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A COMPARISON OF DEFORMATIONS AND ORBIT CLOSURE

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Abstract

The purpose of this research-expository paper is to clarify some of the similarities and differences between “orbit closure” and “deformations” . In Section 1 a common formulation for deformation and degeneration is presented. In particular, it is shown that if the orbit is not (Zariski) closed, then a point in the closure always arises from a deformation. In Section 2, the concept of versal deformation is introduced. It is shown that in the case of degeneration there exists a universal element. In Section 3 two definitions of “rigidity” are discussed. It is proved that in the case of Lie algebras over an algebraically closed field of characteristic 0, these two definitions are equivalent. The role of cohomology is also discussed in deformation and degeneration problems.

Consider the following 2×2 matrices with entries in a field k:

$$A_t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \;,\; A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;.$$

If k is the field of real numbers $\mathbf{R}$, $\lim_{t\to 0} A_t = A_0$ has the standard definition. We can define limits in the set of 2×2 matrices over an arbitrary field k by endowing that set with the Zariski topology (see [20]). Then we say that $\lim_{t\to 0} A_t = A_0$ in the sense that A_0 is in the Zariski closure of the set $\{A_t\}_{t\neq 0}$.

We consider this example from the two different viewpoints of *deformation* and *orbit closure.* In deformation theory, the above example would be written

$$A_t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

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and we would say that $\{A_t, t \in k\}$ is a <u>deformation</u> family of A_0. As an orbit closure example, we would note that $A_t = S_t A_1 S_t^{-1}$ where

$$S_t = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

and so A_0 is in the (Zariski) closure of the orbit of A_1 under the conjugation action of $GL_2(k)$ on the variety of 2×2 matrices. For orbit closure, we say that A_0 is a <u>degeneration</u> of A_1. (Note the duality in viewpoint between “deformation” and “degeneration”.)

The family

$$B_t = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$$

is a deformation of B_0, but B_0 is not a degeneration of B_1. It is easy to verify that

$$C_0 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a degeneration of

$$C_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

but no one-parameter family giving a deformation of C_0 contains C_1. However, there is a one-parameter family giving a deformation of C_0 which contains matrices similar to C_1. In fact, we shall show that this situation is true in general.

Deformation theory of different mathematical objects is one of the oldest disciplines in mathematics. The most natural questions arose in geometry, in attempts to understand which other objects one can get from an original one by "deforming" it, or whether one can get any new, non-isomorphic objects at all. At the same time, the deformations give more information about the structure of the original objects as well.

The notion of local and infinitesimal deformations of a complex analytic manifold first appeared in the works of Kodaira and Spencer (1958) [24]. In particular, they proved that infinitesimal deformations can be parametrized by a related cohomology group. The deformation theory of compact complex manifolds was developed by Kuranishi (1962) [29]. These basic ideas of the deformation theory of analytic structures motivated the deformation theory of algebraic manifolds, which was founded by Artin and Schlessinger (1968) [3], [39]. Deformations of arbitrary rings and associative algebras and the corresponding cohomology questions were first studied by Gerstenhaber (1964-68) [13]. The most actively studied field was cohomology and deformations of commutative local rings, applied to problems in analytic and algebraic geometry (Grauert-Kerner 1964 [16], Illusie 1971 [21], Palamodov 1976 [35], Laudal 1979 [30], etc.)

The deformation theory of Lie algebras has been studied much less. Some general questions in the theory were considered by Nijenhuis and Richardson (1966-68) [34], but not much is known about deformations of a given Lie algebra. In recent years the interest in infinite dimensional Lie algebras has grown in various fields of mathematics and physics (see e.g. [7]), which naturally emphasizes the importance of studying invariants of these objects, such as their cohomology and deformations. This field is now rapidly developing and there are many open problems.

The concept of orbit closure (degeneration) arises in diverse areas of mathematics, specifically in any setting where algebraic or topological transformation groups are considered, e.g. invariant theory (Hilbert 1893 [19], Mumford 1965 [33], representation theory (Kostant 1963 [25]), theory of singularities (Levine 1971 [31]), applied to different classification problems (e.g. Arnold 1972 [2]). Kraft [26] gives a good exposition of the algebraic theory of the techniques commonly employed. In the case of algebraic structures on a fixed finite dimensional vector space, degeneration means the following: the orbits under the "change of basis" action of the general linear group are the isomorphism classes, and so orbit closure coincides with the closure of isomorphism classes. Degenerations of Lie algebras were first introduced in the physics literature by Inönü and Wigner in 1953 [22], and have been studied by physicists and mathematicians since then.

There are many categories in which both deformations and orbit closure may be considered: $n \times n$ matrices (or linear transformations) [12], associative algebras [13], Lie algebras [34], [41], [10], [18], representations of a group or of an algebra ([26], [32]), representations of a quiver [28], linear systems of differential equations [40], etc. (This list is far from complete.)

Until now orbit closure and deformations were developed independently resulting in both confusion and duplication of ideas which were previously thought to be different. The purpose of this paper is to clarify some of the similarities and differences in the two concepts. The above examples of deformations and degenerations of matrices demonstrate some of the differences. In spite of these differences, we will show in fact that many of the ideas are identical. Our main contribution is a larger viewpoint, from which one can much more clearly understand the ideas. In Section 1 we present a common formulation for deformation and degeneration. In particular, we show that if the orbit is not (Zariski) closed, then a point in the closure always arises from a deformation. In Section 2, we discuss the concept of versal deformation, i.e. a deformation which induces all other deformations. We show that in the case of degeneration the picture is more complete: there exists a universal degeneration. In Section 3, we discuss "rigidity", a term which is defined both in terms of deformations and in terms of degenerations. Rigidity with respect to degeneration follows from rigidity with respect to deformation, but the converse does not always hold. We show that in the case of Lie algebras over an algebraically closed field of characteristic 0, the two definitions of rigidity are equivalent. We also discuss the role of cohomology in deformation and degeneration problems.

In order to make the exposition readable, we will concentrate on one category, that of Lie algebras, although occasionally we will use examples from the category of $n \times n$ matrices. In the following, the reader may substitute the category of his or her choice.

