

# Singular deformations of Lie algebras on an example

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## 1. Introduction

We introduce the concept of singular deformations of Lie algebras. As far as we know they have been never considered in the literature. Nevertheless, they are unavoidable in any complete classification of deformations. In this paper we will show that singular deformations occur even for some of the simplest infinite dimensional cases.

Let  $\mathfrak{g}$  be a Lie algebra with the commutator  $[\cdot, \cdot]$ . Consider a formal one-parameter deformation

$$[g, h]_t = [g, h] + \sum_{k \geq 1} \alpha_k(g, h)t^k$$

of  $\mathfrak{g}$ . A deformation is called *non-singular* if there exists a formal one-parameter family of linear transformations

$$\varphi_t(g) = g + \sum_{l \geq 1} \beta_l(g)t^l$$

of  $\mathfrak{g}$  and a formal (not necessarily invertible) parameter change  $u = u(t)$  which transform the deformation  $[g, h]_t$  into a deformation

$$[g, h]'_u = [g, h] + \sum_{k \geq 1} \alpha'_k(g, h)u^k, \quad \varphi_t^{-1}[\varphi_t(g), \varphi_t(h)]_t = [g, h]'_{u(t)}$$

with the cocycle  $\alpha'_1 \in C^2(\mathfrak{g}; \mathfrak{g})$  being not cohomologous to 0. Otherwise the deformation is called *singular*.

The example we present is the following. Consider the complex infinite-dimensional Lie algebra  $L_1$  of polynomial vector fields in  $\mathbb{C}$  with trivial 1-jet at 0. This Lie algebra is spanned by the vector fields  $e_i = z^{i+1} \frac{d}{dz}$ ,  $i = 1, 2, 3, \dots$ , and the commutator is defined by the standard formula

$$[e_i, e_j] = (j - i)e_{i+j}.$$

This Lie algebra proves to be especially interesting from the point of view of the deformation theory, for on one hand its deformations may be completely classified, and on the other hand they behave in a very unusual manner.

The deformations of  $L_1$  were first studied in 1983 by the first author [Fi1]. In [Fi1] three one-parameter deformations of the Lie algebra  $L_1$  were considered:

$$\begin{aligned} [e_i, e_j]_t^1 &= (j-i)(e_{i+j} + te_{i+j-1}); \\ [e_i, e_j]_t^2 &= \begin{cases} (j-i)e_{i+j} & \text{if } i \neq 1, j \neq 1, \\ (j-1)e_{j+1} + tje_j & \text{if } i = 1, j \neq 1; \end{cases} \\ [e_i, e_j]_t^3 &= \begin{cases} (j-i)e_{i+j} & \text{if } i \neq 2, j \neq 2, \\ (j-2)e_{j+2} + tje_j & \text{if } i = 2, j \neq 2. \end{cases} \end{aligned}$$

All the three families of Lie algebras may be realized as families of subalgebras of the Lie algebra  $L_0$  (spanned by  $e_i$  with  $i \geq 0$ ). The first deformation may be defined by the formula  $e_i \mapsto e_i + te_{i-1}$  ( $i \geq 1$ ); in other words, the Lie algebra  $L_1$  of vector fields with a double zero at 0 is deformed into the Lie algebra of vector fields with two zeroes at points 0 and  $t$ . The two other deformations are defined by the formulas

$$\begin{aligned} e_1 &\mapsto e_1 + te_0, \quad e_i \mapsto e_i \text{ if } i \neq 1; \\ e_2 &\mapsto e_2 + te_0, \quad e_i \mapsto e_i \text{ if } i \neq 2. \end{aligned}$$

All the three deformations are pairwise not equivalent. Moreover, if  $L_1^1, L_1^2, L_1^3$  are Lie algebras from the three families corresponding to arbitrary non-zero values of the parameter (up to an isomorphism, they do not depend on the non-zero parameter value), then neither two of  $L_1^1, L_1^2, L_1^3$  are isomorphic to each other. Indeed, obviously

$$\dim(L_1^r/[L_1^r, L_1^r]) = \begin{cases} 2 & \text{if } r = 1, \\ 1 & \text{if } r = 2, 3. \end{cases}$$

On the other hand, the Lie algebra  $M^2 = [L_1^2, L_1^2]$  is spanned by  $e_2, e_3, e_4, \dots$ , and  $M^3 = [L_1^3, L_1^3]$  is spanned by  $e_1, e_3, e_4, \dots$ . It is seen from this that

$$\dim(M^r/[M^r, M^r]) = \begin{cases} 3 & \text{if } r = 2, \\ 2 & \text{if } r = 3. \end{cases}$$

The main result of [Fi1] is the following

**THEOREM 1.1.** *Any formal one-parameter deformation of  $L_1$  may be reduced by a formal parameter change to one of the three deformations above.*

However, the article [Fi1] contains no detailed proof of this result. It is claimed there that the result follows from certain calculations of Lie and Massey products in the cohomology  $H^*(L_1; L_1)$ . These calculations are done in a more detailed paper [Fi2], but still they do not imply Theorem 1.1. Moreover, the description of the miniversal deformation of  $L_1$  given in [Fi1] needs a correction. Namely, the second of these three deformations is *singular* in the above sense. The correct description is the following: the base of the

miniversal deformation of  $L_1$  is the union of two smooth curves and one cuspidal curve passing through 0 with the common tangent.

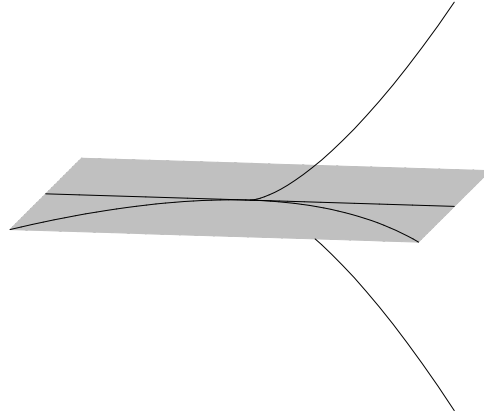


Fig. 1

Hence not only the base of the miniversal deformation of the Lie algebra  $L_1$  is a singular variety, but also one of its irreducible components is singular. Although it is not impossible from the point of view of the general theory, such examples have not been known before. As we have already mentioned, singular one parameter deformations, which appear, certainly, not only in the Lie algebra theory, have never been properly studied. For example, it is a common belief that if no non-trivial infinitesimal one parameter deformation is extendable to a formal deformation, then the Lie algebra is formally rigid. But is it really?\*

There exists a general theory which provides a construction of a local miniversal deformation of a Lie algebra  $\mathfrak{g}$ . This theory is outlined in [Fi3]. The article [Fi3] contains also a conjecture that the base of the miniversal deformation may be described explicitly by a system of algebraic equations in the cohomology space  $H^2(\mathfrak{g}; \mathfrak{g})$ , and that these equations may be derived in a certain way from the Lie-Massey multiplication in this cohomology. This procedure, however, needs some further explanations. We are working on an explicit general construction of formal miniversal deformations of Lie algebras (see [FiFu]).

In this article we give direct proofs of Theorem 1.1 and of the above description of the base of miniversal deformation of  $L_1$ . Both proofs are direct and independent of any general theory; they appear completely reliable.

In Sections 2 and 3 we list results which we regard as known; they concern general theory of deformations of Lie algebras, and the cohomology of the Lie algebra  $L_1$ . Section 4 contains the proof of the singularity of the deformation  $[\cdot, \cdot]_t^2$  and the non-singularity of

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\* Known (at least in the associative case) examples show that the relations between local (formal) and global (smooth) deformations may also be complicated in the infinite dimensional case; for example, the algebra of regular functions on the complex projective line with 4 punctures has a natural deformation with the cross ratio of the punctures as the parameter, but it has no infinitesimal deformations at all (see [Ko]).

the deformations  $[\cdot, \cdot]_t^1, [\cdot, \cdot]_t^3$ . Theorem 1.1 is proved in Section 6 with some technical work done in Section 5.

