EXTENSIONS OF MODULES OVER LOOP ALGEBRAS

ALICE FIALOWSKI
Department of Mathematics, University of California
Davis CA 95616 USA

FYODOR MALIKOV
Department of Mathematics, Faculty of Science, Kyoto University
Kyoto 606 Japan

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Abstract

An algebra of functions on the line taking values in a finite-dimensional simple Lie algebra $\mathfrak{g}$ possesses two families of modules which do not belong to the category $\mathcal{O}$. These are modules of functions on the line taking values in a finite-dimensional simple $\mathfrak{g}-$module, and simple finite-dimensional $\mathfrak{g}-$modules on which the algebra of functions acts by taking value at some fixed point on the line. We classify extensions of modules of the first class under the presence of the Lie algebra of vector fields on the line and all extensions of modules of the second class. As a preliminary step we calculate the cohomology of an algebra of functions with values in $\mathfrak{g}$ with coefficients in such modules. Globally our results lead to a classification of certain sheaves on a complex curve.

Key words: loop algebra, module, cohomology, extension.

1 Introduction

The existence of non-splitting exact short sequences

$$0 \to A \to ? \to B \to 0$$

(or non-trivial extensions of B by A) is one of the origins of homological algebra. However, if A and B are finite-dimensional modules over a simple Lie algebra

$\mathfrak{g}$, then this sequence necessarily splits. If A and B are arbitrary Bernstein-Gelfand-Gelfand (BGG) modules, i.e. modules with the set of weight spaces bounded from above, then this sequence can be non-splitting only in the case when the highest weights of composition factors of A and B belong to the same orbit of the Weyl group. A proper generalization of this result for an arbitrary Kac-Moody Lie algebra can be found in [2].

Loop algebras—an important class of Kac-Moody algebras—possess irreducible modules which are not of the BGG-type. These are modules of functions taking values in finite-dimensional $\mathfrak{g}-$modules, or just finite-dimensional $\mathfrak{g}-$modules on which the corresponding loop algebra acts by taking value at some fixed point. We study extensions of such modules by themselves (Sect. 4), establishing necessary and sufficient conditions for non-trivial extensions and classifying extensions under the presence of the Lie algebra of vector fields on the line. We apply these results to obtain a classification of certain sheaves on a complex curve.

Beside the above mentioned general cohomology theory framework the investigation of such extensions is motivated by the following two constructions.

Vertex Operators. The following is one of the basic constructions of 2-dimensional quantum conformal field theory. Let $L(\mathfrak{g})$ be the loop algebra associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$, i.e.

$$L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{C}[z, z^{-1}],$$

and set $L(\mathfrak{g})_{\geq} = \oplus_{n\geq 0}\mathfrak{g} \otimes z^n$.

We now pass from the loop algebra $L(\mathfrak{g})$ to its unique central extension $\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbf{C} \cdot c$ determined by the cocycle

$$\psi : (x \otimes z^i) \times (y \otimes z^j) \mapsto < x, y > i\delta_{i,-j} \cdot c,$$

where $< ., . >$ is an invariant inner product on $\mathfrak{g}$. This central extension is also well-known as a non-twisted affine Lie algebra. Obviously, $L(\mathfrak{g})_{\geq}$ is a subalgebra of $\tilde{L}(\mathfrak{g})$.

There is a Lie algebra homomorphism $L(\mathfrak{g})_{\geq} \to \mathfrak{g}$ of taking value at the point $z = 0$ and, therefore, each $\mathfrak{g}-$module may be regarded as an $L(\mathfrak{g})_{\geq}-$module. Furthermore, if some complex number k is fixed, then each $\mathfrak{g}-$module appears to be an $L(\mathfrak{g})_{\geq}\oplus\mathbf{C}\cdot c-$module, on which the central element c acts as multiplication by k. If λ is a finite dimensional simple $\mathfrak{g}-$module and $k \in \mathbf{C}$, then denote by $M_{\lambda,k}$ the $\tilde{L}(\mathfrak{g})-$module, induced from the $L(\mathfrak{g})_{\geq} \oplus \mathbf{C} \cdot c-$module λ.

Denote by $\mathcal{G}$ the algebra of rational functions on the complex projective line taking values in $\mathfrak{g}$ and by $\mathcal{G}(a_1, \ldots, a_k)$ its subalgebra consisting of functions regular outside the set $\{a_1, \ldots, a_k\}$. Now one may "attach $M_{\lambda,k}$ to a point t on the projective line", which means to define a $\mathcal{G}-$module $M_{\lambda,k}(t)$ equal to M_λ as a vector space, the action of $\mathcal{G}$ being determined by the Laurent expansion at the point t and the requirement that c is a multiplication by k. Furthermore, one may attach three copies of the "vacuum" module $M_{0,k}$ to three points

0, t, ∞ and consider the $\mathcal{G}(0,t,\infty)-$module $M_{0,k}(0)\otimes M_{0,k}(t)\otimes M_{0,k}(\infty)$. The Frobenius duality implies that

$$\mathrm{Hom}_{\mathcal{G}(0,t,\infty)}(M_{0,k}(0)\otimes M_{0,k}(t)\otimes M_{0,k}(\infty),\ \mathbf{C})=\mathbf{C}.$$

This isomorphism determines a $\mathcal{G}(0,t,\infty)-$homomorphism

$$\Phi: M_{0,k}(t)\rightarrow \mathrm{Hom}(M_{0,k},M_{0,k}^{*})[[t,t^{-1}]].$$

For any $x\in M_0$, $\Phi(x)$ is called a *vertex operator*. (See [1] for a Virasoro algebra analogue and [4] for an axiomatic approach to vertex operator algebras.)

Now observe that $L(\mathfrak{g})=\mathcal{G}(0,\infty)\subset\mathcal{G}(0,t,\infty)$. It implies that any $L(\mathfrak{g})_{\geq}-$ submodule of $M_{0,k}(t)$ determines an $\tilde{L}(\mathfrak{g})-$ submodule in $\mathrm{Hom}(M_0,M_0^*)[[t,t^{-1}]]$. In particular, the highest weight vector v_0 gives the trivial $L(\mathfrak{g})-$module $\mathbf{C}$, the subspace spanned by v_0 and $\mathfrak{g}\otimes z^{-1}\cdot v_0$ gives a module, containing $\mathbf{C}$ as a module and $L(\mathfrak{g})$ as a quotient, i.e. an extension of the form

$$0\rightarrow\mathbf{C}\rightarrow ?\rightarrow L(\mathfrak{g})\rightarrow 0. \tag{1}$$

Moving "down" along $M_{0,k}$ one obtains more and more complicated $L(\mathfrak{g})-$modules, containing modules of functions as subquotients. Moreover, the Sugawara operators (see below) make the algebra of vector fields on the circle act on these modules. Therefore the obtained extensions are actually extensions of modules over the semi-direct product of the loop algebra and the algebra of vector fields on the circle.