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§1. **Deformations and Degenerations.**

First we consider finite dimensional Lie algebras. An n-dimensional Lie algebra may be considered as an element μ of $\mathrm{Hom}(\Lambda^2 V, V)$, where V is an n-dimensional vector space over an algebraically closed field k. The set of Lie algebras is an algebraic subset L of the variety $\mathrm{Hom}(\Lambda^2 V, V)$, and the general linear group $GL_n(k)$ acts on L by:

$$(g \cdot \mu)(x, y) = g(\mu(g^{-1}x, g^{-1}y)).$$

The orbits under this action are the isomorphism classes, and we say that μ_1 degenerates to μ_0, or μ_0 is a degeneration of μ_1, if μ_0 is in $\overline{O(\mu_1)}$, the Zariski closure of the orbit of μ_1. For example, every Lie algebra μ degenerates to an abelian Lie algebra μ_0 via

$$(t^{-1}I \cdot \mu)(x, y) = t^{-1}\mu(tx, ty) = t\mu(x, y).$$

Then $\lim\limits_{t \to 0} t^{-1}I \cdot \mu = \mu_0$ (in the sense that μ_0 is in the Zariski closure of $\{t^{-1}I \cdot \mu\}_{t \neq 0}$), where $\mu_0(x, y) = 0$ for all x, y in V.

The intuitive definition of a formal deformation of a Lie algebra μ_0 over an arbitrary field k is a one- parameter family of Lie algebras $\mu(t)$ in $V \otimes k[[t]]$ over the formal power series ring $k[[t]]$:

$$\mu(t) = \mu_0 + t\phi_1 + t^2\phi_2 + \dots$$

where $\phi_i \in \mathrm{Hom}(\Lambda^2 V, V)$. The example above is a deformation; we have

$$\mu(t) = \mu_0 + t\mu \ .$$

Let us remark that the question of convergence is not covered by this definition.

Note that this definition of deformation does not require the vector space V to be finite dimensional; in fact, deformations of infinite dimensional Lie algebras have been intensively studied (see, for instance, [9] and [11]), especially because of the various applications in physics (see, for instance, [7]).

A trivial connection between deformations and orbit closure is that if $\mu(t)$ is a deformation of μ_0 defined by a convergent power series in t, then $\mu_0 \in \overline{\cup_{t\in k} O(\mu(t))}$.

A deeper connection between deformation and orbit closure arises when one considers deformations more generally. From the viewpoint of deformations, we consider a formal deformation $\mu(t)$ not as a family of Lie algebra structures, but as a Lie algebra over the field $k((t))$. Then a natural generalization is to allow more parameters, i.e. use $k[[t_1, \ldots, t_r]]$ or consider k-algebras other than power series rings. (By "k-algebras" we mean associative, commutative k-algebras with identity.) Examples 1.3, 1.5 and 1.6 illustrate the usefulness of multiple parameters.

Definition. Let the parameter ring A be a local finite dimensional algebra over k and let μ_0 be a Lie algebra on a vector space V over k (not necessarily finite dimensional). If μ_A is a Lie algebra over A, i.e. an element of $\mathrm{Hom}(\Lambda^2(V \otimes A), V \otimes A)$ which satisfies the Jacobi identity, then any morphism $f\colon A \to B$ defines a Lie algebra $\mu_A \otimes_A B$ in $\mathrm{Hom}(\Lambda^2(V \otimes B), V \otimes B)$. A deformation of μ_0 parametrized by A is a Lie algebra μ_A over A on $V \otimes_k A$ such that

$$\mu_A \otimes_A k = \mu_0$$

where the tensor product is defined by the residue map $A \to A/m_A = k$.

More generally, if A is a complete local k- algebra (i.e. $A = \underleftarrow{\lim} A/m_A^n$) such that A/m_A^n is finite dimensional for all n, then a deformation of the Lie algebra μ_0 parametrized by A is a Lie algebra μ_A over A such that $\mu_A = \underleftarrow{\lim} \mu_n$, where μ_n is a deformation of μ_0 parametrized by A/m_A^{n+1}.

Two deformations μ_A and μ'_A of μ_0 parametrized by A are *equivalent* if there is a Lie algebra isomorphism $\mu_A \cong \mu'_A$ which induces the identity map on $\mu_A \otimes k = \mu_0$. A deformation μ_A of μ_0 is *trivial* if it is equivalent to $\mu_0 \otimes A$.

In the case that the parametrization algebra A is $k[[t]]$, this definition coincides with Gerstenhaber's concept of formal deformation [13].

In the following, we denote the scheme of Lie algebras of a given dimension by $\mathcal{L}$ (i.e. the subscheme of $\mathrm{Hom}(\Lambda^2 V, V)$ defined by the Jacobi identity).

Remark. Another analogous definition of the deformations of a finite dimensional Lie algebra is the following. Let $\mathcal{A}$ be the category of complete local k- algebras with residue field k and let Λ be the category of schemes $\{\mathrm{spec} A | A \in \mathcal{A}\}$. For a Lie algebra μ_0 over k, define the functor $\mathrm{Def}(\mu_0, \cdot)$ from Λ to the category of sets by

$$\mathrm{Def}(\mu_0, \mathrm{spec}A) = \{\tau\colon \mathrm{spec}A \to \mathcal{L} | \tau \text{ is a morphism of schemes and}$$

$$\mathrm{im}(\tau \circ \pi_A^*) = \mu_0\}$$

where $\pi_A\colon A \to k$ is the residue map. The elements of the set $\mathrm{Def}(\mu_0, \mathrm{spec}A)$ are called deformations of μ_0 with the base spec$A$. Two deformations $\tau_1\colon \mathrm{spec}A \to \mathcal{L}$ and $\tau_2\colon \mathrm{spec}A \to \mathcal{L}$ are equivalent if there exist an analytic map $\sigma\colon \mathrm{spec}A \to GL(V)$ such that $\sigma \circ \tau_1 = \tau_2$. Easy to show that in this case there exist an analytic map $\sigma'\colon \mathrm{spec}A \to GL(V)$ such that $\sigma' \circ \tau_2 = \tau_1$. The ring of sections $\Gamma(\mathcal{L}, \mathcal{O})$ of $\mathcal{L}$ is isomorphic to $k[X_{ijk}]/I$, where X_{ijk} are the coordinate functions of the structure constants and I is the ideal generated by the anticommutativity and Jacobi relations. As τ defines a k-algebra homomorphism $\overline{\tau}^*\colon k[X_{ijk}]/I \to A$, it may be identified with the Lie algebra μ_A over A which has structure constants $a_{ijk} = \overline{\tau}^*(X_{ijk} + I)$.

An analogous viewpoint in the theory of orbit closure is the following characterization of orbit closure given by Grunewald and O'Halloran [18]. We present it here for the category of n- dimensional Lie algebras, although it holds for many algebraic group actions on varieties.

