

# Cohomology of the infinite-dimensional Lie algebra $L_1$ with nontrivial coefficients

B. L. Feigin and Alice Fialowski

**1.** Let  $\mathcal{L}$  be the Lie algebra of vector fields on the circle of the form  $f(\phi)d/d(\phi)$  where  $f$  is a function having finite Fourier expansion ( $\phi$  is the angular parameter on the circle). In  $\mathcal{L}$  we can choose the basis

$$e_n = x^{n+1} \frac{d}{dx}, \quad n \in \mathbf{Z},$$

with the bracket  $[e_i, e_j] = (j - i)e_{i+j}$ . The Lie algebra  $\mathcal{L}$  is naturally graded, the degree of  $e_i$  being  $i$ . The most natural modules over  $\mathcal{L}$  are the so-called tensor field modules. A tensor field on the circle is of the form  $g(\phi)(d/d\phi)^\lambda$ . A vector field acts on this by infinitesimally changing the coordinate  $\phi$ , where  $g(\phi)$  is a section of some line bundle on the circle  $S^1$  with a flat connection. In the space of tensor fields we choose a basis  $f_i$ ,  $i \in \mathbf{Z}$ , such that  $e_i(f_j) = (-\lambda(i + 1) + \mu + j) \cdot f_{i+j}$ . Here  $\lambda, \mu \in \mathbf{C}$  are the invariants characterizing the module, i.e., the power of  $d/d\phi$  and the logarithm of the monodromy of the flat connection. We denote such a module by  $\mathcal{F}_{\lambda, \mu}$  (see [3]).

Denote by  $L_1$  the subalgebra of  $\mathcal{L}$  with basis  $(e_1, e_2, e_3, \dots)$ . It is easy to see that  $L_1$  is isomorphic to the Lie algebra of vector fields on the line, with polynomial coefficients, having a two-fold zero at the origin. The strategy of the cohomology computation for  $L_1$  with coefficients in the adjoint module is the following: we first compute the cohomology of  $L_1$  with coefficients in  $\mathcal{F}_{\lambda, \mu}$ , and then remark that the adjoint representation of  $L_1$  is a submodule of such an  $\mathcal{F}_{\lambda, \mu}$ . After this the spaces  $H^i(L_1, L_1)$  can easily be determined. The computations of  $H^1(L_1, L_1)$  and  $H^2(L_1, L_1)$  are contained in [3]. Deformations of  $L_1$  are studied in [5]. In this paper we shall describe a more general method for the computation of  $H^*(L_1, \mathcal{F}_{\lambda, \mu})$ .

It will be more convenient for us to deal with homology instead of cohomology. It is easy to see that  $H^*(L_1, \mathcal{F}_{\lambda, \mu})$  is dual to  $H_*(L_1, \mathcal{F}_{-1-\lambda, -\mu})$ . Then, using the fact that  $L_1^*$  is the factor of some  $\mathcal{F}_{\lambda, \mu}$ , we can compute  $H_i(L_1, L_1^*)$ . Notice that for almost every  $(\lambda, \mu)$  the module  $\mathcal{F}_{\lambda, \mu}$  is an irreducible representation of  $\mathcal{L}$ , and  $L_1$  is the maximal nilpotent subalgebra in  $\mathcal{L}$ . That means that the problem of determining  $H_*(L_1, \mathcal{F}_{\lambda, \mu})$  is analogous to that of determining the cohomology of the maximal nilpotent subalgebra of a complex semisimple Lie

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algebra with coefficients in an irreducible representation. We call theorems of this type Bott–Kostant theorems [1], [8]. (Notice that the representations  $\mathcal{F}_{\lambda,\mu}$  are reminiscent of the Harish–Chandra modules rather than of representations in the category  $\mathcal{O}$ ; see [1].) A method analogous to that used in this paper was applied to the current algebras in [2].

**2.** We first recall some pertinent facts. Introduce another Lie algebra  $L(0, 1)$  which consists of polynomial vector fields on the line with zeros at the points  $0, 1$ . In  $L(0, 1)$  we can choose the basis