## 2. Deformations and cohomology

Here we recall very briefly the classical theory of deformations of a Lie algebra structure (see [Fu] and [Fi3] for details). Let  $\mathfrak{g}$  be a (complex) Lie algebra with the bracket  $[\cdot, \cdot]$ . A formal one-parameter deformation of  $\mathfrak{g}$  is a power series

$$[g, h]_t = [g, h] + \sum_{k \geq 1} \alpha_k(g, h)t^k,$$

where  $\alpha_k \in \text{Hom}_{\mathbb{C}}(\Lambda^2 \mathfrak{g}, \mathfrak{g}) = C^2(\mathfrak{g}; \mathfrak{g})$ , the latter refers to the standard cochain complex  $\{C^i(\mathfrak{g}; \mathfrak{g}), \delta\}$  of  $\mathfrak{g}$  with coefficients in the  $\mathfrak{g}$ -module  $\mathfrak{g}$  (that is in the adjoint representation) – see [Fu]. The Jacobi identity for  $[\cdot, \cdot]_t$  is equivalent to the sequence of relations

$$\delta \alpha_k = -\frac{1}{2} \sum_{i=1}^{k-1} [\alpha_i, \alpha_{k-i}], \quad (1)$$

where  $[\cdot, \cdot]$  denotes the usual product in the standard graded differential Lie algebra structure of the complex  $\{C^i(\mathfrak{g}; \mathfrak{g}), \delta\}$ : if  $\beta \in C^p(\mathfrak{g}; \mathfrak{g})$  and  $\gamma \in C^q(\mathfrak{g}; \mathfrak{g})$  then  $[\beta, \gamma] \in C^{p+q-1}(\mathfrak{g}; \mathfrak{g})$ ,

$$\begin{aligned} & [\beta, \gamma](g_1, \dots, g_{p+q-1}) \\ &= \sum_{1 \leq i_1 < \dots < i_q \leq p+q-1} (-1)^{i_1 + \dots + i_q - \frac{q(q+1)}{2}} \beta(\gamma(g_{i_1}, \dots, g_{i_q}), g_1, \dots, \hat{g}_{i_1} \dots \hat{g}_{i_q}, \dots, g_{p+q-1}) \\ &+ (-1)^{pq+p+q} \sum_{1 \leq j_1 < \dots < j_p \leq p+q-1} (-1)^{j_1 + \dots + j_p - \frac{p(p+1)}{2}} \gamma(\beta(g_{j_1}, \dots, g_{j_p}), \\ & \qquad \qquad \qquad g_1, \dots, \hat{g}_{j_1} \dots \hat{g}_{j_p}, \dots, g_{p+q-1}). \end{aligned}$$

In particular,

$$\delta \alpha_1 = 0, \quad \delta \alpha_2 = -\frac{1}{2} [\alpha_1, \alpha_1], \quad \delta \alpha_3 = -[\alpha_1, \alpha_2]. \quad (2)$$

Two deformations  $[g, h]_t, [g, h]'_t$  are called equivalent if there exists a formal one-parameter family  $\{\varphi_t\}$  of linear transformations of  $\mathfrak{g}$ ,

$$\varphi_t(g) = g + \sum_{k \geq 1} \beta_k(g)t^k,$$

such that

$$[g, h]'_t = \varphi_t^{-1}[\varphi_t(g), \varphi_t(h)]_t.$$

It is easy to see that

$$\alpha'_1 - \alpha_1 = \delta \beta_1 \quad (3)$$

and, more generally, if  $\beta_1 = \dots = \beta_{s-1} = 0$ , then

$$\alpha'_s - \alpha_s = \delta\beta_s$$

(here  $\alpha'_i$  corresponds to  $\alpha_i$  for  $[\cdot, \cdot]_t'$ ). The first of equalities (2) shows that  $\alpha_1$  is a cocycle, and the equality (3) shows that the cohomology class of  $\alpha_1$  depends only on the equivalence class of the deformation. Cohomology classes from  $H^2(\mathfrak{g}; \mathfrak{g})$  are called *infinitesimal deformations of  $\mathfrak{g}$* , and the cohomology class of  $\alpha_1$  is called the *differential* of the formal deformation  $[\cdot, \cdot]_t$ . An infinitesimal deformation  $\alpha \in H^2(\mathfrak{g}; \mathfrak{g})$  is not necessarily the differential of any formal deformation: the second and the third of the equalities (2) provide a necessary condition for it: the Lie square and the Massey-Lie cube should be equal to 0. The other relations (1) give more necessary conditions for an infinitesimal deformation being a differential (which comprise all together a sufficient condition for this). Usually these conditions are formulated in terms of higher Massey-Lie products (see [R], [Fu] and [FL]), but we use another method in this article.

### 3. Cohomology of $L_1$

Here we recall the necessary information from [FeFu] and [Fil] about the (continuous) cohomology  $H^*(L_1; L_1)$ . First of all, the Lie algebra  $L_1$  is  $\mathbb{Z}_{>0}$ -graded,  $\deg e_i = i$ , and this gives rise to a  $\mathbb{Z}$ -grading in  $C^*(L_1; L_1)$  and  $H^*(L_1; L_1)$ :  $\deg \alpha = k$  for  $\alpha \in C^q(L_1; L_1)$  if

$$\alpha(e_{i_1}, \dots, e_{i_q}) \in \mathbb{C}e_{i_1 + \dots + i_q - k} \text{ for all } i_1, \dots, i_q.$$

One has  $H^q(L_1; L_1) = \bigoplus_k H_{(k)}^q(L_1; L_1)$ . The following is a corollary from a more general result of [FeFu] (see [Fil]):

**THEOREM 3.1.**

$$H_{(k)}^q(L_1; L_1) \cong \begin{cases} \mathbb{C} & \text{if } \frac{3(q-1)^2 + (q-1)}{2} \leq k < \frac{3q^2 - q}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

*In particular,*

$$H^2(L_1; L_1) \cong \bigoplus_{k=2}^4 H_{(k)}^2(L_1; L_1), \quad H^3(L_1; L_1) \cong \bigoplus_{k=7}^{11} H_{(k)}^3(L_1; L_1),$$

*all the summands in these two sums being isomorphic to  $\mathbb{C}$ .*

Lie and Massey-Lie -products in  $H^*(L_1; L_1)$  have been calculated in [Fi2]. We need the following result.

**THEOREM 3.2.** *Let  $0 \neq b \in H_{(3)}^2(L_1; L_1)$ ,  $0 \neq c \in H_{(4)}^2(L_1; L_1)$ . Then*

$$\begin{aligned} 0 &\neq [b, c] \in H_{(7)}^3(L_1; L_1), \\ 0 &\neq [c, c] \in H_{(8)}^3(L_1; L_1), \\ 0 &\neq \langle b, b, b \rangle \in H_{(9)}^3(L_1; L_1). \end{aligned}$$

Here  $\langle b, b, b \rangle$  is the Massey-Lie cube of  $b$ ; the inequality  $\langle b, b, b \rangle \neq 0$  means that if  $\beta \in b$  is a cocycle, and  $[\beta, \beta] = \delta f$ , then the cocycle  $[\beta, f]$  is not cohomologous to 0.

We do not prove here Theorems 3.1 and 3.2. Theorem 3.2 may be obtained also from the results of Section 5 below.

#### 4. The three deformations of $L_1$

Consider the three deformations of the Lie algebra  $L_1$  defined in the introduction.