Theorem 1.1. *Let μ_0 and μ_1 be n-dimensional Lie algebras over k. The Lie algebra μ_0 is a degeneration of μ_1 (i.e. $\mu_0 \in \overline{O(\mu_1)}$) if and only if there is a discrete valuation k-algebra A with residue field k whose quotient field K is finitely generated over k of transcendence degree one, and there is an n-dimensional Lie algebra μ_A over A such that*

$$\mu_A \otimes K \cong \mu_1 \otimes K$$

and

$$\mu_A \otimes k = \mu_0.$$

The orbit closure example given at the beginning of this section can be characterized as follows. Let $A = k[t]_{<t>}$ be the polynomial ring localized at the prime ideal $< t >$, and let $\mu_A = t\mu$. Then μ is $k(t)$-isomorphic to μ_A via the isomorphism $t^{-1}I$ and $\mu_A \otimes k = \mu_0$.

Remark. Consider the Lie algebra μ_A as a family of Lie algebras over spec$A$, i.e. as a morphism of schemes from spec$A$ to $\mathcal{L}$. The following is a natural geometric reformulation of the theorem:

There is a morphism ψ from an irreducible curve D to $\mathcal{L}$ and a point $y_0 \in D$ such that $\psi(y_0) \approx \mu_0$ and $\psi(D\backslash\{y_0\}) \subset O(\mu_1)$.

Even though Theorem 1.1 specifies that the quotient field K of the k-algebra A has transcendence degree one over k (one parameter), the proof of the theorem does not require such a restriction. (The theorem is stated in this way to establish that the degeneration *can* be realized by such an A, not that is *must* be.) Just as one may generalize Gerstenhaber's concept to include k-algebras like $k[[t_1,\ldots,t_r]]$, one may also realize orbit closure by more general k-algebras.

With the help of Theorem 1.1 we can give a more algebraic definition of degeneration of a Lie algebra.

Definition. Let $\mathcal{D}$ be the category of k- algebras which are integral domains, and let $\Gamma = \{\mathrm{spec} A | A \in \mathcal{D}\}$. For an n-dimensional Lie algebra μ_1 over k, define the functor Degen $(\mu_1, \cdot)$ from Γ to the category of sets by

$$\mathrm{Degen}(\mu_1, \mathrm{spec}A) = \{\sigma\colon \mathrm{spec}A \to \mathcal{L} | \sigma \text{ is a morphism}$$
$$\text{of schemes and } \mathrm{im}(\sigma \circ i_A^*) \in \overline{O(\mu_1)}\},$$

where $i_A\colon A \to K$ is the inclusion of A into its field of fractions.

The elements of the set $\mathrm{Degen}(\mu_1, \mathrm{spec}A)$ are called degenerations of μ_1 represented by $\mathrm{spec}A$. Two elements σ_1 and σ_2 of $\mathrm{Degen}(\mu_1, \mathrm{spec}A)$ are defined to be isomorphic if there is a morphism $\gamma\colon \mathrm{spec}A \to GL_n(k)$ such that $\gamma(x) \cdot \sigma_1(x) = \sigma_2(x)$ for all $x \in \mathrm{spec}A$.

In terms of Theorem 1.1, a degeneration from μ_1 to μ_0 can be considered as an element of the set $\mathrm{Def}(\mu_0, \mathrm{spec}A) \cap \mathrm{Degen}(\mu_1, \mathrm{spec}A)$ for some complete local algebra A which is an integral domain.

Note that the definition of deformation requires the parametrizing k-algebra to be a complete local k-algebra, and the conditions of Theorem 1.1 require A to be a discrete valuation k-algebra. However, if we consider the completion $\hat{A}$ of the discrete valuation k-algebra A from Theorem 1.1, then we see that every degeneration can be realized by a deformation. If μ_A is the Lie algebra over A defining the degeneration (i.e. $\mu_A \otimes k = \mu_0$ and $\mu_A \otimes K \cong \mu_1 \otimes K$), let $\mu_n = \mu_A \otimes A/m_A^{n+1}$ and let $\mu_{\hat{A}} = \varprojlim \mu_n$. Then $\mu_n \otimes k = \mu_0$ for all n. Thus $\mu_{\hat{A}}$ is a formal deformation of μ_0 and so we have:

Theorem 1.2. *If μ_0 is in the boundary of the orbit of μ_1, then this degeneration defines a non-trivial deformation of μ_0.*

This theorem establishes that for finite dimensional Lie algebras, every degeneration can be realized as a deformation. This motivates the following generalization of degeneration to the infinite dimensional case:

Definition. Let $A = k[[t_1,\ldots,t_r]]$ and let μ_A be a Lie algebra over A with structure constants $(c_{ijk}(t_1,\ldots,t_r))$. For $x \in k^r$, define $\mu_A(x)$ to be the Lie algebra over k with structure constants $(c_{ijk}(x))$ (when this is defined). Let μ_A

be a deformation of an infinite dimensional Lie algebra μ_0. If there is a Zariski open subset S of k^r such that $\mu_A(x) \cong \mu_A(y)$ for all $x, y \in S$, then we say that μ_0 is a *degeneration* of $\mu_A(x)$ (for any $x \in S$).

It follows that in both the finite and infinite dimensional case, the existence of non-trivial degenerations to μ_0 implies the existence of non-trivial deformations of μ_0.

Remark. Of course, not every deformation arises from a degeneration; we see this already in case of 3-dimensional Lie algebras. Consider the 3-dimensional Lie algebra M given by $[e_1, e_2] = e_1$, $[e_1, e_3] = [e_2, e_3] = 0$, and the family $L(\alpha)$ with Lie brackets $[e_1, e_2] = e_1$, $[e_3, e_2] = \alpha e_3$, $[e_1, e_3] = 0$. For any pair of distinct elements α_1, α_2 with $\alpha_1 \cdot \alpha_2 \neq 1$, the Lie algebras $L(\alpha_1)$ and $L(\alpha_2)$ are not isomorphic. It is easy to show that the family $L(\alpha)$ $(\alpha \neq 1)$ is a deformation family of M, but M is not a degeneration of any Lie algebra from $L(\alpha)$ $(\alpha \neq 1)$.