For some values of k such towers of extensions split, which gives differential equations on correlation functions. The latter being of an extreme importance in conformal field theory.

Remark The following is an explicit description of the passage from an $L(\mathfrak{g})_{\geq}-$ submodule to an $\tilde{L}(\mathfrak{g})-$ submodule of $\mathrm{Hom}(M_0,M_0^*)[[t,t^{-1}]]$. Our definitions imply that

$$[f\otimes z^m,\Phi(w)]=\sum_{i=0}^{\infty}\frac{(z^m)^{(i)}}{i!}t^i\Phi(f\otimes z^i\cdot w).$$

For example, if one defines $\Phi_n(g)$ so that

$$\Phi(g\otimes z^{-1}\cdot v_0)=\sum_{i=-\infty}^{+\infty}\Phi_i(g)t^{-i-1},$$

then the above formula shows that

$$[f\otimes z^m,\Phi_n(g)]=\Phi_{n+m}([f,g])+\delta_{n,-m}m<f,g>k,$$

recovering the definition of an affine Lie algebra. In other words, the Fourier components of the vertex operators related to the space $\mathfrak{g}\otimes z^{-1}\cdot v_0$ generate the above mentioned extension (1), while the space $\mathfrak{g} \otimes z^{-1} \cdot v_0 \oplus \mathbf{C}v_0$ as an $L(\mathfrak{g})_{\geq}$−module is an extension of the trivial module by the adjoint representation of $\mathfrak{g}$.

Sugawara Operators. Let $\{g^i\}$,$\{g_i\}$ be two dual bases of $\mathfrak{g}$ with respect to $< .,>$. Following Sugawara let us consider the following series

$$T_n = \frac{1}{2} \sum_{i+j=n} : g^\alpha \otimes z^i : \cdot g_\alpha \otimes z^j,$$

where the normal ordering :: means as usual that the terms with positive powers of z are moved to the right. One may regard these series as elements of an appropriate completion of the universal enveloping algebra $U(\tilde{L}(\mathfrak{g}))$, which acts on BGG-modules.

The following formulas are well-known:

$$[g \otimes z^m, T_n] = (c+h) \cdot m \cdot g \otimes z^{m+n}, \tag{2}$$

$$[T_n, T_m] = (c+h)\{(m-n)T_{n+m} + \delta_{i,-j}\frac{m^3-m}{12}c \text{ dim } \mathfrak{g}\}, \tag{3}$$

where h is some non-zero constant, independent of n, m, g. (Under a standard normalization of $< .,. >$ it appears to be a dual Coxeter number of $\mathfrak{g}$.) Let c act on a BGG-module M as a multiplication by $k \in \mathbf{C}$. Formulas (2, 3) imply that the Sugawara operators determine an action of the Virasoro algebra (the algebra of vector fields on the circle if $k = 0$) if $k \neq -h$, and an infinite family of $\tilde{L}(\mathfrak{g})$−homomorphisms if $k = -h$. Both observations have been intensively exploited recently (see [8, 11, 13, 12]).

From our point of view formula (2) means that the Sugawara operators generate an $L(\mathfrak{g})$−module V which is included in the following exact sequence

$$0 \to L(\mathfrak{g}) \to V \to L(\mathbf{C}) \to 0 \tag{4}$$

where $L(\mathbf{C})$ is a module of functions taking values in the trivial $\mathfrak{g}$−module. We will show (see Sect. 4.4) that a simple generalization of this construction exhausts in a sense extensions splitting with respect to the algebra of vector fields on the line.

Remark. One can also describe the Sugawara construction in terms of vertex operators. In order to obtain this description set

$$\phi : \mathfrak{g}^{\otimes 2} \to L(\mathfrak{g})^{\otimes 2},$$

$$\phi : f \otimes g \mapsto (f \otimes z^{-1}) \otimes (g \otimes z^{-1}).$$

It is easy to see that the Fourier components of vertex operators related to $\phi(g^\alpha \otimes g_\alpha) \cdot v_0$ generate the module V included into the above-mentioned sequence (4). As an $L(\mathfrak{g})_{\geq}$−module $\phi(g^\alpha \otimes g_\alpha) \cdot v_0 \subset M_{0,0}$ generate an extension of the trivial module in the symmetric power of the adjoint representation of $\mathfrak{g}$ by the adjoint representation.

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2 Some Algebras and Modules

Throughout this paper we keep the following notations:

$\mathfrak{g}$ is a finite dimensional complex simple Lie algebra;

$L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{C}[z, z^{-1}]$ is a loop algebra;

$L(\mathfrak{g})_{\geq} = \mathfrak{g} \otimes \mathbf{C}[z]$ and $L(\mathfrak{g})_{>} = \mathfrak{g} \otimes z\mathbf{C}[z]$ are two subalgebras of $L(\mathfrak{g})$;

V_λ is a finite dimensional simple $\mathfrak{g}-$module with the highest weight λ;

$L(V_\lambda) = V_\lambda \otimes \mathbf{C}[z, z^{-1}]$ is an $L(\mathfrak{g})-$module, the action of $L(\mathfrak{g})$ being defined pointwise: if $g \in \mathfrak{g}$, $v \in V_\lambda$ then

$$g \otimes z^m \cdot v \otimes z^n = g \cdot v \otimes z^{m+n};$$

$L(V_\lambda)_{\geq} = V_\lambda \otimes \mathbf{C}[z]$ is a subspace of $L(V_\lambda)$ closed with respect to the action of $L(\mathfrak{g})_{\geq}$.

We remark that the algebras $L(\mathfrak{g}), L(\mathfrak{g})_{\geq}, L(\mathfrak{g})_{>}$, as well as the module $L(V_\lambda)$ are spaces of functions taking values in $\mathfrak{g}$ or in V_λ. Using physical terminology, we will sometimes call such functions *currents*. We also observe that the above definitions have many variations, depending on the class of functions in question, which may vary from all smooth functions to formal power series.