**THEOREM 4.1.** *Of the three deformations  $[\cdot, \cdot]_t^r$ ,  $r = 1, 2, 3$ , of  $L_1$ , the first and the third are non-singular, while the second is singular.*

**PROOF.** The three deformations have the form

$$[g, h]_t^r = [g, h] + \alpha_1^r(g, h)t, \quad r = 1, 2, 3,$$

where

$$\begin{aligned} \alpha_1^1(e_i, e_j) &= (j - i)e_{i+j-1}; \\ \alpha_1^2(e_i, e_j) &= \begin{cases} je_j & \text{if } i = 1, j \neq 1, \\ 0 & \text{if } i \neq 1, j \neq 1; \end{cases} \\ \alpha_1^3(e_i, e_j) &= \begin{cases} je_j & \text{if } i = 2, j \neq 2, \\ 0 & \text{if } i \neq 2, j \neq 2. \end{cases} \end{aligned}$$

Obviously,  $\alpha_1^1, \alpha_1^2 \in C_{(1)}^2(L_1; L_1)$ ,  $\alpha_1^3 \in C_{(2)}^2(L_1; L_1)$ . The cocycle  $\alpha_1^3$  is not cohomologous to 0. To prove this, we calculate its value on a non-trivial cycle in  $C_{(2)}^2(L_1; L_1^*)$ ,  $L_1^*$  being the  $L_1$ -module dual to  $L_1$ . This cycle may be chosen as

$$a^{(2)} = e_3^* \otimes (e_1 \wedge e_4 - 3e_2 \wedge e_3) + \frac{1}{2}e_2^* \otimes (e_1 \wedge e_3) + 3e_1^* \otimes (e_1 \wedge e_2)$$

(the fact that it is a cycle is checked by a direct calculation, the fact that it is not homol-  
ogous to 0 follows from the calculation below). We have:

$$\alpha_1^3(e_1, e_4) = 0, \alpha_1^3(e_2, e_3) = 3e_3, \alpha_1^3(e_1, e_3) = 0, \alpha_1^3(e_1, e_2) = -e_1,$$

$$\langle \alpha_1^3, a^{(2)} \rangle = 0 - 9 + 0 - 3 = -12 \neq 0.$$

Hence the deformation  $[\cdot, \cdot]_t^3$  is non-singular.

The cocycles  $\alpha_1^1, \alpha_1^2$  are cohomologous to 0 (because  $H_{(1)}^2(L_1; L_1) = 0$ ). More precisely,  $\alpha_1^r = \delta\beta^r$ ,  $r = 1, 2$ , where  $\beta^1, \beta^2 \in C_{(1)}^1(L_1; L_1)$  are given by the formulas

$$\beta^1(e_i) = \frac{i-1}{2}e_{i-1}, \quad \beta^2(e_i) = \begin{cases} \frac{i+1}{2}e_{i-1} & \text{for } i \neq 1, \\ 0 & \text{for } i = 1. \end{cases}$$

Put  $\varphi_t^r(e_i) = e_i + \beta^r(e_i)t$ , and compute

$$\gamma^r(e_i, e_j) = (\varphi_t^r)^{-1}[\varphi_t^r(e_i), \varphi_t^r(e_j)]_t$$

modulo  $t^4$ . It follows from  $\alpha_1^r = \delta\beta^r$  that

$$\gamma^r(e_i, e_j) = [e_i, e_j] + \sum_{k \geq 2} \gamma_k^r(e_i, e_j)t^k,$$

where  $\gamma_k^r \in C_{(k)}^2(L_1; L_1)$ , and a direct calculation shows that

$$\begin{aligned} \gamma_2^1(e_1, e_2) &= 0, & \gamma_2^1(e_1, e_3) &= e_2, & \gamma_2^1(e_1, e_4) &= 3e_3, & \gamma_2^1(e_2, e_3) &= \frac{3}{2}e_3, \\ \gamma_3^1(e_1, e_2) &= 0, & \gamma_3^1(e_1, e_3) &= e_1, & \gamma_3^1(e_1, e_4) &= -3e_2, & \gamma_3^1(e_2, e_3) &= -e_2, \\ \gamma_2^2(e_1, e_2) &= 0, & \gamma_2^2(e_1, e_3) &= 4e_2, & \gamma_2^2(e_1, e_4) &= \frac{15}{2}e_3, & \gamma_2^2(e_2, e_3) &= \frac{15}{2}e_3, \\ \gamma_3^2(e_1, e_2) &= 0, & \gamma_3^2(e_1, e_3) &= -6e_1, & \gamma_3^2(e_1, e_4) &= -15e_2, & \gamma_3^2(e_2, e_3) &= -9e_2. \end{aligned}$$

We see, in particular, that

$$\langle \gamma_2^1, a^{(2)} \rangle = 0 + \frac{1}{2} + 3 - \frac{9}{2} = -1 \neq 0, \quad (4)$$

$$\langle \gamma_2^2, a^{(2)} \rangle = 2 + \frac{15}{2} - \frac{45}{2} = -13 \neq 0. \quad (5)$$

A non-trivial cycle in  $C_2^{(3)}(L_1; L_1^*)$  may be chosen as

$$a^{(3)} = e_2^* \otimes (e_1 \wedge e_4 - 3e_2 \wedge e_3)$$

(again it is a cycle in virtue of a direct calculation, and it is not homologous to 0 in virtue of the calculation below), and we see that

$$\langle \gamma_3^1, a^{(3)} \rangle = -3 + 3 = 0, \quad (6)$$

$$\langle \gamma_3^2, a^{(3)} \rangle = -15 + 27 = 12 \neq 0. \quad (7)$$

The inequalities (5) and (7) show that the deformation  $[\cdot, \cdot]_t^2$  is singular: one-parameter families of transformations of  $L_1$  cannot make cohomology classes of  $\gamma_2^2, \gamma_3^2$  collinear, and no parameter change can transform this deformation into a deformation with a non-zero infinitesimal deformation.

On the contrary, the deformation  $[\cdot, \cdot]_t^1$  is non-singular. Indeed, having applied an appropriate transformation  $\varphi_t(g) = g + \lambda(g)t^3$ , we kill  $\gamma_3^1$ . Then we will have

$$\delta\gamma_5^1 = -[\gamma_1^1, \gamma_4^1] - [\gamma_2^1, \gamma_3^1] = 0;$$

the cocycle  $\gamma_{(5)}^1 \in C_{(5)}^2(L_1; L_1)$  is cohomologous to zero (because  $H_{(5)}^2(L_1; L_1) = 0$ ), and we can kill  $\gamma_{(5)}^1$  by an appropriate transformation  $\varphi_t(g) = g + \mu(g)t^5$ . Proceeding in this way, we kill all  $\gamma_k^1$  with odd  $k$ , after which we apply the parameter change  $u(t) = t^2$ . We get the deformation

$$[e_i, e_j]_u' = [e_i, e_j] + \sum_{l \geq 1} \gamma_{2l}(e_i, e_j)u^l$$

with  $\gamma_2$  being not cohomologous to 0.

This completes the proof of Theorem 4.1.

### 5. Some remarkable cochains of $L_1$

Let  $W$  be the  $L_1$ -module spanned by  $e_j$  with all  $j \in \mathbb{Z}$  and with the  $L_1$ -action  $e_i(e_j) = (j - i)e_{i+j}$ . It is an extension of the adjoint representation. Define a cochain

$$\mu_k \in C_{(k)}^1(L_1; W), \quad k \geq 2,$$

by the formula

$$\mu_k(e_i) = (-1)^{i+1} \binom{k-1}{i-2} e_{i-k}.$$

Thus  $\mu_k(e_i) = 0$  if  $i = 1$  or  $i > k + 1$ , and  $\mu_k(e_2) = -e_{2-k}$ ,  $\mu_k(e_{k+1}) = (-1)^k e_1$ .