However, there is a special class of deformations which do define degenerations. These are the jump deformations, which remain constant for generic parameter $t \neq 0$, first introduced for complex manifolds by Griffiths [17], and for algebras by Gerstenhaber [14]. A formal deformation μ_t of an n- dimensional Lie algebra μ_0 parametrized by $k[[t]]$ is a *jump deformation* if there is an element $\Phi_{s,t} \in GL_n(k((s,t)))$ such that

$$\Phi_{s,t} = 1 + s\phi_1(t) + s^2\phi_2(t) + \cdots$$

where each ϕ_i is a power series whose coefficients are k-linear maps $V \to V$, and

$$\mu_{(1+s)t} = \Phi_{s,t} \cdot \mu_t \tag{1}$$

(here the action of $GL_n(k((s,t)))$ is the usual one). In other words, the deformation μ_t is equivalent to $\mu_{(1+s)t}$.

A necessary condition for the existence of a non-trivial jump deformation is that $H^1(\mu_0; \mu_0) \neq 0$ (see Gerstenhaber [14]).

A consequence of (1) is that the jump deformation μ_t obtained by a convergent power series defines a degeneration.

The examples below illustrate the distinction between deformation and degeneration.

Example 1.3. Let the multiplicative group $\mathbf{R}^*$ act on $\mathbf{R} \times \mathbf{R}$ by: $x \cdot (a, b) = (a, xb)$. Then the 1-dimensional orbits are punctured vertical lines and the 0-dimensional orbits are the points on the x-axis. In some sense, the deformation family $t(1,0)+s(0,1)$ of the origin $(0,0)$ includes all possible deformations of $(0,0)$. Only the case $s=0$ gives a realization of a degeneration.

Example 1.4. The orbits under the conjugacy action of $GL_n(k)$ on the variety of $n \times n$ matrices are given by Jordan forms. We know from [12] that a matrix with one Jordan block for each eigenvalue is a degeneration of no other non-equivalent matrix, but every matrix has non-trivial deformations.

Example 1.5. Consider the family of 7-dimensional Lie algebras $\mathcal{G}(\beta,\gamma,\delta,\mu)$ which are described in [41]. The non-zero Lie products of $\mathcal{G}(\beta,\gamma,\delta,\mu)$ are:

$$\begin{array}{ll} [e_1,e_i]=e_{i+1}\ ,\ 2\le i\le 6 & [e_2,e_5]=(\beta-\mu)e_7 \\ [e_2,e_3]=\beta e_5+\gamma e_6+\delta e_7 & [e_3,e_4]=\mu e_7 \\ [e_2,e_4]=\mu e_6+\gamma e_7\ . & \end{array}$$

Consider the deformation $\mathcal{G}(1,0,0,t)$ of $\mathcal{G}(1,0,0,0)$ parametrized by $k[[t]]$. If $\mathcal{G}(\beta,\gamma,\delta,\mu)$ is isomorphic to $\mathcal{G}(\beta',\gamma',\delta',\mu')$, then $\frac{\beta'-\mu'}{\beta'}=\frac{\beta-\mu}{\beta}$. It follows that $\mathcal{G}(1,0,0,\mu)$ is not isomorphic to $\mathcal{G}(1,0,0,\mu')$ if $\mu\ne\mu'$, so this deformation of $\mathcal{G}(1,0,0,0)$ is not a realization of any degeneration to $\mathcal{G}(1,0,0,0)$. On the other hand, the family $\mathcal{G}(\alpha,\alpha,\alpha,\alpha)$ consists of Lie algebras isomorphic to $\mathcal{G}(1,1,1,1)$, and so the Lie algebra $\mathcal{G}(t,t,t,t)$ over $k[[t]]$ is a deformation parameterized by $k[[t]]$ which realizes the degeneration of $\mathcal{G}(1,1,1,1)$ to $\mathcal{G}(0,0,0,0)$.

Example 1.6. Consider the countably parametrized family $\mathcal{G}(\alpha_k,\beta_k)_{k\ge 2}$ of infinite dimensional Lie algebras with basis $\{e_1,e_2,e_3,\cdots\}$ (see [8]):

$$[e_i,e_j]=0 \text{ if either } i \text{ or } j \text{ are even integers other than } 2,$$

$$[e_1,e_{4k-1}]=\alpha_k e_{4k}\ ,\ [e_2,e_{4k-2}]=\beta_k e_{4k}\ ,\quad k\ge 3.$$

The remaining Lie products are uniquely determined from the above products by the Jacobi identity. This family can be considered as a deformation parameterized by $k[[\alpha_k,\beta_k]]_{k\ge 2}$ of the infinite dimensional Lie algebra $\mathcal{G}(0,0)$ whose non- zero Lie products are:

$$[e_1,e_2]=e_3\ ,\ [e_1,e_3]=e_4,$$

$$[e_2,e_{4k-1}]=e_{4k+1}\ ,\ [e_2,e_{4k}]=e_{4k+2}\ ,\ [e_2,e_{4k+1}]=e_{4k+3}\ ,\ k\ge 1.$$

It is straightforward to show that $\mathcal{G}(\alpha_k,\beta_k)_{k\ge 2}$ is isomorphic to $\mathcal{G}(\alpha'_k,\beta'_k)_{k\ge 2}$ if and only if there are non-zero constants λ_k such that $(\alpha_k,\beta_k)=(\lambda_k a'_k,\lambda_k\beta'_k)$ for all $k\ge 2$. Then, as in Example 1.5, some (α_k,β_k)-parameter families define a degeneration to $\mathcal{G}(0,0)$ while some do not.

§2. **Versal Deformation and Universal Degeneration.**

The most important concept in deformation theory is that of a (uni)versal deformation, that is, one deformation which induces all others. This deformation is generally not unique, as in the category of deformations a universal element does not exist. However, there exists a so-called versal (more precisely, mini-versal) element.

Let us consider, for a given $\mu_0 \in \mathcal{L}$ and $\tau \in \mathrm{Def}(\mu_0, \mathrm{spec}A)$, the morphism of functors

$$\theta_{(\mathrm{spec}A,\tau)}\colon \mathrm{Mor}(Y, \mathrm{spec}A) \to \mathrm{Def}(\mu_0, Y)$$

defined by composition with τ (Y is an element of Λ). Then the pair $(\mathrm{spec}A, \tau)$ is defined to be *universal* if $\theta_{(\mathrm{spec}A,\tau)}$ is an isomorphism for any object Y. The pair $(\mathrm{spec}A, \tau)$ is said to be *versal* if (a) the map $\theta_{(\mathrm{spec}A,\tau)}$ is surjective for any $Y \in \Lambda$ and (b) $\theta_{(\mathrm{spec}A,\tau)}$ is an isomorphism if $Y = \mathrm{spec}\ k[t]/t^2$.