The above defined $L(\mathfrak{g})-$ and $L(\mathfrak{g})_{\geq}-$modules are infinite dimensional. Finite dimensional $L(\mathfrak{g})_{\geq}-$ (as well as $L(\mathfrak{g})-$) modules can be obtained as follows. For any $y \in \mathbf{C}$ there is a Lie algebra homomorphism (of "taking value at the point y")

$$\begin{aligned} L(\mathfrak{g})_{\geq} &\rightarrow \mathfrak{g} \\ g \otimes z^m &\mapsto y^m g. \end{aligned}$$

Therefore V_λ is endowed with an $L(\mathfrak{g})_{\geq}-$module structure. We denote by $V_{\lambda,y}$ the $L(\mathfrak{g})_{\geq}-$module, obtained this way.

We will also deal with the Lie algebra of vector fields on the line $\mathcal{L}$ and a family of $\mathcal{L}-$modules of $a-$differentials $\mathcal{F}_a = \mathbf{C}[z, z^{-1}]dz^{-a}$. Under the following choice of bases of $\mathcal{L}$ and $\mathcal{F}_a$:

$$\mathcal{L} = \oplus_{i \geq 0} \mathbf{C} L_i, \text{ where } L_i = z^{i+1}\frac{d}{dz},$$

$$\mathcal{F}_a = \oplus_{i \geq} \mathbf{C} f_i, \text{ where } f_i = z^i dz^{-a}$$

the commutation relations of $\mathcal{L}$ and the $\mathcal{L}-$module structure of $\mathcal{F}_a$ are given by

$$[L_i, L_j] = (j-i)L_{i+j},$$

$$L_i f_j = (j - a(i+1))f_{i+j}.$$

3 Cohomology of Current Algebras with coefficients in Modules of Functions and finite-dimensional Modules

The results of this section are based on H. Garland's computation of the homology $H_*(L(\mathfrak{g})_>, \mathbf{C})$ (see [6]). To recall his result we introduce the following notations. Let W_a be the affine Weyl group, associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$ and let Δ (Δ_+, Δ_- resp.) be the corresponding affine root system (the system of positive, negative roots resp.). To each root $\alpha \in \Delta$ we associate a root space $L(\mathfrak{g})_\alpha \subset L(\mathfrak{g})$ in a standard way (see [12]). We denote by $\Delta^1_+ \subset \Delta_+$ the set of all roots such that the corresponding root spaces belong to $L(\mathfrak{g})_>$. For $\sigma \in W_a$ we set $\mathfrak{a}_\sigma = \sigma\Delta_- \cap \Delta_+$. Let W^1 be the set of all $\sigma \in W_a$ such that $\mathfrak{a}_\sigma \subset \Delta^1_+$. We denote by $< \mathfrak{a}_\sigma >$ an arbitrary element (determined uniquely up to a non-zero factor) of the highest exterior power of the vector space, spanned by $\mathfrak{a}_\sigma$. Finally, note that $\mathfrak{g}$ naturally acts on the homology group $H_i(L(\mathfrak{g})_>, \mathbf{C})$ for any i.

Theorem 1 (H. Garland) *(i) For each $\sigma \in W^1$, $< \mathfrak{a}_\sigma >$ is a cycle, which determines a non-trivial element of $H_{l(\sigma)}(L(\mathfrak{g})_>, \mathbf{C})$, where $l(\sigma)$ is the length of σ.*

(ii) For any $\sigma \in W^1$, $< \mathfrak{a}_\sigma >$ is the highest weight vector of a certain $\mathfrak{g}-$module.

(iii) The map, sending σ to $< \mathfrak{a}_\sigma >$ is a bijection between W^1 and the set of highest weight vectors of $H_{l(\sigma)}(L(\mathfrak{g})_>, \mathbf{C})$.

(iv) Each $\mathfrak{g}-$module appears in $H_*(L(\mathfrak{g})_>, \mathbf{C})$ with multiplicity ≤ 1.

Now let V_λ be an irreducible finite-dimensional $\mathfrak{g}-$module with the highest weight λ. Theorem 1 suggests the following definition:

$$c(\lambda) := \begin{cases} i & , \text{ if } V_\lambda \text{ appears in } H_i(L(\mathfrak{g})_>, \mathbf{C}) \text{ for some } i \\ -1 & , \text{ otherwise.} \end{cases}$$

Proposition 2

$$H^i(L(\mathfrak{g})_\geq, V_{\lambda,y}) = \begin{cases} 0 & , \ \textit{if } c(\lambda) < 0 \\ H^{i-c(\lambda)}(\mathfrak{g}, \mathbf{C}) & , \ \textit{otherwise.} \end{cases}$$

Proof Evidently, the kernel of the evaluation map $L(\mathfrak{g})_{\geq} \to \mathfrak{g}$ is isomorphic to $L(\mathfrak{g})_{>}$. There is a Lindon-Serre-Hochschild spectral sequence (see [10] Ch. II.6)

$$\{E_k^{p,q}\} \Longrightarrow H^{p+q}(L(\mathfrak{g})_{\geq}, V_{\lambda,y})$$

such that

$$E_2^{p,q} = H^p(\mathfrak{g}, H^q(L(\mathfrak{g})_{>}, V_{\lambda,y})).$$

The algebra $L(\mathfrak{g})_{>}$ acts on $V_{\lambda,y}$ trivially; therefore, $H^q(L(\mathfrak{g})_{>}, V_{\lambda,y}) = H^q(L(\mathfrak{g})_{>}) \otimes V_{\lambda,y}$. On the other hand, since $\mathfrak{g}$ is simple, one has

$$H^p(\mathfrak{g}, H^q(L(\mathfrak{g})_{>}, V_{\lambda,y})) = H^p(\mathfrak{g}) \otimes (H^q(L(\mathfrak{g})_{>}, V_{\lambda,y}))^{\mathfrak{g}}.$$

The combination of these isomorphisms together with Theorem 1 implies that, firstly, $E_2^{p,q}$ is the right-hand side of the statement of Proposition 2 and, secondly, $E_2^{p,q}$ is concentrated on the vertical line $q = c(\lambda)$ and hence all higher differentials are equal to 0. □

Remark If V_λ is the adjoint representation then $c(\lambda) = 1$ and Proposition 2 implies that $H^1(L(\mathfrak{g})_{\geq}, V_{\lambda,y}) = \mathbf{C}$. It is well-known that the first cohomology group with coefficients in a module is in one-to-one correspondence with extensions of the trivial module by the module in question. In our case the unique extension stipulated by Proposition 2 is realized as an $L(\mathfrak{g})_{\geq}-$submodule of the vacuum module $M_{0,k}(y)$ (see the discussion of the Sugawara construction in the Introduction) generated by $g^\alpha \otimes z^{-1} g_\alpha \otimes z^{-1} \cdot v_0$. The extension dual to this one is generated by $\mathfrak{g} \otimes z^{-1} \cdot v_0$ (see the discussion of the vertex construction in the Introduction).