LEMMA 5.1.  $\delta\mu_k(e_1, e_i) = 0$  for all  $i, k$ .

PROOF.

$$\begin{aligned} \delta\mu_k(e_1, e_i) &= \mu_k((i-1)e_{i+1}) - [e_1, \mu_k(e_i)] \\ &= \left( (-1)^i (i-1) \binom{k-1}{i-1} - (i-k-1)(-1)^{i+1} \binom{k-1}{i-2} \right) e_{i-k+1} \\ &= (-1)^i \left( \frac{(i-1)(k-1)!}{(k-i)!(i-1)!} - \frac{(k-i+1)(k-1)!}{(k-i+1)!(i-2)!} \right) e_{i-k+1} = 0. \end{aligned}$$

LEMMA 5.2. The cochains  $\delta\mu_2, \delta\mu_3, \delta\mu_4$  are cocycles in  $C^2(L_1; L_1) \subset C^2(L_1; W)$  not cohomologous to 0.

PROOF. Since  $\delta\mu_k(e_i, e_j) \in \mathbb{C}e_{i+j-k}$ , and if  $j > i \geq 2$  and  $k \leq 4$  then  $i + j - k \geq 1$ , then  $\delta\mu_2, \delta\mu_3, \delta\mu_4 \in C^2(L_1; L_1)$ . These cochains are obviously cocycles, and they are not cohomologous to 0, because no non-zero coboundary in  $C_{(k)}^2(L_1; L_1)$  with  $k \geq 2$  is annihilated by  $e_1$ : if  $\lambda \in C_{(k)}^1(L_1; L_1)$ ,  $k \geq 2$ , then  $\lambda(e_i) = a_i e_{i-k}$ ,  $a_i = 0$  if  $i \leq k$ , and if  $\delta\lambda(e_1, e_i) = ((i-1)a_{i+1} - (i-k-1)a_i)e_{i+1} = 0$ , then  $(k-1)a_{k+1} = 0, ka_{k+2} = 0, (k+1)a_{k+3} - a_{k+2} = 0, (k+2)a_{k+4} - 2a_{k+3} = 0, \dots$ , which implies successively that  $a_i = 0$  for  $i = k+1, k+2, k+3, k+4, \dots$

COROLLARY. Any cocycle in  $C^2(L_1; L_1)$  is cohomologous to a linear combination of  $\delta\mu_2, \delta\mu_3, \delta\mu_4$ .

Define the cochains  $\delta_k \in C_{(k)}^2(L_1; L_1)$  by the formula

$$\delta_k(e_i, e_j) = \begin{cases} \delta\mu_k(e_i, e_j) & \text{if } i + j - k \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

(in particular,  $\delta_k = \delta\mu_k$  if  $k = 2, 3, 4$ ), and put

$$\delta_{k,l} = [\delta_k, \delta_l] \in C_{(k+l)}^3(L_1; L_1).$$

LEMMA 5.3. For  $k, l, s, t$  fixed and  $N > 1 + \max(k, l, k + l - s, k + l - t)$

$$\delta_{k,l}(e_s, e_t, e_N) = a(k, l, s, t)(N + k + l - s - t)e_{N+s+t-k-l}.$$

where  $a(k, l, s, t)$  depends on  $k, l, s, t$ , but not on  $N$ .

PROOF. Obviously, for  $N > k + 1$

$$\delta\mu_k(e_s, e_N) = (-1)^s \binom{k-1}{s-2} (N + k - s)e_{N+s-k}.$$

By definition,  $\delta_{k,l}(e_s, e_t, e_N)$  is the sum of 6 summands. One of them is  $\delta_k(\delta_l(e_s, e_t), e_N) = \delta\mu_k(\delta_l(e_s, e_t), e_N)$ , which is either 0, or

$$\begin{aligned} \delta\mu_k(\delta\mu_l(e_s, e_t), e_N) &= a\mu_k(e_{s+t-l}, e_N) \\ &= ab(N + k + l - s - t)e_{N+s+t-k-l}, \end{aligned}$$

where  $a$  depends only on  $l, s, t$ , and  $b$  depends only on  $k, l, s, t$ . A similar formula is true for  $\delta_k(\delta_l(e_s, e_t), e_N)$ . Two more summands are

$$\delta\mu_k(\delta\mu_l(e_t, e_N), e_s) + \delta\mu_l(\delta\mu_k(e_N, e_s), e_t).$$

This sum is  $e_{N+s+t-k-l}$  times

$$\begin{aligned} &- (-1)^{s+t} \binom{l-1}{t-2} \binom{k-1}{s-2} (N + l - t)(N + k - l + t - s) \\ &+ (-1)^{s+t} \binom{k-1}{s-2} \binom{l-1}{t-2} (N + k - s)(N + l - k + s - t) \\ &= (-1)^{s+t} \binom{l-1}{t-2} \binom{k-1}{s-2} (k - t - l + s)(N + k + l - t - s). \end{aligned}$$

And the last two summands

$$\delta\mu_k(\delta\mu_l(e_N, e_s), e_t) + \delta\mu_l(\delta\mu_k(e_t, e_N), e_s)$$

are treated similarly.

LEMMA 5.4. The quantity  $a(k, l, s, t)$  from Lemma 5.3 is equal to 0 if  $s + t - k - l > 1$ .

PROOF. If  $s + t > k + l + 1$ , then all the summands comprising  $a(k, l, s, t)$  are equal to 0 separately (see Proof of Lemma 5.3).

Define the cochains  $\lambda_{k,l} \in C_{(k+l)}^2(L_1; W)$  by the formula

$$\lambda_{k,l}(e_s, e_t) = a(k, l, s, t)e_{s+t-k-l}.$$

Lemma 5.4 shows that the cochain  $\lambda_{k,l}$  takes values in the subspace of  $W$  spanned by  $e_i$  with  $i \leq 1$ .

DEFINITION. We say that two cochains  $\alpha, \beta \in C^q(L_1; W)$  are *commensurable*,  $\alpha \sim \beta$ , if the equality

$$\alpha(e_{i_1}, \dots, e_{i_q}) = \beta(e_{i_1}, \dots, e_{i_q})$$

holds for all but finitely many sets  $i_1, \dots, i_q$ . In particular,

$$\lambda_{k,l} \sim 0 \text{ for all } k, l.$$

LEMMA 5.5.  $\delta\lambda_{k,l} \sim \delta_{k,l}$  for all  $k, l$ .

PROOF. Lemma 5.3 implies that for any  $s, t$

$$\delta\lambda_{k,l}(e_s, e_t, e_N) = \delta_{k,l}(e_s, e_t, e_N)$$

when  $N$  is large. It is also true that if  $k \leq l, l+1 < s < t < u$ , then

$$\delta\lambda_{k,l}(e_s, e_t, e_u) = \delta_{k,l}(e_s, e_t, e_u) = 0.$$

The following is the most important property of the commensurability relation.

LEMMA 5.6. If  $\alpha, \beta \in C_{(k)}^q(L_1; W)$ ,  $\alpha \sim \beta$ , and  $\delta\alpha = \delta\beta = 0$ , then  $\alpha = \beta$ .

PROOF. Let  $\gamma = \alpha - \beta$ , let  $\gamma(e_{i_1}, \dots, e_{i_q}) \neq 0$ , and let  $\gamma(e_{j_1}, \dots, e_{j_q}) = 0$  if  $j_q \geq N$ . Take  $j \geq N, j \neq i_1 + \dots + i_q - k$ ; then

$$\delta\gamma(e_{i_1}, \dots, e_{i_q}, e_j) = \pm[e_j, \gamma(e_{i_1}, \dots, e_{i_q})] \neq 0$$

which contradicts to  $\delta\gamma = 0$ .