The following definition holds for arbitrary (also infinite dimensional) Lie algebras.

Definition. A deformation μ_R of a Lie algebra μ_0 parameterized by a complete local K-algebra R is called versal if for any deformation μ_A of μ_0 parameterized by a complete local k-algebra A, there is a morphism $f\colon R \to A$ such that the induced map $m_R/m_R^2 \to m_A/m_A^2$ is unique and $\mu_R \otimes_R A$ is equivalent to μ_A.

Remark. Of course, one can state a similar definition in other categories. For instance, the deformation family given in Example 1.3 is a versal one.

The following theorem of Fialowski [11] establishes the existence of a versal deformation in the case that the second cohomology group with coefficients in the adjoint representation is finite dimensional.

Theorem 2.1. *Let μ_0 be a Lie algebra over k (not necessarily finite dimensional). If $H^2(\mu_0, \mu_0)$ is finite dimensional, then there is a formal versal deformation of μ_0.*

This theorem was established by applying a theorem of Schlessinger [39, 2.11] to the category of Lie algebras. Schlessinger's construction of a versal deformation is based on the fact that the parameter ring A is Artinian.

Remark. Of course, the condition in Theorem 2.1 always holds for finite dimensional Lie algebras. In this case we can give a geometric interpretation of a versal deformation (see [10]). Let $\tau\colon S \to \mathcal{L}$ be a deformation of μ_0 and $\sigma\colon T \to S$ be an analytic map of the germs. The composition $\tau \circ \sigma\colon T \to \mathcal{L}$ is called a deformation of μ_0 over T, induced by the map σ. Let us show that there exist a deformation $\theta\colon R \to \mathcal{L}$ such that for any deformation $\tau\colon S \to \mathcal{L}$

there exist a map $\alpha: S \to V$ which induces on S a deformation, equivalent to the original one. In other words, there exist a versal deformation $\theta: R \to \mathcal{L}$.

For R we choose the germ of the subset in $\mathcal{L}$, transversal with respect to the action of $GL(V)$, and for θ the embedding of R into $\mathcal{L}$. Transversality means that the tangent space to $\mathcal{L}$ is the direct sum of the tangent space to $GL(V) \cdot \mu_0$ and of the tangent space to $\theta(R)$. Consider the deformation $\tau: S \to \mathcal{L}$. From the transversality of $\mathbf{R}$ on the $GL(V)$ action follows that in a small enough neighborhood of the marked point of $\mathcal{L}$, for every $\tau(s)$, $s \in S$ from this neighborhood, there exists a unique element $g_{\tau(s)} \in GL(V)$ such that $\beta(s) = g_{\tau(s)} \cdot \tau(s) \in \theta(R)$. So we get a map $\alpha = \theta^{-1}\beta: S \to R$. Easy to show that α is analytic. Then $\psi = \theta\alpha: S \to \mathcal{L}$ is a deformation, induced by the map α. Let us note that $s \mapsto g_{\tau(s)}$ gives a map (also analytic) $\sigma: S \to GL(V)$. We have

$$\sigma(s) \circ \tau(s) = g_{\varphi(s)} \circ \tau(s) = \alpha\beta = \theta\alpha.$$

That means that the deformation τ is equivalent to the induced one. This proves the versality of θ.

Remark. The analogous set-up for deformation which exploits the orbit closure is considered in the case of $n \times n$ matrices over $\mathbf{C}$ by Arnold [1]. A deformation of a matrix A_0 is an $n \times n$ matrix $A(\lambda)$ with entries in $\mathbf{C}[[t_1, \ldots, t_r]]$, convergent in a neighborhood of $\lambda = 0$, with $A(0) = A_0$. The deformation $A(\lambda)$ may be considered as a mapping $A: \mathbf{C}^r \to M$, where M is the space of matrices. Then the deformation $A(\lambda)$ is versal if and only if the mapping A is transversal to the orbit N of A_0 at $\lambda = 0$ (N is the conjugacy class of A_0):

$$T_{A(0)}(M) = A_* T_0(\mathbf{C}^r) + T_{A_0}(N).$$

Schlessinger's construction for the universal deformation does not provide a method for computing it, and for a given Lie algebra, it remains a difficult problem to compute a versal deformation. The following example demonstrates the complexity of the problem.

Example 2.2 (see [11]). Let L_1 be the "maximal nilpotent subalgebra" of the Virasoro algebra. This infinite dimensional Lie algebra of vector fields on the line has a basis

$$e_i = x^{i+1}\frac{d}{dx}\ ,\ i = 1, 2, 3, \ldots$$

and bracket operation

$$[e_i, e_j] = (j - i)e_{i+j}.$$

The space $H^2(L_1; L_1)$ is 3-dimensional, so there exists a versal deformation. If the 2-cocycles α, β and γ define a basis for $H^2(L_1; L_1)$, then a versal deformation is given by the finite sum

$$\mu_{t_1,t_2,t_3} = \mu_0 + \alpha t_1 + \beta t_2 + \gamma t_3 + \sum_{1\le i,j\le 3} \varphi_{ij} t_i t_j + \sum_{1\le i,j,k\le 3} \psi_{ijk} t_i t_j t_k$$

where μ_0 is the Lie product in L_1 and φ_{ij}, ψ_{ijk} are 2-cochains chosen so that the bilinear map μ_{t_1,t_2,t_3} satisfies the Jacobi identity. In general, a formal versal deformation may not be a finite sum. In this case the vanishing Massey products ([11]) insure that any formal versal deformation will be of order 3.

One might ask what kind of characteristic object exists in the case of orbit closure. In the following we will see that for degenerations a universal object always exists.

An element σ in $\mathrm{Degen}(\mu_1, \mathrm{spec}R)$ is *universal* if the morphism of functors

$$\theta_{(\mathrm{spec}R,\sigma)} \colon \mathrm{Mor}(Y, \mathrm{spec}R) \to \mathrm{Degen}(\mu_1, Y),$$

defined by composition, is an isomorphism for all $Y \in \Gamma$. In other words:

Definition. Let μ_1 be an n-dimensional Lie algebra. A universal degeneration of μ_1 is an n-dimensional Lie algebra μ_R over a k- algebra $R \in \mathcal{D}$ such that for any n-dimensional Lie algebra μ_A over a k-algebra $A \in \mathcal{D}$ which defines a degeneration μ_1 (i.e. $\mu_A \otimes K \approx \mu_1 \otimes K$, where K is the quotient field of A) there is a morphism $f\colon R \to A$ such that $(\mu_R \otimes A) \otimes K$ and $\mu_A \otimes K$ are isomorphic over K.