Proposition 2 has a natural analogue for the module $L(V_\lambda)_{\geq}$. To make the statement simpler we extend the algebra $L(\mathfrak{g})_{\geq}$ by the degree derivation $z\frac{d}{dz}$. Evidently, $L(\mathfrak{g})_{\geq} \oplus \mathbf{C}z\frac{d}{dz}$ acts on $L(V_\lambda)_{\geq}$ as well.

Proposition 3 *For all i and any $y \in \mathbf{C}$*

$$H^i(L(\mathfrak{g})_{\geq} \oplus \mathbf{C}z\frac{d}{dz}\,,\, L(V_\lambda)_{\geq}) = H^i(L(\mathfrak{g})_{\geq}\,,\, V_{\lambda,y}) \oplus H^{i-1}(L(\mathfrak{g})_{\geq}\,,\, V_{\lambda,y}).$$

Proof The element $z\frac{d}{dz}$ acts in the natural way on the standard cochain complex $C^*(L(\mathfrak{g})_{\geq} \oplus \mathbf{C}z\frac{d}{dz}\,,\, L(V_\lambda)_{\geq})$, the latter being graded by its eigenspaces: $C^*(L(\mathfrak{g})_{\geq} \oplus \mathbf{C}z\frac{d}{dz}\,,\, L(V_\lambda)_{\geq}) = \oplus_i C_i^*$. This grading is preserved by the differential and only C_0^* can make non-zero contribution to the cohomology (see [5] Ch. I.3). One can easily see that for all p there is an isomorphism commuting with the differentials

$$C_0^p(L(\mathfrak{g})_{\geq} \oplus \mathbf{C}z\frac{d}{dz}\,,\, L(V_\lambda)_{\geq}) = C^p(L(\mathfrak{g})_{\geq}, V_{\lambda,y}) \oplus C^{p-1}(L(\mathfrak{g})_{\geq}, V_{\lambda,y}).$$

This completes the proof. □

4 Extensions of Modules of Functions and of finite-dimensional Modules

4.1 *The Case of finite-dimensional Modules*

It is well-known that the equivalence classes of the following short exact sequences of $L(\mathfrak{g})_{\geq}-$modules

$$0 \to V_{\lambda,x} \to ? \to V_{\mu,y} \to 0$$

are labelled by the elements of the group

$$\mathrm{Ext}^1(V_{\mu,y}, V_{\lambda,x}) = H^1(L(\mathfrak{g})_{\geq}, \mathrm{Hom}(V_{\mu,y}, V_{\lambda,x})).$$

Theorem 4 *Suppose that the modules V_λ, V_μ are non-trivial. Then*

(i)(B.L.Feigin) $\qquad \mathrm{Ext}^1(V_{\mu,y}, V_{\lambda,x}) = 0 \qquad$ *if $x \neq y$,*

(ii) $\qquad \mathrm{Ext}^1(V_{\mu,x}, V_{\lambda,x}) \approx \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Hom}(V_\mu, V_\lambda)).$

Proof Evidently, $\mathrm{Hom}(V_{\mu,y}, V_{\lambda,x})$ is a $\mathfrak{g} \oplus \mathfrak{g}-$module and its $L(\mathfrak{g})_{\geq}-$module structure arises from the homomorphism

$$L(\mathfrak{g})_{\geq} \to \mathfrak{g} \oplus \mathfrak{g}$$

$$g \otimes z^m \mapsto x^m g \oplus y^m g.$$

(This homomorphism is also an evaluation map.) Let $\mathcal{K}$ be its kernel. One immediately concludes that $\mathcal{K} \approx \mathfrak{g} \otimes (z-x)(z-y)\mathbf{C}[z]$.

In the Lindon-Serre-Hochschild spectral sequence

$$\{E_k^{p,q}\} \Longrightarrow H^{p+q}(L(\mathfrak{g})_{\geq}, \mathrm{Hom}(V_{\mu,y}, V_{\lambda,x}))$$

the second term is given by

$$E_2^{p,q} = H^p(\mathfrak{g} \oplus \mathfrak{g}, H^q(\mathcal{K}) \otimes \mathrm{Hom}(V_{\mu,y}, V_{\lambda,x}))$$

(cf. the proof of Proposition 2). In particular, $H^1(L(\mathfrak{g})_{\geq}, \mathrm{Hom}(V_{\mu,y}, V_{\lambda,x}))$ is a subquotient of $[H^1(\mathcal{K}) \otimes \mathrm{Hom}(V_{\mu,y}, V_{\lambda,x})]^{\mathfrak{g}\oplus\mathfrak{g}}$. The definition of $\mathcal{K}$ implies that $H^1(\mathcal{K}) = \mathfrak{g} \otimes \mathbf{C} \oplus \mathbf{C} \otimes \mathfrak{g}$ as a $\mathfrak{g} \oplus \mathfrak{g}$-module. But non-trivial $\mathfrak{g} \oplus \mathfrak{g}-$elements in $(\mathfrak{g} \otimes \mathbf{C}) \otimes \mathrm{Hom}(V_{\mu,y}, V_{\lambda,x})$ can only exist if $V_\mu = \mathfrak{g}$, $V_\lambda = \mathbf{C}$. Similarly, non-trivial $\mathfrak{g} \oplus \mathfrak{g}-$elements in $(\mathbf{C} \otimes \mathfrak{g}) \otimes \mathrm{Hom}(V_{\lambda,y}, V_{\mu,x})$ can only exist if $V_\lambda = \mathfrak{g}$, $V_\mu = \mathbf{C}$. This concludes the proof of (i). Part (ii) is a particular case of the following result.