With the exception of  $\lambda_{2,2} \in C_{(4)}^2(L_1; L_1)$ , the cochains  $\lambda_{k,l}$  are not contained in  $C^2(L_1; L_1)$ . But in some cases we can force them into  $C^2(L_1; L_1)$  by adding an appropriate multiple of  $\delta\mu_{k+l}$  (which would not affect the coboundaries).

LEMMA 5.7.

$$\begin{aligned} \lambda_{2,2} &\in C_{(4)}^2(L_1; L_1) \\ 13\lambda_{2,3} - \delta\mu_5 &\in C_{(5)}^2(L_1; L_1) \\ 7\lambda_{2,4} + 2\delta\mu_6 &\in C_{(6)}^2(L_1; L_1) \\ 21\lambda_{3,3} - 10\delta\mu_6 &\in C_{(6)}^2(L_1; L_1) \\ 65\lambda_{2,5} + 119\lambda_{3,4} - 2\delta\mu_7 &\in C_{(7)}^2(L_1; L_1) \\ 151\lambda_{2,6} - 105\lambda_{3,5} + 60\delta\mu_8 &\in C_{(8)}^2(L_1; L_1) \\ 20\lambda_{2,6} + 7\lambda_{4,4} + 6\delta\mu_8 &\in C_{(8)}^2(L_1; L_1) \end{aligned}$$

PROOF: direct calculation. Each of the cochains  $\lambda_{k,l}, \delta\mu_m$  has finitely many values outside  $L_1$ , and it is very easy to compute them all. For example,

$$\lambda_{2,3}(e_2, e_3) = 2e_0, \quad \delta\mu_5(e_2, e_3) = 26e_0,$$

all other values of these cochains are in  $L_1$ . Hence  $13\lambda_{2,3} - \delta\mu_5 \in C^2(L_1; L_1)$ . Similarly

$$\begin{aligned}\lambda_{2,4}(e_2, e_3) &= -12e_{-1}, & \lambda_{3,3}(e_2, e_3) &= 20e_{-1}, & \delta\mu_6(e_2, e_3) &= 42e_{-1}, \\ \lambda_{2,4}(e_2, e_4) &= 12e_0, & \lambda_{3,3}(e_2, e_4) &= -20e_0, & \delta\mu_6(e_2, e_4) &= -42e_0,\end{aligned}$$

all other values of these cochains are in  $L_1$ . Hence  $7\lambda_{2,4} + 2\delta\mu_6 \in C^2(L_1; L_1)$ ,  $21\lambda_{3,3} - 10\delta\mu_6 \in C^2(L_1; L_1)$ . In the remaining cases the computations are similar (though longer).

LEMMA 5.8. *If a linear combination of  $\delta_{2,5}$  and  $\delta_{3,4}$  is commensurable with a coboundary, then this linear combination is a multiple of  $65\delta_{2,5} + 119\delta_{3,4}$ . If a linear combination of  $\delta_{2,6}$ ,  $\delta_{3,5}$  and  $\delta_{4,4}$  is commensurable with a coboundary, then this linear combination is a linear combination of  $151\delta_{2,6} - 105\delta_{3,5}$  and  $20\delta_{2,6} + 7\delta_{4,4}$ . In other words, if  $a\delta_{2,6} + b\delta_{3,5} + c\delta_{4,4}$  is commensurable with a coboundary, then  $105a + 151b - 300c = 0$ .*

PROOF. Lemma 5.7 implies that  $65\delta_{2,5} + 119\delta_{3,4}$ ,  $151\delta_{2,6} - 105\delta_{3,5}$  and  $20\delta_{2,6} + 7\delta_{4,4}$  are commensurable with coboundaries, and Lemma 5.6 and Theorem 3.2 imply that  $\delta_{3,4}$  and  $\delta_{4,4}$  are not commensurable with coboundaries. Lemma follows.

In conclusion we remark, that adding linear combinations of  $\lambda$ 's to  $\delta\mu$ 's does not affect essentially the Lie products. For example,

$$[13\lambda_{2,3} - \delta\mu_5, 7\lambda_{2,4} + 2\delta\mu_6] \sim -2\delta_{5,6},$$

and so on.

## 6. Proof of Theorem 1.1

We consider formal one-parameter deformations

$$[g, h]_t = [g, h] + \sum_{k \geq 1} \alpha_k(g, h)t^k$$

of  $L_1$ ; recall that the relation (1) from Section 2 should hold for all the cochains  $\alpha_k \in C^2(L_1; L_1)$ .

The general theory of Section 2 combined with Theorem 3.1 and Corollary to Lemma 5.2 implies that the following construction gives all possible formal deformations of  $L_1$  up to the equivalence described in Section 2.

Suppose that  $\alpha_1, \dots, \alpha_{k-1}$  has already been defined. Fix an arbitrary cochain  $\alpha_k^0 \in C^2(L_1; L_1)$  with

$$\delta\alpha_k^0 = -\frac{1}{2} \sum_{i=1}^{k-1} [\alpha_i, \alpha_{k-i}], \quad (8)$$

then choose three complex numbers  $c_{k2}, c_{k3}, c_{k4}$ , and put inductively

$$\alpha_k = \alpha_k^0 + c_{k2}\delta\mu_2 + c_{k3}\delta\mu_3 + c_{k4}\delta\mu_4.$$

Notice that we do not vary the cochains  $\alpha_k^0$ , their choice is arbitrary, but after having been chosen, they are fixed. On the contrary, the numbers  $c_{ij}$  are varied, but the existence of  $\alpha_k^0$

imposes a condition on all  $c_{ij}$  with  $i < k$ . We will always choose  $\alpha_k^0$  under the condition: if the right hand side of (8) lies in  $\oplus_{q \leq m} C_{(q)}^3(L_1; L_1)$ , then  $\alpha_k^0 \in \oplus_{q \leq m} C_{(q)}^2(L_1; L_1)$ .

Below we use the notation:  $\oplus_{q \leq m} C_{(q)}^r(L_1; L_1) = C_{(\leq m)}^r(L_1; L_1)$ .

Our goal is to show that the conditions imposed on  $c_{ij}$  by the solvability of the equations (8) leave only a few possibilities for the deformation.

LEMMA 6.1.  $c_{13} = c_{14} = 0$ .

This follows from Theorem 3.2. The argumentation is well known (see [Fu]), but we give it here for the completeness sake. The degree 8 component of  $[\alpha_1, \alpha_1]$  is  $c_{14}^2[\delta\mu_4, \delta\mu_4]$ , and since it belongs to  $\text{Im } \delta$  and  $[c, c] \neq 0$ , we must have  $c_{14} = 0$ . Let  $\beta$  be the component of degree 6 of  $\alpha_2$ ; then  $\delta\beta = -\frac{1}{2}c_{13}^2[\delta\mu_3, \delta\mu_3]$ . Since the degree 7 component of  $\alpha_2$  is 0, the degree 9 component of  $[\alpha_1, \alpha_2]$  is  $c_{13}[\delta\mu_3, \beta]$ , and since  $\langle b, b, b \rangle \neq 0$ , the latter belongs to  $\text{Im } \delta$  only if  $c_{13} = 0$ .

It remains to consider two cases:  $c_{12} \neq 0$  and  $c_{12} = 0$ . The parameter change  $u = c_{12}t$  would reduce the case  $c_{12} \neq 0$  to the case  $c_{12} = 1$ . Consider this case.

