficients φ_{ij} satisfy the identity

$$-2\sum_{i,j}d\varphi_{ij}t_it_j=\sum_{i,j}[\varphi_i,\varphi_j]t_it_j \mod I$$

(here $\varphi_1=\overline{\alpha}$, $\varphi_2=\overline{\beta}$ and $\varphi_3=\overline{\gamma}$).

5. Examples of deformations

Let us now define three real deformations of the Lie algebra L_1 with the brackets

$$[e_i,e_j]_t^1=(j-i)(e_{i+j}+te_{i+j-1});$$
$$[e_i,e_j]_t^2=\begin{cases}(j-i)e_{i+j} & \text{if } i,j>1,\\ (j-i)e_{i+j}+tje_j, & \text{if } i=1;\end{cases}$$
$$[e_i,e_j]_t^3=\begin{cases}(j-i)e_{i+j} & \text{if } i,j\neq 2\\ (j-i)e_{i+j}+tje_j, & \text{if } i=2.\end{cases}$$

These deformations have infinitesimal deformations of weight $-1,-1$ and -2. Denote the three Lie algebra families by $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$. They can be realized as families of subalgebras of L_0. In the first deformation e_i deforms into e_i+te_{i-1}, $i>0$. In other words, $L_1^{(1)}$ consists of the vector fields on the line which vanish at 0 and t. In the second one the e_i-s, $i > 1$ remain and e_1 deforms into $e_1 + te_0$, while in the third one e_2 turns into $e_2 + te_0$ and the rest elements remain.

Theorem. *The Lie algebra families $L_1^{(1)}$, $L_1^{(2)}$ and $L_1^{(3)}$ are nontrivial and pairwise nonisomorphic.*

Proof. The commutant $[L_1^{(1)}, L_1^{(1)}]$ consists of vector fields, vanishing at 0 and t together with their first derivative. From this it follows that $\operatorname{codim}[L_1^{(1)}, L_1^{(1)}] = 2$. At the same time $\operatorname{codim}[L_1^{(2)}, L_1^{(2)}] = \operatorname{codim}[L_1^{(3)}, L_1^{(3)}] = 1$ since $[L_1^{(2)}, L_1^{(2)}]$ is isomorphic to L_2 and $[L_1^{(3)}, L_1^{(3)}]$ is isomorphic to $\mathbb{C}e_1 \oplus e_3$. Finally $[L_1^{(2)}, L_1^{(2)}]$ is not isomorphic to $[L_1^{(3)}, L_1^{(3)}]$, because the first Lie algebra has three generators: e_2, e_3, e_4, while the second one has only two: e_1, e_3. So the three families are pairwise nonisomorphic and the last two ones are nontrivial. The nontriviality of the first family is obvious. □

Remark 1. In the third family the infinitesimal deformation is $\overline{\alpha}$. More exactly, the cocycle $\alpha_3(e_2, e_j) = je_j$, $\alpha_3(e_i, e_j) = 0$ for $i \neq 2$ is cohomological to $\overline{\alpha}$, as $\overline{\alpha} - \alpha_3$ is the coboundary of the one-cochain $\kappa(e_3) = e_1$, $\kappa(e_i) = 0$ for $i \neq 3$.

Remark 2. The first two families have trivial infinitesimal deformations (the coefficient of t is a coboundary). If we change the bracket with adding a trivial cocycle, we can get an equivalent deformation with vanishing first term (see [4]). Let us investigate the coefficient of the t^2-term.

For the first family the cocycle $\omega_1 : (e_i, e_j) \to (j - i)e_{i+j-1}$ is trivial, as there exists a κ_1 1-cochain for which $d\kappa_1 = \omega_1$: $\kappa_1(e_{2i+1}) = ie_{2i}$, $i \geqq 0$; $\kappa_1(e_{2i}) = \frac{2i-1}{2}e_{2i-1}$, $i \geqq 1$. With the transformation $\phi_t(x) = x + t\kappa_1(x)$ we get an equivalent deformation $[\widetilde{e_i, e_j}]_t^1 = \phi_t^{-1}([\phi_t(e_i), \phi_t(e_j)])$ without t-term, where the coefficient φ_2 of the t^2-term is a nontrivial cocycle. A straightforward calculation shows that $\overline{\alpha} - \frac{1}{12}\varphi_2$ is a trivial cocycle.

In the second family for the cocycle $\omega_2 : (e_1, e_j) \to je_j$, $(e_i, e_j) \to 0$ if $i \neq 1$ there exists a κ_2 1-cochain for which $d\kappa_2 = \omega_2 : \kappa_2(e_1) = 0$, $\kappa_2(e_{2i+1}) = (i+1)e_{2i}$, $i \geqq 1$; $\kappa_2(e_{2i}) = \frac{2i+1}{2}e_{2i+1}$, $i \geqq 1$. With the $x \to x + t\kappa_2(x)$ transformation we get an equivalent deformation without a t-term, where the coefficient φ_2' of the t^2-term is a nontrivial cocycle and $\overline{\alpha}$ is cohomological to $\frac{12}{13}\varphi_2'$.

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