Theorem 5

$$\mathrm{Ext}^*_{L(\mathfrak{g})_{\geq}}(V_{\lambda,x}, V_{\mu,x}) \approx H^*(\mathfrak{g}, \mathbf{C}) \otimes [\mathrm{Hom}_{\mathfrak{g}}(H^n(L(\mathfrak{g})_{>}, \mathbf{C}),\ \mathrm{Hom}(V_\lambda, V_\mu))].$$

Proof If $\mathrm{Hom}(V_\mu, V_\lambda) = \oplus_i V_i$ then $\mathrm{Hom}(V_{\mu,x}, V_{\lambda,x}) = \oplus_i V_{i,x}$. Now the desired result follows from Proposition 2. In particular, since $H^1(L(\mathfrak{g})_>, \mathbf{C}) = \mathfrak{g}$ (see Theorem 1), one has that $\mathrm{Ext}^1(V_{\mu,x}, V_{\lambda,x}) \approx \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Hom}(V_\mu, V_\lambda))$. This completes the proof of Theorem 4.$\Box$

Remarks

(i) One can easily write out a cocycle ϕ_p, corresponding to the element $p \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Hom}(V_\lambda, V_\mu))$:

$$\phi_p(g \otimes z^i) = i x^{i-1} p(g)$$

for any $g \in \mathfrak{g}$.

(ii) Theorem 4 remains valid if one passes from $L(\mathfrak{g})_>$ to the loop algebra $L(\mathfrak{g})$, provided both x, $y \neq 0$. ($V_{\lambda,0}$ is not an $L(\mathfrak{g})-$module.)

4.2 *The Case of Modules of Functions*

Here we consider equivalence classes of short exact sequences of $L(\mathfrak{g})_\geq-$modules

$$0 \to L(V_\lambda)_\geq \to ? \to L(V_\mu)_\geq \to 0 \tag{5}$$

or, equivalently, the group $\mathrm{Ext}^1(L(V_\mu)_\geq, L(V_\lambda)_\geq)$. Our first observation is that the Lie algebra of vector fields on the line $\mathcal{L}$ naturally acts by differentiations on the modules in question as well as on $L(\mathfrak{g})_\geq$. Moreover, since the space of functions on the line is naturally isomorphic to the space of a-differentials

$$\mathbf{C}[z] \approx \mathbf{C}[z] \otimes dz^{-a},$$

A one-parameter family of actions of $\mathcal{L}$ arises on each of the modules. We start with those short sequences (5) which split with respect to $\mathcal{L}$. Such sequences (up to an equivalence) are in one to one correspondence with elements of the relative cohomology group

$$\mathrm{Ext}^1_{(L(\mathfrak{g})_{\geq_\geq} \oplus \mathcal{L}, \mathcal{L})}(L(V_\mu)_\geq, L(V_\lambda)_\geq) = H^1(L(\mathfrak{g})_\geq \oplus \mathcal{L}, \mathcal{L}; \mathrm{Hom}(L(V_\mu)_\geq, L(V_\lambda)_\geq))$$

where $L(\mathfrak{g})_\geq \oplus \mathcal{L}$ is a semi-direct sum.

Remark The algebraic condition of $\mathcal{L}-$splitting of the sequence (5) has a transparent geometrical meaning. It simply means that a cocycle Φ associated to (5), which is a map

$$\Phi : L(\mathfrak{g})_\geq \otimes L(V_\mu)_\geq \to L(V_\lambda)_\geq$$

is invariant with respect to changes of the variable z.

We now construct a family of invariant extensions of the type (5). For any $p \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Hom}(V_\mu, V_\lambda))$ define a map

$$\Phi_p : L(\mathfrak{g})_\geq \to \mathrm{Hom}(L(V_\mu)_\geq, L(V_\lambda)_\geq)$$

by setting

$$\Phi_p(g \otimes z^m)(v \otimes z^k) = m \cdot p(g)(v) \otimes z^{m+k-1} \tag{6}$$

where $g \in \mathfrak{g},\ v \in V_\mu$. The following lemma is proved by direct calculation.

Lemma 6 *For any $p \neq 0$ and any complex number a, the map Φ_p determines a non-zero element of the group $H^1(L(\mathfrak{g})_{\geq} \oplus \mathcal{L}, \mathcal{L};\ \mathrm{Hom}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}))$, where $L(V_\mu)_{\geq}$ ($L(V_\lambda)_{\geq}$ resp.) as an $\mathcal{L}-$module is a module of $a-$differentials taking values in V_μ (($a-1$)$-$differentials taking values in V_λ resp.).*

Theorem 7

(i) $\mathrm{Ext}^1_{L(\mathfrak{g})_{\geq}}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}) = 0$ *if and only if* $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\ \mathrm{Hom}(V_\mu, V_\lambda)) = 0$.

(ii) $\mathrm{Ext}^1_{(L(\mathfrak{g})_{\geq} \oplus \mathcal{L}, \mathcal{L})}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}) = 0$ *unless $L(V_\mu)_{\geq}$ ($L(V_\lambda)_{\geq}$ resp.) as an $\mathcal{L}-$module is the module of $a-$ (($a-1$)$-$ resp.) differentials taking values in $L(V_\mu)$ ($L(V_\lambda)-$ resp.). If the latter condition holds then*

$$\mathrm{Ext}^1_{(L(\mathfrak{g})_{\geq} \oplus \mathcal{L}, \mathcal{L})}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}) \approx \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\ \mathrm{Hom}(V_\mu, V_\lambda)),$$

the isomorphism being given by the map $p \to \Phi_p$ (see formula (6)).

Proof One can easily see that the sequence (5) is $\mathfrak{g}-$splitting: both $L(V_\mu)_{\geq}$ and $L(V_\lambda)_{\geq}$ are direct sums of finite-dimensional irreducible $\mathfrak{g}-$modules; each exact short sequence consisting of irreducible finite-dimensional $\mathfrak{g}-$modules splits. Therefore (5) actually determines an element of the relative cohomology group $H^1(L(\mathfrak{g})_{\geq}, \mathfrak{g}; \mathrm{Hom}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}))$.