LEMMA 6.2. *Let  $\alpha_1 = \delta\mu_2$ . Then  $\alpha_k \in C_{(\leq 2k)}^2(L_1; L_1)$ .*

PROOF. Let  $k \geq 2$  and suppose that  $\alpha_l \in C_{(\leq 2l)}^2(L_1; L_1)$  for  $l < k$ . Then the right hand side of (8) lies in  $C_{(\leq 2k)}^3(L_1; L_1)$ , and hence  $\alpha_k^0 \in C_{(\leq 2k)}^2(L_1; L_1)$ . But since  $k \geq 2$ , then  $\delta\mu_2, \delta\mu_3, \delta\mu_4 \in C_{(\leq 2k)}^2(L_1; L_1)$ . Thus  $\alpha_k \in C_{(\leq 2k)}^2(L_1; L_1)$ .

Consider now the ‘‘homogeneous’’ case.

LEMMA 6.3. *There exist (up to an equivalence) at most two deformations with  $\alpha_1 = \delta\mu_2$  and  $\alpha_k \in C_{(2k)}^2(L_1; L_1)$ .*

PROOF. First,  $\delta\alpha_2 = -\frac{1}{2}[\alpha_1, \alpha_1] = -\frac{1}{2}[\delta\mu_2, \delta\mu_2] = -\frac{1}{2}\delta_{2,2}$ . Since  $\delta_{2,2} = \delta\lambda_{2,2}$  (Lemmas 5.5 and 5.6), we can take  $\alpha_2^0 = -\frac{1}{2}\lambda_{2,2}$ , and then we have  $\alpha_2 = -\frac{1}{2}\lambda_{2,2} + x\delta\mu_4$ . Notice, that in the whole construction  $x$  is the only constant to choose: if  $k \geq 3$ , then  $\deg \alpha_k > 4$ , and  $\alpha_k$  does not involve either of  $\delta\mu_2, \delta\mu_3, \delta\mu_4$ . Next we have  $\delta\alpha_3 = -[\alpha_1, \alpha_2] = -\left[\delta\mu_2, -\frac{1}{2}\lambda_{2,2} + x\delta\mu_4\right] \sim -x\delta_{2,4}$ , and we may put  $\alpha_3 = -x\lambda_{2,4} - \frac{2}{7}x\delta\mu_6$  (Lemmas 5.7 and 5.6). Further we have

$$\begin{aligned} \delta\alpha_4 &= -[\alpha_1, \alpha_3] - \frac{1}{2}[\alpha_2, \alpha_2] \\ &= -\left[\delta\mu_2, -x\lambda_{2,4} - \frac{2}{7}x\delta\mu_6\right] - \frac{1}{2}\left[-\frac{1}{2}\lambda_{2,2} + x\delta\mu_4, -\frac{1}{2}\lambda_{2,2} + x\delta\mu_4\right] \\ &\sim \frac{2}{7}x\delta_{2,6} - \frac{1}{2}x^2\delta_{4,4}. \end{aligned}$$

According to Lemma 5.8 the latter should be a linear combination of  $151\delta_{2,6} - 105\delta_{3,5}$  and

$20\delta_{2,6} + 7\delta_{4,4}$ , that is

$$105 \cdot \frac{2}{7}x - 300 \cdot \frac{1}{2}x^2 = 210x(5x - 1) = 0.$$

Hence  $x = 0$  or  $-\frac{1}{5}$ , which proves the Lemma.

Actually, the two deformations do exist: the deformation  $[\cdot, \cdot]_t^3$  and the deformation constructed in Section 4 from  $[\cdot, \cdot]_t^1$ . We do not check it now: it will be easier to see later.

LEMMA 6.4. *Up to a parameter change, there are no deformations with  $\alpha_1 = \delta\mu_2$ , other than those in Lemma 6.3.*

PROOF. First we notice that using an appropriate parameter change, we may kill  $\delta\mu_2$  in  $\alpha_k = \alpha_k^0 + c_{k2}\delta\mu_2 + c_{k3}\delta\mu_3 + c_{k4}\delta\mu_4$  for all  $k \geq 2$ . Hence we can write:

$$\begin{aligned}\alpha_1 &= \delta\mu_2, \\ \alpha_2 &= -\frac{1}{2}\lambda_{2,2} + y_2\delta\mu_3 + x_2\delta\mu_4, \\ \delta\alpha_3 &\sim -y_2\delta_{2,3} - x_2\delta_{2,4} \\ \alpha_3 &= -y_2\lambda_{2,3} + \frac{1}{13}y_2\delta\mu_5 - x_2\lambda_{2,4} - \frac{2}{7}x_2\delta\mu_6 + y_3\delta\mu_3 + x_3\delta\mu_4, \\ \delta\alpha_4 &\sim -\frac{1}{2}y_2^2\delta_{3,3} - x_2y_2\delta_{3,4} - \frac{1}{2}x_2^2\delta_{4,4} - \frac{2}{13}y_2\delta_{2,5} + \frac{2}{7}x_2\delta_{2,6} - y_3\delta_{2,3} - x_3\delta_{2,4}.\end{aligned}$$

As in the previous proof,  $-\frac{1}{2}x_2^2\delta_{4,4} + \frac{2}{7}x_2\delta_{2,6}$  being in  $\text{Im } \delta$  implies (by Lemma 5.8) that either  $x_2 = 0$  or  $x_2 = -\frac{1}{5}$ . Consider these two cases.

*Case 1:  $x_2 = 0$ .* In this case

$$\delta\alpha_4 \sim -\frac{1}{2}y_2^2\delta_{3,3} - \frac{2}{13}y_2\delta_{2,5} - y_3\delta_{2,3} - x_3\delta_{2,4},$$

and since  $-\frac{2}{13}y_2\delta_{2,5} \in \text{Im } \delta$ , Lemma 5.8 implies that  $y_2 = 0$ . Thus  $\alpha_1 = \delta\mu_2, \alpha_2 = \frac{1}{2}\lambda_{2,2}, \alpha_3 = y_3\delta\mu_3 + x_3\delta\mu_4$ , and hence

$$\begin{aligned}\alpha_4 &= -y_3\lambda_{2,3} + \frac{1}{13}y_3\delta\mu_5 - x_3\lambda_{2,4} - \frac{2}{7}x_3\delta\mu_6 + y_4\delta\mu_3 + x_4\delta\mu_4, \\ \delta\alpha_5 &\sim -\frac{1}{13}y_3\delta_{2,5} + \frac{2}{7}x_3\delta_{2,6} - y_4\delta_{2,3} - x_4\delta_{2,4},\end{aligned}$$

which implies, in virtue of Lemma 5.8, that  $y_3 = 0, x_3 = 0$ . Similarly, for  $k > 4$  we have inductively

$$\begin{aligned}\alpha_k &= -y_{k-2}\lambda_{2,3} + \frac{1}{13}y_{k-1}\delta\mu_5 - x_{k-1}\lambda_{2,4} - \frac{2}{7}x_{k-1}\delta\mu_6 + y_k\delta\mu_3 + x_k\delta\mu_4, \\ \delta\alpha_{k+1} &\sim -\frac{1}{13}y_{k-1}\delta_{2,5} + \frac{2}{7}x_{k-2}\delta_{2,6} - y_k\delta_{2,3} - x_k\delta_{2,4},\end{aligned}$$

which implies that  $y_{k-2} = 0, x_{k-2} = 0$ .