Theorem 2.3. *For each finite dimensional Lie algebra $\mu_1 \in \mathcal{L}$, a universal degeneration of μ_1 exists.*

Proof. Let R be the coordinate ring of $\overline{O(\mu_1)}$ and let $\sigma\colon \mathrm{spec}R \to \mathcal{L}$ be given by the inclusion $\overline{O(\mu_1)} \subset \mathcal{L}$. Because σ is an inclusion, the morphism $\theta_{(\mathrm{spec}R,\sigma)}$ is injective. If $\eta \in \mathrm{Degen}(\mu_1, \mathrm{spec}A)$, then η is defined by a homomorphism $\overline{\eta}$ from $\Gamma(\mathcal{L}, \mathcal{O})$ into A. The ring of sections $\Gamma(\mathcal{L}, \theta)$ is $k[X_{ijk}]/I$, where X_{ijk} are the coordinate functions of the structure constants and I is the ideal generated by the anti-commutativity and Jacobi relations. Since R is the coordinate ring of $\overline{O(\mu_1)}$, $R = k[X_{ijk}]/J$ for some ideal J ($J = \sqrt{J}$) containing I. For each triple (i, j, k) let $a_{ijk} = \overline{\eta}(X_{ijk} + I)$, and define $\varphi\colon R \to A$ by $\varphi(X_{ijk} + J) = a_{ijk}$. (The map φ is well defined because $\mathrm{im}(\eta \circ i_A^*) \in \overline{O(\mu_1)}$ where i_A is the inclusion of A into its field of fractions.) Then we have $\sigma \circ \varphi^* = \eta$, and so we see that $\theta_{(\mathrm{spec}R,\sigma)}$ is surjective for all $Y \in \Gamma$.

Remark. Using the notation introduced in the proof, let μ_R be the Lie algebra over R defined by the structure constants $(\overline{X}_{ijk})$ where $\overline{X}_{ijk} = X_{ijk} + J$, that is, for $e_i = (0, \ldots, 1, \ldots, 0)$ in R^n, let

$$\mu_R(e_i, e_j) = \sum_k \overline{X}_{ijk} e_k.$$

Suppose that μ_A defines a degeneration of μ_1, i.e. $\mu_A \otimes K \approx \mu_1 \otimes K$, where K is the field of fractions of A. Let $f_1: R \to k$ be given by $f_1(\overline{X}_{ijk}) = c_{ijk}$, where (c_{ijk}) are the structure constants for μ_1, and let $f = i \circ f_1$ where i is the inclusion of k into A. It follows that

$$(\mu_R \otimes_f A) \otimes K = \mu_1 \otimes K \approx \mu_A \otimes K.$$

Thus μ_R is a universal degeneration of μ_1.

In terms of the original definition of degeneration as orbit closure, an element μ_0 of the algebraic set $\overline{O(\mu_1)}$ defines the evaluation morphism

$$\epsilon_0: R \to k \text{ given by } \overline{X}_{ijk} \to b_{ijk},$$

where μ_0 has structure constants (b_{ijk}) relative to a fixed basis of k^n. Then μ_R is a degeneration of μ_1 to μ_0:

$$\mu_R \otimes_{\epsilon_0} k = \mu_0.$$

Although the universal degeneration μ_R which we constructed is not defined over a local ring (one of the conditions in Theorem 1.2), for a given degeneration μ_0 of μ_1 we can choose a localization R_M of R such that

$$\mu_{R_M} \otimes k = \mu_0,$$

where $\mu_{R_M} = \mu_R \otimes R_M$. Simply let M be the maximal ideal of R corresponding to μ_0 ($M = ker \epsilon_0$). A natural question is: under what conditions does μ_{R_M} define a degeneration of μ_1 to μ_0? That is, when do we have

$$\mu_{R_M} \otimes K \approx \mu_1 \otimes K, \tag{2}$$

where K is the quotient field of R?

Geometrically, property (2) is satisfied if and only if the following deformations are isomorphic:

$$\varphi: \text{spec} K \to \text{spec} R_M \to \text{spec} R \overset{\mu_R}{\to} \mathcal{L}$$

$$\text{spec} K \overset{\mu_1}{\to} \mathcal{L}.$$

In other words, is there a morphism $\sigma: \text{spec} K \to GL_n(k)$ such that $\rho_{\mu_1} \circ \sigma = \varphi$, where ρ_{μ_1} is the orbit map ($\rho_{\mu_1}(g) = g \cdot \mu_1$)? Since R is the coordinate ring of $\overline{O(\mu_1)}$, then the morphism φ maps onto the generic point of $O(\mu_1)$. Thus property (2) is equivalent to the existence of a lifting of φ to $G = GL_n(k)$:

$$\begin{tikzcd}
G \arrow[rr, "\rho_{\mu_1}"] & & \mathcal{L} \\
& \text{spec } K \arrow[ul] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Proposition 2.4. The following three statements are equivalent:

a) The principal bundle $G \to G/H$ is locally trivial, where H is the stabilizer of μ_1.

b) Every morphism $\psi: X \to O(\mu_1)$ lifts to G locally.

c) Every morphism $\eta: \text{spec} K \to O(\mu_1)$ lifts to G.

Proof. $a \Rightarrow b$). Let $x \in X$. Choose $g \in G$ such that $\rho(g) = \psi(x)$. Since the quotient map $\pi: G \to G/H$ is locally trivial, there is a neighborhood $U \subseteq G$ of g, a neighborhood $V \subseteq G/H$ of gH and a homeomorphism $\sigma: V \times H \to U$ such that the following diagram is commutative:

$$\begin{tikzcd}
V \times H \arrow[r, "\sigma", "\approx"'] \arrow[d, "p_1"'] & U \arrow[d, "\pi"] \\
V \arrow[r, phantom, "\subseteq"] & G/H
\end{tikzcd}$$

Then $\rho(U) = (\rho \circ \sigma)(V \times H) = (\rho \circ \sigma)(V \times \{e\})$ and $\rho \circ \sigma$ is injective on $V \times \{e\}$. For any $y \in \psi^{-1}(\rho(U))$, define $\varphi(y) = \sigma((\rho \circ \sigma)^{-1}\psi(y))$. The map φ is a local lifting of ψ.