The spaces $L(V_\mu)_{\geq},\ L(V_\lambda)_{\geq},\ L(\mathfrak{g})_{\geq}$ are graded by the power of z. This enables us to define a grading of the standard cochain complex as follows:

$C^n(L(\mathfrak{g})_{\geq}, \mathfrak{g}; \mathrm{Hom}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}))_k$ consists of functions

$$f \in C^n(L(\mathfrak{g})_{\geq}, \mathfrak{g}; \mathrm{Hom}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}))$$

such that for all $s, i_j \in \mathbf{Z},\ g_i \in \mathfrak{g}$

$$f(g_1 \otimes z^{i_1}, \ldots, g_n \otimes z^{i_n}) \text{ maps } V_\lambda \otimes z^s \text{ to } V_\lambda \otimes z^{s+k+i_1+\cdots+i_n}.$$

This grading is preserved by the differential and, therefore, the cohomology is also graded. The definition implies that each 1-cocycle is a collection of maps

$$\mathfrak{g} \to \mathrm{Hom}(V_\lambda, V_\mu).$$

Since the action of $\mathfrak{g}$ does not change the power of z and cocycles in question are $\mathfrak{g}-$relative, this map is a $\mathfrak{g}-$homomorphism. This proves the implication $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\ \mathrm{Hom}(V_\mu, V_\lambda)) = 0 \Longrightarrow \mathrm{Ext}^1_{L(\mathfrak{g})_{\geq}}(L(V_\mu)_{\geq}, L(V_\lambda)_{\geq}) = 0$. The inverse implication follows from Lemma 6. Part (i) is proved.

In order to prove (ii) we observe that under the assumptions of (ii) the sequence (5) is still $\mathfrak{g}-$splitting and, therefore, the corresponding cocycle is cohomologically equivalent to an $\mathcal{L}\oplus\mathfrak{g}-$invariant map

$$(\mathfrak{g}\otimes\mathcal{F}_0)\otimes(\mathcal{F}_a\otimes V_\mu)\to\mathcal{F}_b\otimes V_\lambda,$$

vanishing on constant currents, i.e. satisfying

$$(\mathfrak{g}\otimes\mathbf{C})\otimes(\mathcal{F}_a\otimes V_\mu)\mapsto 0,$$

where $\mathcal{F}_s$ stands for the $\mathcal{L}-$module of $s-$differentials (see sect. 2). In other words, each cocycle is determined by an element of the space

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\mathrm{Hom}(V_\mu,V_\lambda))\otimes\mathrm{Hom}_{\mathcal{L}}(\mathcal{F}_0\otimes\mathcal{F}_a,\mathcal{F}_b).$$

P. Groman's classification of bilinear invariant differential operators (see [9]) implies that $\mathrm{Hom}_{\mathcal{L}}(\mathcal{F}_0\otimes\mathcal{F}_a,\mathcal{F}_b)$ is 0, unless $b=a-1$ or a and if $b=a$ or $a-1$ then $\mathrm{Hom}_{\mathcal{L}}(\mathcal{F}_0\otimes\mathcal{F}_a,\mathcal{F}_b)=\mathbf{C}$. Moreover, it follows that if $b=a$ then the corresponding invariant map is the ordinary multiplication and, therefore, does not vanish on constants. It implies that $b=a-1$. Hence we have obtained that there is a surjection

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\mathrm{Hom}(V_\mu,V_\lambda))\to\mathrm{Ext}^1_{(L(\mathfrak{g})_{\geq}\oplus\mathcal{L},\mathcal{L})}(L(V_\mu)_{\geq},L(V_\lambda)_{\geq}).$$

Lemma 6 implies that this map is actually a bijection. □

In order to calculate the absolute cohomology group we recall a result of L. V. Goncharova (see [7]).

Theorem 8 (L. V. Goncharova) *Let $\mathcal{F}_a$ be an $\mathcal{L}-$module of $a-$differentials and $\mathcal{F}'_a$ be its restricted dual. If $a=-(3r^2\pm r)/2$ for some $r\in\{0,1,2\ldots\}$ then*

$$H_q(\mathcal{L}\,,\,\mathcal{F}'_a)=\begin{cases}1 & , \quad \textit{if } q=r,r+1\\ 0 & , \quad \textit{otherwise.}\end{cases}$$

If a is not equal to $-(3r^2\pm r)/2$ for any nonnegative integer r, then $H_*(\mathcal{L}\,,\,\mathcal{F}'_a)=0$.

Goncharova's theorem implies in particular that

$$\dim\,H_1(\mathcal{L}\,,\,\mathcal{F}'_a)=\begin{cases}1 & , \quad \text{if } a\in\{-2,-1,0\}\\ 0 & , \quad \text{otherwise.}\end{cases}\tag{7}$$

In order to use Goncharova's result, all algebras and modules in the following theorem will be understood (conflicting the assumptions throughout the rest of the paper) as algebras and modules over formal series in one indeterminate z.

That determines a non-trivial topology on each of them and the corresponding cohomology will be understood as a continuous cohomology. Under these assumptions on has, in particular,

$$H^q(\mathcal{L}, \mathcal{F}_a)' = H_q(\mathcal{L}, \mathcal{F}'_a)'.$$

We are now in the position to state the central result of the paper.

Theorem 9 *Let $L(V_\mu)_\geq$ ($L(V_\lambda)_\geq$ resp.) be the module of $a-$differentials ($b-$differentials resp.) as an $\mathcal{L}-$module. Then*

$$\mathrm{Ext}^1_{L(\mathfrak{g})_\geq \oplus \mathcal{L}}(L(V_\mu)_\geq, L(V_\lambda)_\geq) =$$
$$\begin{cases} \mathrm{Ext}^1_{(L(\mathfrak{g})_\geq \oplus \mathcal{L}, \mathcal{L})}(L(V_\mu)_\geq, L(V_\lambda)_\geq) & , \ \text{if } \mu \neq \lambda \\ \mathrm{Ext}^1_{(L(\mathfrak{g})_\geq \oplus \mathcal{L}, \mathcal{L})}(L(V_\mu)_\geq, L(V_\lambda)_\geq) \oplus H_1(\mathcal{L}\,, \mathcal{F}'_{b-a}) & , \ \text{if } \mu = \lambda. \end{cases} \tag{8}$$

This theorem shows that unless $\mu = \lambda$ and $b-a \in \{-2,-1,0\}$, all extensions

$$0 \to L(V_\lambda)_\geq \to ? \to L(V_\mu) \to 0$$

are $\mathcal{L}-$splitting and therefore are given by Theorem 7 (ii). If however, the latter condition holds, then only one non-splitting extension arises and it is given by the generator of the homology group $H_1(\mathcal{L}, \mathcal{F}'_{b-a})$.