*Case 2:*  $x_2 = -\frac{1}{5}$ . In the calculations below we skip for all  $\alpha$ 's the terms of degree  $\leq 8$  and for all  $\delta\alpha$ 's the terms of degree  $\leq 9$ . We have:

$$\begin{aligned}\alpha_1 &= \delta\mu_2, \\ \alpha_2 &= -\frac{1}{2}\lambda_{2,2} - \frac{1}{5}\delta\mu_4, \\ \alpha_3 &= -\frac{1}{5}\lambda_{2,4} + \frac{2}{35}\delta\mu_6 + y_3\delta\mu_3 + x_3\delta\mu_4, \\ \delta\alpha_4 &\sim -\frac{1}{50}\delta_{4,4} + \frac{2}{35}\delta_{2,6} - y_3\delta_{2,3} - x_3\delta_{2,4}, \\ \alpha_4 &= \dots - y_3\lambda_{2,3} + \frac{1}{13}y_3\delta\mu_5 - x_3\lambda_{2,4} - \frac{2}{7}x_3\delta\mu_6 + y_4\delta\mu_3 + x_4\delta\mu_4, \\ \delta\alpha_5 &\sim \dots - \frac{1}{13}y_3\delta_{2,5} + \frac{2}{7}x_3\delta_{2,6} - y_4\delta_{2,3} - x_4\delta_{2,4} + \frac{1}{5}y_3\delta_{3,4} + \frac{1}{5}x_3\delta_{4,4}.\end{aligned}$$

The latter implies that  $x_3 = 0, y_3 = 0$ : otherwise  $-\frac{1}{13}y_3\delta_{2,5}$  and  $\frac{2}{7}x_3\delta_{2,6} + \frac{1}{5}x_3\delta_{4,4}$  belonging to  $\text{Im } \delta$  contradicts to Lemma 5.8. Further, for  $k > 4$  we have inductively

$$\begin{aligned}\alpha_k &= \dots - y_{k-1}\lambda_{2,3} + \frac{1}{13}y_{k-1}\delta\mu_5 - x_{k-1}\lambda_{2,4} - \frac{2}{7}x_{k-1}\delta\mu_6 + y_k\delta\mu_3 + x_k\delta\mu_4, \\ \delta\alpha_{k+1} &\sim \dots - \frac{1}{13}y_{k-1}\delta_{2,5} + \frac{2}{7}x_{k-1}\delta_{2,6} - y_k\delta_{2,3} - x_k\delta_{2,4} + \frac{1}{10}y_{k-1}\delta_{3,4} + \frac{1}{10}x_{k-1}\delta_{4,4},\end{aligned}$$

which implies  $x_{k-1} = 0, y_{k-1} = 0$ .

Thus in both cases we have no deformations different from those of Lemma 6.3, and Lemma 6.4 is proved.

The case  $\alpha_1 \neq 0$  is over; consider the case  $\alpha_1 = 0$ . In this case we are interested only in *singular* deformations (See Section 4).

First of all,  $\delta\alpha_2 = \delta\alpha_3 = 0$ , and we can assume that

$$\begin{aligned}\alpha_2 &= z_2\delta\mu_2 + y_2\delta\mu_3 + x_2\delta\mu_4, \\ \alpha_3 &= z_3\delta\mu_2 + y_3\delta\mu_3 + x_3\delta\mu_4.\end{aligned}$$

The degree 8 component of  $\delta\alpha_4$  should be  $-\frac{1}{2}x_2^2[\delta\mu_4, \delta\mu_4]$ , which is possible only if  $x_2 = 0$  (Theorem 3.2). Now, the degree 9 component of  $\delta\alpha_6$  will be a cocycle representing the

Massey-Lie cube of the cohomology class of  $\delta\mu_3$  times some non-zero factor times  $y_2^2$ , which implies that  $y_2 = 0$  (again Theorem 3.2). After it, applying Theorem 3.2 to the degree 8 component of  $\alpha_6$  we find that  $x_3 = 0$ .

Thus  $\alpha_2 = z_2\delta\mu_2, \alpha_3 = z_3\delta\mu_2 + y_3\delta\mu_3$ . As before, two essentially different cases are  $z_2 = 0$  and  $z_2 \neq 0$ . Suppose that  $z_2 \neq 0$ . Then using an appropriate parameter change, we can make  $z_2 = 1, z_3 = 0$ . Thus we have  $\alpha_2 = \delta\mu_2, \alpha_3 = y\delta\mu_3$  (here  $y = y_3$ ); also we have  $\alpha_k \in C_{(\leq k)}^2(L_1; L_1)$ , which is similar to Lemma 6.2. As before, we begin with the ‘‘homogeneous’’ case.

LEMMA 6.5. *There exist (up to an equivalence) at most one singular deformation with  $\alpha_1 = 0, \alpha_2 = \delta\mu_2$  and  $\alpha_k \in C_{(k)}^2(L_1; L_1)$ .*

PROOF. We have  $\alpha_1 = 0, \alpha_2 = \delta\mu_2, \alpha_3 = y\delta\mu_3$ ,

$$\delta\alpha_4 = -\frac{1}{2}\delta_{2,2}, \alpha_4 = -\frac{1}{2}\lambda_{2,2} + x\delta\mu_4 \sim x\delta\mu_4,$$

and  $y$  and  $x$  are the only parameters of which our deformation can depend. Next we have

$$\begin{aligned} \delta\alpha_5 &= -y\delta_{2,3}, \alpha_5 = -\lambda_{2,3} + \frac{1}{13}\delta\mu_5 \sim \frac{1}{13}y\delta\mu_5, \\ \delta\alpha_6 &\sim -x\delta_{2,4} - \frac{1}{2}y^2\delta_{3,3}, \alpha_6 \sim \left(\frac{5}{21}y^2 - \frac{2}{7}x\right)\delta\mu_6, \\ \delta\alpha_7 &\sim -\frac{1}{13}y\delta_{2,5} - xy\delta_{3,4}. \end{aligned}$$

According to Lemma 5.8, the latter implies the relation

$$\frac{1}{13}y : xy = 65 : 119,$$

which may mean either  $y = 0$  or  $x = \frac{119}{13 \cdot 65}$ . But the first would imply  $\alpha_3 = 0$  and successively  $\alpha_5 = 0, \alpha_7 = 0, \dots$ , which means that the deformation is actually non-singular. This leaves us with the second possibility. Next we have:

$$\delta\alpha_8 \sim \left(-\frac{5}{21}y^2 + \frac{2}{7}x\right)\delta_{2,6} - \frac{1}{13}y^2\delta_{3,5} - \frac{1}{2}x^2\delta_{3,4},$$

and we again apply Lemma 5.8 to get the second relation:

$$105 \left(\frac{5}{21}y^2 - \frac{2}{7}x\right) + 151 \cdot \frac{1}{13}y^2 - 300 \cdot \frac{1}{2}x^2 = 0,$$

which determines  $y$  up to the sign:  $y^2 = \frac{2 \cdot 6^3}{13^3}$ . The sign of  $y$  is irrelevant, for it may be changed by the parameter change  $t \mapsto -t$ . This completes the proof of Lemma 6.5.

LEMMA 6.6. *There are no more one-parameter formal deformations of  $L_1$ . In other words, any one-parameter formal deformation of  $L_1$  is reduced by an equivalence and a parameter change to one of the deformations of Lemmas 6.3 and 6.5.*

PROOF. First we consider the case of a non-homogeneous deformation with  $\alpha_1 = 0, \alpha_2 = \delta\mu_2, \alpha_3 = y\delta\mu_3, \alpha_4 = y_4\delta\mu_3 + x\delta\mu_4$ . In this case  $\alpha_5 \sim \frac{1}{13}y\delta\mu_5 + x_5\delta\mu_4 + y_5\delta\mu_3$ , and the following formulas give the components of the degree less by one than the maximal one:

$$\begin{aligned}\alpha_6 &\sim \dots - y_4\delta_{2,3} \dots, \\ \delta\alpha_6 &\sim \dots + \frac{1}{13}\delta\mu_5 \dots, \\ \delta\alpha_7 &\sim \dots - x_5\delta_{2,4} - yy_4\delta_{3,3} \dots, \\ \alpha_7 &\sim \dots \left( -\frac{2}{7}x_5 + \frac{10}{21}yy_4 \right) \delta\mu_6 \dots, \\ \delta\alpha_8 &\sim \dots - \frac{1}{13}y_4\delta_{2,5} - xy_4\delta_{3,4} - x_5y\delta_{3,4} \dots, \\ \delta\alpha_9 &\sim \dots \left( \frac{2}{7}x_5 - \frac{10}{21}yy_4 \right) \delta_{2,6} - \frac{2}{13}yy_4\delta_{3,5} - xx_5\delta_{4,4}.\end{aligned}$$