$b \Rightarrow a$). Let $\overline{\rho}: G/H \to O(\mu_1)$ be the map defined by $\overline{\rho}(gH) = g \cdot \mu_1$. Applying b) to the map $\overline{\rho}$, we have, for each $gH \in G/H$, a neighborhood V of gH in G/H and a morphism $\varphi: V \to G$ such that $\rho \circ \varphi = \overline{\rho}$ on V. The following map defines a local trivialization of the principal bundle $G \to G/H$ in a neighborhood of gH. Let $\sigma: V \times H \to G$ be given by $\sigma(v, h) = \varphi(v)h$. Suppose

$\varphi(v_1)h_1 = \varphi(v_2)h_2$. Then $\overline{\rho}(v_1) = \rho(\varphi(v_1)) = \rho(\varphi(v_1)h_1) = \rho(\varphi(v_2)h_2) = \rho(\varphi(v_2)) = \overline{\rho}(v_2)$. Since $\overline{\rho}$ is injective, it follows that $v_1 = v_2$. Then $h_1 = h_2$ and so σ is injective. From dimension argument it follows that $\sigma(V \times H)$ is an open subset in G which proves a). $b \Rightarrow c$) is obviously true.

$c \Rightarrow b$). For $x \in X$, define $\eta_x\colon \mathrm{spec} K \to O(\mu_1)$ to be the constant morphism with value $\psi(x)$. From c), we have a lifting of η_x to $\overline{\eta}_x\colon \mathrm{spec} K \to G$. The morphism $\overline{\eta}_x$ is defined by functions in K, where K is the quotient field of the coordinate ring of $\overline{O(\mu_1)}$. Let V be the open subset of $O(\mu_1)$ on which these functions are regular. Then $\overline{\eta}_x$ induces a morphism $\tau\colon V \to G$. The morphism $\tau \circ \psi\colon \psi^{-1}(V) \to G$ gives a local lifting.

Remark. We use the proposition in this special set up, but note that the equivalence is obviously true for any group action.

If the stablilizer of the Lie algebra μ_1 is either $GL(n)$, $SL(n)$, $Sp(n)$, a product or an extension of those, or a connected solvable group, then every principal G-bundle is locally trivial, and so the Lie algebra μ_{R_M} defines a degeneration of μ_1 to μ_0.

§3. Rigidity and Cohomology.

Much of the motivation for the theory of deformations of Lie algebras is the construction of new Lie algebras from known ones. Thus it is important to have criteria for identifying those Lie algebras which have no non-trivial deformations. Such Lie algebras are called rigid.

A related concept in the orbit closure setting over an algebraically closed field arises from the idea of "generic" Lie algebras. If the orbit of a Lie algebra μ is open, then $O(\mu)$ is dense in the component of the variety L in which it lies. It follows that all Lie algebras in that component are degenerations of μ and so μ is the "generic" Lie algebra in that component. Lie algebras with open orbits are also referred to as rigid. Of course, if μ is a Lie algebra which is rigid in this sense, then there are no Lie algebras which degenerate to μ.

Over any algebraically closed field k, it immediately follows that the orbit of μ is open if μ has no nontrivial deformations.

The two concepts of rigidity are equivalent if k has characteristic 0. In the case of one parameter deformations, the following theorem was proved by Gerstenhaber and Schack [15]. Here we present a simpler proof for deformations parametrized by $k[[t_1, \ldots, t_r]]$.

Theorem 3.1. *Let char $k = 0$. If the orbit of the n-dimensional Lie algebra μ_0 is open, then every deformation of μ_0 parameterized by $k[[t_1, \ldots, t_r]]$ is trivial.*

<u>Proof</u>. Recall that $\mu_{t_1,\ldots,t_r}$ is a trivial deformation of μ_0 if it is equivalent to μ_0, i.e. if there is an element $a_{t_1,\ldots,t_r}$ in $GL_n(k[[t_1,\ldots,t_r]])$ such that

$$a_{t_1,\ldots,t_r}\cdot\mu_{t_1,\ldots,t_r}=\mu_0.$$

Consider the Zariski tangent space $T_{\mu_0}(O(\mu_0))$ to the orbit $O(\mu_0)$ of μ_0 (it is defined to be $T_{\mu_0}(\overline{O(\mu_0)})$). For any Lie algebra μ, let $\rho_\mu\colon GL_n(k)\to O(\mu)$ be the orbit map $(g\to g\cdot\mu)$ and let $d\rho_\mu$ be the differential of this map. In general (for k of any characteristic), we have

$$B^2(\mu,\mu)=\mathrm{im}d\rho_\mu\subset T_\mu(O(\mu))\subset T_\mu(L)\subset Z^2(\mu,\mu) \tag{3}$$

where $T_\mu(L)$ is the tangent space to the variety L and $Z^2(\mu,\mu)$ is the space of 2-cocycles.

Because $O(\mu_0)$ is open, $T_{\mu_0}(O(\mu_0))=T_{\mu_0}(L)$. The map $d\rho_{\mu_0}$ is surjective if and only if $\mathrm{Ker}d\rho_{\mu_0}=T_e(\mathrm{Stab}(\mu_0))$, see [27, AI.5.5]. Since k has characteristic 0, this condition is satisfied [4, Ch. 3,1.5]. Therefore we have

$$B^2(\mu_0;\mu_0)=T_{\mu_0}(L). \tag{4}$$

We will construct, for each integer $m\geq 0$ a polynomial of degree m

$$a^m_{t_1,\ldots,t_r}=1+\sum_{i=1}^{r}a_it_i+\sum_{1\leq i\leq j\leq r}a_{ij}t_it_j+\ldots+\sum_{1\leq i_1\leq\ldots\leq i_m\leq r}a_{i_1\ldots i_m}t_{i_1}\ldots t_{i_m}$$

such that $a^m_{t_1,\ldots,t_r}\cdot\mu_{t_1,\ldots,t_r}$ is a Lie algebra and

$$a^m_{t_1,\ldots,t_r}\cdot\mu_{t_1,\ldots,t_r}=\mu_0+\text{terms of degree}\geq m+1. \tag{5}$$

We proceed by induction. The case $m=0$ is trivial. Assume that $a^m_{t_1,\ldots,t_r}$ satisfies condition (5) above. We have

$$a^m_{t_1,\ldots,t_r}\cdot\mu_{t_1,\ldots,t_r}=\mu_0+\sum_{1\leq i_1\leq\ldots\leq i_{m+1}\leq r}\varphi_{i_1,\ldots,i_{m+1}}t_{i_1}\ldots t_{i_{m+1}}$$

$$+\text{ higher degree terms.}$$

We may assume that some $\varphi_{i_1,\ldots,i_{m+1}}$ is non-zero; otherwise we let $a^{m+1}_{t_1,\ldots,t_r}=a^m_{t_1,\ldots,t_r}$.