Proof of Theorem 9 We consider the Serre-Hochschild spectral sequence

$$\{E^{pq}_k\} \Rightarrow H^{p+q}(L(\mathfrak{g})_\geq \oplus \mathcal{L}, \mathrm{Hom}(L(V_\lambda)_\geq, L(V_\mu)_\geq))$$

applied to an algebra-subalgebra pair: $L(\mathfrak{g})_\geq \oplus \mathcal{L} \supset L(\mathfrak{g})_\geq$. Since $L(\mathfrak{g})_\geq$ is an ideal, one has (see [5] Ch. I.5):

$$E^{pq}_2 = H^p(\mathcal{L}, H^q(L(\mathfrak{g})_\geq, \mathrm{Hom}(L(V_\lambda)_\geq, L(V_\mu)_\geq)).$$

In particular,

$$E^{10}_2 = H^1(\mathcal{L}, \mathrm{Hom}_{L(\mathfrak{g})_\geq}(L(V_\lambda)_\geq, L(V_\mu)_\geq)),$$

$$E^{01}_2 = H^1(L(\mathfrak{g})_\geq \oplus \mathcal{L}, \mathcal{L}; \mathrm{Hom}_{L(\mathfrak{g})_\geq}(L(V_\lambda)_\geq, L(V_\mu)_\geq)).$$

The $L(\mathfrak{g})_\geq-$modules of the type $L(V_\lambda)_\geq$ are "almost irreducible": their only $L(\mathfrak{g})_\geq-$ endomorphisms are multiplications by a function. It implies that

$$\mathrm{Hom}_{L(\mathfrak{g})_\geq}(L(V_\lambda)_\geq, L(V_\mu)_\geq) = 0, \ \text{if } \lambda \neq \mu$$

and, as an $\mathcal{L}-$module,

$$\mathrm{Hom}_{L(\mathfrak{g})_\geq}(L(V_\lambda)_\geq, L(V_\mu)_\geq) = \mathcal{F}_{b-a}, \ \text{if } \lambda = \mu.$$

It implies that

$$E_2^{10} = H^1(\mathcal{L}, \mathcal{F}_{b-a}) \text{ if } \lambda = \mu.$$

We now observe that:

(i) the term E_2^{01} was discussed in Theorem 7, in particular, it was shown to be equal to $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Hom}(L(V_\lambda), L(V_\mu)))$; it also follows from Theorem 7 that all higher differentials vanish when applied to, or mapping to $E_k^{01}, k \geq 2$;

(ii) the term E_2^{10} is calculated in (7); the consideration of dimensions shows that again all higher differentials vanish when applied to, or mapping to $E_k^{10}, k \geq 2$. □

4.3 *Globalization*

The purely local assertion of Theorem 7 can be easily transformed into information on geometric quantities associated with a complex curve. We start with the case of the projective line (though not mentioning it explicitly) and then pass to the case of an arbitrary curve.

A. Consider, in precise analogy with (5), the following sequence of $L(\mathfrak{g})-$modules:

$$0 \to L(V_\lambda) \to ? \to L(V_\mu) \to 0. \tag{9}$$

Such sequences are in one-to-one correspondence with the elements of the group $H^1(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda)))$.

Now an analogue of the $\mathcal{L}-$invariance is given as follows:

Suppose that $L(V_\mu)_\geq$ ($L(V_\lambda)_\geq$ resp.) is the space of $a-$ ($b-$ resp.) differentials taking values in V_μ (V_λ resp.); then a cocycle Φ associated to (9) is a differential operator, mapping from $L(\mathfrak{g}) \otimes L(V_\mu)$ to $L(V_\lambda)$. From the homological algebra point of view this means that one has to replace the standard cochain complex $C^*(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda)))$ by its subcomplex

$$C^*_{\mathcal{D}}(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda))),$$

consisting of differential operators D

$$D:\ \Lambda^* L(\mathfrak{g}) \otimes L(V_\mu) \ \to\ L(V_\lambda)$$

and consider the corresponding cohomology $H^*_{\mathcal{D}}(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda)))$. It is easy to see that

$$H^1_{\mathcal{D}}(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda))) \subset H^1(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda))).$$

Theorem 10

$$(i)\ \ H^1(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda))) = 0$$

if and only if

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g},\ \mathrm{Hom}(V_\mu, V_\lambda)) = 0.$$

$(ii)\ H^1_{\mathcal{D}}(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda))) \approx \begin{cases} \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Hom}(V_\mu, V_\lambda)) & \text{if } b = a-1 \\ 0 & \text{otherwise} \end{cases}$.

Note that the isomorphism of (ii) in Theorem 10 is again given by the map $p \to \widehat{\Phi_p}$, where $\widehat{\Phi_p}$ is a natural continuation of Φ_p (see formula (6)) on the entire $L(\mathfrak{g})$.

Proof of Theorem 10 Part (i) can be proved in exactly the same fashion as Theorem 7 (i). As to (ii) we first observe that the elements of

$$C^1_{\mathcal{D}}(L(\mathfrak{g}),\ \mathrm{Hom}(L(V_\mu), L(V_\lambda)))$$

are determined locally. Therefore locally cocycles are determined by the elements of the group

$$C^1(L(\mathfrak{g})_{\geq}\ \mathrm{Hom}(L(V_\mu), L(V_\lambda))).$$

In order to be glued together these elements must be invariant with respect to changes of local coordinates or, equivalently, belong to the space of relative cochains

$$C^1(L(\mathfrak{g})_{\geq} \oplus \mathcal{L}, \mathcal{L};\ \mathrm{Hom}(L(V_\mu), L(V_\lambda))).$$

If these elements do not vanish on constant currents (equivalently, on the elements of $\mathfrak{g}$) then the Grozman classification of invariant differential operators (c.f. proof of Theorem 7) implies that $b = a$ and such a cocycle is a differential operator of order zero, i.e. pointwise multiplication. In particular, the action of constant currents is independent of the point. But locally our extension is necessarily $\mathfrak{g}-$splitting. Therefore, locally our cocycle may be made vanishing on constant currents. This transformation may be performed globally due to the above description of the action of constant currents. Therefore, our cocycle is cohomologically equivalent to the cocycle, which is, firstly, multiplication and, secondly, vanishing on constants. Hence it is 0. We deduce that $b = a - 1$ and in this case, as it follows from Theorem 7, the cocycle is given by (6). □

B. Let C be a smooth complex curve, $\mathcal{G}$ be a sheaf of regular functions on C with values in $\mathfrak{g}$, $\mathcal{V}^a_\lambda$ be a sheaf of regular a-differentials on C with values in V_λ. Evidently, $\mathcal{G}$ is a sheaf of Lie algebras and $\mathcal{V}^a_\lambda$ is a sheaf of $\mathcal{G}-$modules.