The last two equalities, combined with Lemma 5.8 give two linear relations between  $y_4$  and  $x_5$ :

$$\begin{aligned}\left( \frac{119}{13} - 65x \right) y_4 + 65yx_5 &= 0, \\ \frac{8 \cdot 119}{13}yy_4 - (30 + 300x)x_5 &= 0.\end{aligned}$$

The calculations made above leave us with only three possibilities for  $x$  and  $y$ :

$$y = x = 0; \quad y = 0, x = -\frac{1}{5}; \quad y = \frac{12}{13}\sqrt{\frac{3}{13}}, x = \frac{119}{13 \cdot 65};$$

in each of these cases the determinant of the above system is not equal to zero, and hence  $y_4 = x_5 = 0$ . Then we consider the components of degree 7 and 8 for  $\delta\alpha_9$  and  $\delta\alpha_{10}$ , and we obtain precisely the same system for  $y_5$  and  $x_6$ ; hence they are also equal to zero. Proceeding in this way we prove that our deformation is actually homogeneous and is covered by Lemma 6.5.

And the last case is  $\alpha_1 = 0, \alpha_2 = 0$ . Suppose that  $\alpha_3 = 0, \dots, \alpha_{p-1} = 0, \alpha_p \neq 0$  and that the deformation is not equivalent to a deformation with  $\alpha_1 = 0, \dots, \alpha_p = 0$ . In this case  $\delta\alpha_i = 0$  and  $\alpha_i - z_i\delta\mu_2 + y_i\delta\mu_3 + x_i\delta\mu_4$  for  $p \leq i < 2p - 1$ . If  $x_i = 0$  for  $i < p + r$  for some  $r, 0 \leq r < p$ , then the degree 8 component first appears in  $\delta\alpha_{2p+2r}$  and is equal to  $-\frac{1}{2}x_{p+r}^2\delta_{4,4}$ , which implies that  $x_{p+r} = 0$ . Hence all  $x_i = 0$ , and  $\alpha_i = z_i\delta\mu_2 + y_i\delta\mu_3$  for  $p \leq i < 2p - 1$ . Now, the degree 9 component first appear in  $\delta\alpha_{3p}$  and is  $\sim \frac{5}{21}y_p^3\delta_{3,6}$ , which

implies in virtue of Lemma 5.8, that  $y_p = 0$ . This equality shifts the first appearance of the degree 9 component to  $\delta\alpha_{3p+3}$ , and Lemma 5.8 yields  $y_{p+1} = 0$ . Proceeding in the same way, we prove that  $y_i = 0$  for  $i < \frac{3}{2}p$ . A parameter change makes  $\alpha_p = \delta\mu_2$  and kills

$\delta\mu_2$  term in all  $\alpha_i$ 's with  $i > p$ ; in particular, it makes  $\alpha_i = 0$  for  $p < i < \frac{3}{2}p$ .

In the remaining part of the proof the cases of even and odd  $p$  are slightly different. If  $p = 2q$ , then we have  $\alpha_{2q} = \delta\mu_2$ ,  $\alpha_{3q} = y\delta\mu_3$ ,  $\alpha_i = y_i\delta\mu_3$  if  $3q < i < 4q$ ,  $\alpha_{4q} = y_{4q}\delta\mu_3 + x\delta\mu_4$ ,  $y_{4q+1} = y_{4q+1}\delta\mu_3 + x_{4q+1}\delta\mu_4$ . Then we consider the degree 7 component of  $\delta\alpha_{7q}$  and the degree 8 component of  $\delta\alpha_{8q}$ , and, using Lemma 5.8, obtain the same system of equations for  $y$  and  $x$  as in the proof of Lemma 6.5; we already know all the solutions of this system. Next we consider the degree 7 component of  $\delta\alpha_{7q+1}$  and the degree 8 component of  $\delta\alpha_{8q+1}$ , and obtain for  $y_{3q+1}$  and  $x_{4q+1}$  the same equations as for  $y_4$  and  $x_5$  in the first part of this proof; these equations imply that  $y_{3q+1} = x_{4q+1} = 0$ . Proceeding in the same way, we annihilate all terms with  $\delta\mu_3$  and  $\delta\mu_4$ , with the exception of  $y\delta\mu_3$  in  $\alpha_{3q}$  and  $x\delta\mu_4$  in  $\alpha_{4q}$ , and finally we get  $\alpha_i = 0$  if  $i$  is not divisible by  $q$ . The parameter change  $t^q \mapsto t$  makes our deformation the deformation of Lemma 6.5 if  $y \neq 0$  and one of the deformations of Lemma 6.3 if  $y = 0$ .

If  $p$  is odd, then we proceed as above and get  $\alpha_p = \delta\mu_2$ ,  $\alpha_i = 0$  if  $p+1 \leq i < \frac{3p+1}{2}$ ,  $\alpha_i = y_i\delta\mu_3$  if  $\frac{3p+1}{2} \leq i < 2p$ ,  $\alpha_{2p} = y_{2p}\delta\mu_3 + x\delta\mu_4$ ,  $\alpha_i = y_i\delta\mu_3 + x_i\delta\mu_4$  if  $2p < i < \frac{5p+1}{2}$ ,  $\alpha_i = \frac{1}{13}y_{i-p}\delta\mu_5 + y_i\delta\mu_3 + x_i\delta\mu_4$  if  $\frac{1}{13} \leq i < 3p$ ,  $\alpha_{3p} = -\frac{2}{7}x\delta\mu_6 + \frac{1}{13}y_{2p}\delta\mu_5 + y_i\delta\mu_3 + x_i\delta\mu_4$ ,  $\alpha_{3p+1} = -\frac{2}{7}x_{2p+1}\delta\mu_6 + \frac{1}{13}y_{2p+1}\delta\mu_5$ . The degree 8 component of  $\delta\alpha_{4p}$  is  $\sim \frac{2}{7}x\delta_{2,6} - \frac{1}{2}x^2\delta_{4,4}$ , and Lemma 5.8 implies  $x = 0$  or  $-\frac{1}{5}$  (compare the proof of Lemma 6.3). Further the degree 7 component of  $\delta\alpha_{\frac{7p+1}{2}}$  is  $\sim -\frac{1}{13}y_{\frac{3p+1}{2}}\delta_{2,5} - \frac{1}{2}xy_{\frac{3p+1}{2}}\delta_{3,4}$ . For the both values of  $x$  Lemma 5.8 implies  $y_{\frac{3p+1}{2}} = 0$ . Further we consider the degree 8 component of  $\delta\alpha_{8p+1}$ , the degree 7 component of  $\delta\alpha_{\frac{7p+3}{2}}$ , the degree 8 component of  $\delta\alpha_{8p+2}$ , and so on, and Lemma 5.8. implies that  $x_{2p+1} = 0$ ,  $y_{\frac{3p+3}{2}} = 0$ ,  $x_{2p+2} = 0$ , and so on. Finally we get  $\alpha_i = 0$  if  $i$  is not divisible by  $p$ . The parameter change  $t^p \mapsto t$  makes our deformation the deformation of Lemma 6.3.

This completes the proof of Lemma 6.6.

Lemmas 6.3 – 6.6 show that there are at most three different formal one parameter deformations of  $L_1$ , and we already know, that three such deformations exist. This ends the proof of Theorem 1.1.

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