Let I be the ideal of $k[X_{ijk}]$ consisting of the polynomials which vanish on L. Because $a^m_{t_1,\ldots,t_r}\cdot\mu_{t_1,\ldots,t_r}$ is a Lie algebra, $f(a^m_{t_1,\ldots,t_r}\cdot\mu_{t_1,\ldots,t_r})=0$ for all $f\in I$. By straightforward computation, it follows that the coefficients $\varphi_{i_1,\ldots,i_{m+1}}$ of the non-vanishing terms of lowest positive degree are in the tangent space $T_{\mu_0}(L)$ (i.e. $\varphi_{i_1,\ldots,i_{m+1}}$ defines a point derivation of μ_0 which vanishes on I). It follows from the equality (4) that each $\varphi_{i_1,\ldots,i_{m+1}}$ is a 2-coboundary. Then there exist 1-cochains $x_{i_1,\ldots,i_{m+1}}$ such that $\delta(x_{i_1,\ldots,i_{m+1}}) = \varphi_{i_1,\ldots,i_{m+1}}$. The differential map δ defines an action of any 1-cochain x on the space of alternating bilinear maps:

$$(\delta x)(a,b) = (x \cdot \eta)(a,b) = -x(\eta(a,b) + \eta(a,xb) - \eta(b,xa).$$

Then

$$(1 - \sum_{1 \leq i_1 \leq \ldots \leq i_{m+1} \leq r} t_{i_1} \ldots t_{i_{m+1}} x_{i_1,\ldots,i_{m+1}}) \cdot (a^m_{t_1,\ldots,t_r} \cdot \mu_{t_1,\ldots,t_r}) =$$

$$= \mu_0 + \text{terms of degree} > m+1.$$

Then the polynomial

$$a^{m+1}_{t_1,\ldots,t_r} = (1 - \sum_{1 \leq i_1 \leq \ldots \leq i_{m+1} \leq r} t_{i_1} \ldots t_{i_{m+1}} x_{i_1,\ldots,i_{m+1}}) \cdot a^m_{t_1,\ldots,t_r}$$

satisfies the condition (5).

The resulting formal power series $a_{t_1,\ldots,t_r}$ is an infinite product which is defined because there are only finitely many terms of a given order. It follows that $\mu_{t_1,\ldots,t_r}$ is a trivial deformation via $a_{t_1,\ldots,t_r}$.

Remark. If the field k has nonzero characteristic, then the differential map d_{ρ_μ} may fail to be surjective. Then a Lie algebra μ with an open orbit could have non-trivial deformation. For associative algebras such examples are known [15].

Remark. We see from (3) in the proof of Theorem 3.1 that in characteristic 0, if $Z^2(\mu;\mu) = T_\mu(L)$, then the property $H^2(\mu;\mu) = 0$ is equivalent to the property that the orbit of μ is open. Over a field of any characteristic, it is easy to show that $T_\mu(\mathcal{L}) = Z^2(\mu;\mu) = T_\mu(L)$ if and only if the scheme $\mathcal{L}$ is reduced at the point μ (i.e. the local ring O_μ satisfies $\sqrt{0} = 0$). In both characteristic 0 and characteristic p, there are examples of algebras μ such that $H^2(\mu;\mu) \neq 0$ although μ has no non-trivial deformations ([37],[15]).

A sufficient condition for rigidity for any Lie algebra μ is that $H^2(\mu;\mu) = 0$ [13]. For instance, if a finite dimensional Lie algebra μ over a field k of characteristic 0 is semisimple or if μ is its Borel subalgebra, then $H^2(\mu;\mu) = 0$ and so μ is rigid [6, 24.1].

The following diagram summarizes the above results.

$$\begin{array}{l} H^2(\mu;\mu) = 0 \\ \qquad \Downarrow \\ \mu \text{ has no non-trivial deformations} \\ \qquad \Downarrow \ (\Uparrow \text{ if char } k = 0) \\ O(\mu) \text{ is open.} \end{array}$$

In the category of semisimple commutative k-algebras, rigidity with respect to deformation is equivalent to the vanishing of the symmetric 2-cohomology space $H^2(\mu;\mu)^s$ [23].

A more direct relationship between deformations and cohomology is given by the concept of infinitesimal deformations.

Definition. A deformation μ_A of μ_0 parameterized by A is of order r if the local algebra A satisfies $m_A^{r+1} = 0$. A deformation of order 1 is called an infinitesimal deformation.

From Section 1, recall the definition of a formal deformation parameterized by a complete local ring. A deformation of μ_0 parametrized by A is a projective limit $\underleftarrow{\lim}\mu_n$, where μ_n is a deformation of μ_0 parametrized by A/m_A^{n+1}. If μ_A is a deformation parametrized by a complete local ring, then the Lie algebra μ_r is a deformation of order r.

For instance, if $A = k[[t_1, \ldots, t_s]]$, and μ_A is a deformation of μ_0, then μ_1 can be written

$$\mu_1 = \mu_0 + \sum_{i+1}^{s} t_i \varphi_i .$$

It follows from the Jacobi identity that φ_i is a 2-cocyle for all i. If φ_i is a 2-coboundary for some i, then there is an equivalent deformation where the t_i-term is zero and at least one of the non-zero terms of lowest degree involving t_i has a coefficient which is a 2-cocycle not cohomologous to zero [13]. It follows that if $H^2(\mu_0;\mu_0) = 0$, then every infinitesimal deformation of μ_0 is trivial.

If $H^2(\mu_0;\mu_0) \neq 0$, then the maximal set of non- trivial pairwise non-equivalent infinitesimal deformations forms a basis of $H^2(\mu_0;\mu_0)$. (For $\varphi \in Z^2(\mu_0;\mu_0)$, $\mu_0 + t\varphi$ is an infinitesimal deformation of μ_0.)

The elements of the 3-cohomology space $H^3(\mu_0;\mu_0)$ can be interpreted as obstructions to extending an infinitesimal deformation to a higher order deformation. These obstructions are closely connected with the Massey operations in the cohomology space (see [11]).

In the case of deformations of Lie algebras over **R** or **C** there is an analytic map from $H^2(\mu;\mu)$ to $H^3(\mu;\mu)$ whose zeroes parametrize a neighborhood of μ

[38]. In particular, if this map is injective, then the orbit of μ is open (see e.g. [36]). Notice that the converse is not true (see [5]).

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