Now the global analogue of (5) is the following exact sequence of sheaves of $\mathcal{G}-$modules:

$$0 \to \mathcal{V}^b_\lambda \to ? \to \mathcal{V}^a_\mu \to 0. \tag{10}$$

Theorem 11 *The exact sequence (10) splits unless $b = a-1$. If $b = a-1$ then the equivalence classes of exact sequences (10) are in one-to-one correspondence with the vector space*

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{Hom}(V_\mu, V_\lambda)).$$

Proof repeats the steps of the proof of Theorem 10. We briefly sketch it. The action of a sheaf of algebras on a sheaf of vector spaces is defined locally. In order to be glued together these local actions should be invariant under changes of variable. The Grozman classification implies that from the "coordinate" point of view it is either multiplication or given by (6). The first possibility is showed to give rise to a splitting sequence in exactly the same way it was done in the proof of Theorem 10.□

4.4 *Extensions and the Sugawara Construction*

To any element $g_1 \otimes \cdots \otimes g_k$ of the tensor algebra $T^*(\mathfrak{g})$ over the vector space $\mathfrak{g}$ we associate a series by setting

$$\Psi_n(g_1 \otimes \cdots \otimes g_k) = \sum_{i_1+i_2+\cdots+i_k=n} : g_1 \otimes z^{i_1} g_2 \otimes z^{i_2} \cdots g_k \otimes z^{i_k} : \tag{11}$$

where : : means, as usual, that all terms with positive powers of z are moved to the right.

We now define a completion $\tilde{U}$ of the universal enveloping algebra $U(L(\mathfrak{g}))$ which contains such series. The algebra $U(L(\mathfrak{g}))$ is naturally filtered by subspaces $U(L(\mathfrak{g}))$: $\mathbf{C} = U^0 \subset U^1 \subset U^2 \subset \ldots, \cup_i U^i = U(L(\mathfrak{g}))$, where U^i is the linear span of the monomials $g_1 \cdot g_2 \cdots g_i$, for all $g_k \in L(\mathfrak{g})$. On the other hand the algebra $L(\mathfrak{g})$, as well as $L(\mathfrak{g})_>$, is graded by powers of the indeterminate z. Therefore, the universal enveloping algebra is also graded:

$$U(L(\mathfrak{g})) = \oplus_{i \in \mathbf{Z}} U(L(\mathfrak{g}))_i,$$

$$U(L(\mathfrak{g})_>) = \oplus_{i>0} U(L(\mathfrak{g})_>)_i.$$

For each U^i define a family of subsets $\{M_k^i,\ k \geq 0\}$ as follows:

$$M_k^i = U^i \cap \sum_{j \geq k} U(L(\mathfrak{g})) \cdot U(L(\mathfrak{g})_>)_j.$$

Define a topology $\mathcal{T}$ on each U^i to be determined by $\{M_k^i,\ k \geq 0\}$ as a family of fundamental neighborhoods of 0. Finally we define the algebra $\tilde{U}$ to be a completion of $U(L(\mathfrak{g}))$ with respect to $\mathcal{T}$. One can easily see that the series (11) is convergent in $\mathcal{T}$.

The definition shows that the above defined filtration of $U(L(\mathfrak{g}))$ is inherited by $\tilde{U} = \bigcup_{k \geq 0} \tilde{U}^k$. Another evident consequence of the definition is that the adjoint action of $L(\mathfrak{g})$ on $U(L(\mathfrak{g}))$, as well as the action of $\mathcal{L}$ by differentiations is continuous in $\mathcal{T}$ and, therefore, can be uniquely continued to an action on the entire $\tilde{U}$.

Let $S^*(\mathfrak{g}) \subset T^*(\mathfrak{g})$ be a symmetric algebra over the vector space $\mathfrak{g}$.

Lemma 12 *Let $p \in S^k(\mathfrak{g})$ and $e \in \mathfrak{g}$.*

(i) In $\tilde{U}_k / \tilde{U}_{k-2}$ one has

$$[e \otimes z^m, \Psi_{n+m}(p)] = \Psi_{n+m}([e, p]) + m\Psi_{n+m}(q)$$

where $q \in S^{k-1}(\mathfrak{g})$ *depends only on* e, p.

(ii) The vector space $\oplus_{n\geq 0}\mathbf{C}\Psi_n(p)$ *is an* $\mathcal{L}-$*module, isomorphic to the module of* $(k-1)-$*differentials.*

Lemma 12 can be proved by a simple and direct calculation and, besides, (ii) is proved in [13] (1990).

Remark If $p \in [S^*(\mathfrak{g})]^{\mathfrak{g}}$ then q is proportional to $\frac{\partial}{\partial e}\cdot p$, where $\frac{\partial}{\partial e}$ is an element dual to e with respect to the Killing form (see [8, 13]). If $p \in [S^2(\mathfrak{g})]^{\mathfrak{g}}$ then our construction is the well-known Sugawara construction.

Now let $p \in S^*(\mathfrak{g})$ (q resp.) generate the $\mathfrak{g}-$module V (W resp.). Lemma 12 shows that the $L(\mathfrak{g})_{\geq}-$module $\widehat{L(V)}_{\geq}$ generated by $\Psi_n(p)$, $n \geq 0$, within $\tilde{U}_k/\tilde{U}_{k-2}$, is included into the following exact sequence

$$0 \to L(W)_{\geq} \to \widehat{L(V)}_{\geq} \to L(V)_{\geq} \to 0,$$

splitting with respect to $\mathcal{L}$, and the corresponding cocycle coincides with (6). This gives a realization of the extensions of Theorem 7 (as well as of Theorem 10 if one allows n to be negative). Moreover if one passes from $L(\mathfrak{g})$ to its (unique) 1-dimensional central extension then this sequence will be no longer $\mathcal{L}-$ splitting which gives one of the extensions of Theorem 9, that is the one corresponding to $b = a - 1$. However it is not clear whether all possible pairs of $\mathfrak{g}-$modules (V, W) (even among those appearing in $S^*(\mathfrak{g})$) can be obtained this way.

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