$$\bar{e}_i = x^i(x-1)\frac{d}{dx}, \quad i \in \mathbf{Z},$$

such that  $[\bar{e}_i, \bar{e}_j] = (j-i)(\bar{e}_{i+j} - \bar{e}_{i+j-1})$ . There exists a family of one-dimensional  $L(0, 1)$ -modules  $M(\alpha, \beta)$ ;  $\bar{e}_i$  acts on  $M(\alpha, \beta)$  as multiplication by  $\beta - \alpha$  if  $i = 1$  and by  $\beta$  if  $i > 1$ . The significance of  $M(\alpha, \beta)$  is the following. The commutator of  $L(0, 1)$  consists of vector fields on the line, which, together with their first derivative, vanish at  $0, 1$ . Therefore the character of  $M(\alpha, \beta)$  takes the value  $\alpha f'(0) + \beta f'(1)$  on the vector field  $f(x)d/dx \in L(0, 1)$ . Observe that  $M(\alpha, \beta) = M(\alpha, 0) \otimes M(0, \beta)$ , where  $M(\alpha, 0)$  is the module on which the vector field  $f(x)d/dx$  acts by multiplication on  $f'(0)$ , and  $M(0, \beta)$  is the one on which it acts by multiplication on  $f'(1)$ . Recall that  $H_i(L_1)$  is two-dimensional for  $i > 0$ , and that the weight of the two homogeneous basis elements of  $H_i(L_1)$  are

$$\frac{3i^2 + i}{2} \quad \text{and} \quad \frac{3i^2 - i}{2}.$$

This result is proved in [7]. Further, the cohomology of  $L(0, 1)$  is also two-dimensional in every positive dimension. In [5] it is proved that  $L(0, 1)$  is a deformation of  $L_1$ . Namely, there exists a Lie algebra family  $L(0, t)$  with the basis  $\bar{e}_i$  and the bracket  $[\bar{e}_i, \bar{e}_j] = (j-i)(\bar{e}_{i+j} + t\bar{e}_{i+j-1})$ . It is clear that  $L(0, 0) \cong L_1$ . In [4] it is proved that the cohomology spaces for  $t = 0$  and  $t \neq 0$  are isomorphic as graded vector spaces (although the multiplication and the Massey operations in them are different).

We now describe the algebra structure of  $H^*(L(0, 1))$ . It turns out that  $H^*(L(0, 1))$  is free and is generated by three generators, two of degree 1 and one of degree 2. The one-dimensional generators correspond to the cochains  $f(x)d/dx \rightarrow f'(0)$  and  $f(x)d/dx \rightarrow f'(1)$ . The two-dimensional generator corresponds to the cochain

$$f(x)\frac{d}{dx} \wedge g(x)\frac{d}{dx} \rightarrow (f'(x)g''(x) - f''(x)g'(x))dx.$$

The cohomology space  $H^*(L(0, 1))$  can be computed in the following way. The algebra  $L(0, 1)$  is the intersection of two algebras of vector fields,  $L(0)$  and  $L(1)$ . Here  $L(0)$  and  $L(1)$  consist of vector fields on the line with polynomial coefficients vanishing at  $0$  and  $1$  respectively. Their sum  $L(0) + L(1) = W_1$  is

the algebra of all polynomial vector fields. We get a diagram of inclusions

$$\begin{array}{ccc}
 & L(0) & \\
 L(0,1) & \nearrow & \\
 & L(1) & \searrow \\
 & & W_1
 \end{array}$$

From this it follows that, as a differential algebra, the standard cohomology complex  $C^*(L(0,1))$  is the tensor product of the differential algebras  $C(L(0))$  and  $C(L(1))$  over  $C^*(W_1)$ . (Observe that  $C^*(L(0))$  and  $C^*(L(1))$  are modules over  $C^*(W_1)$  as there exist inclusions  $L(0) \rightarrow W_1$  and  $L(1) \rightarrow W_1$ .) This means that there exists an Eilenberg–Moore spectral sequence (see [9]) whose second term is  $\text{Tor}_{H^*(W_1)}(H^*(L(0)), H^*(L(1)))$ , and which converges to  $H^*(L(0,1))$ . Further,  $H^*(L(0)) = 0$  for  $i \neq 0, 1$  and  $H^0(L(0)) \cong H^1(L(0)) \cong \mathbf{C}$ ,  $L(1) \cong L(0)$ . For  $W_1$  we have  $H^i(W_1) = 0$  for  $i \neq 0, 3$  and  $H^0(W_1) \cong H^3(W_1) \cong \mathbf{C}$ . The action of  $H^*(W_1)$  on  $H^*(L(0))$  and on  $H^*(L(1))$  is obviously trivial. This means that

$$\text{Tor}_{H^*(W_1)}(H^*(L(0)), H^*(L(1))) \cong H^*(L(0)) \otimes \text{Tor}_{H^*(W_1)}(\mathbf{C}, \mathbf{C}).$$

According to [9],  $\text{Tor}_{H^*(W_1)}(\mathbf{C}, \mathbf{C})$  is a free algebra with one two-dimensional generator. The differentials in the spectral sequence are zero and we get the desired result.

In the following proposition we state the result about the homology of  $L(0,1)$  with coefficients in  $M(\alpha, \beta)$ . It will be more convenient for us to describe the structure of the dual cohomology space. It is clear that  $H_*(L(0,1), M(\alpha, \beta))$  is dual to  $H^*(L(0,1), M(-\alpha, -\beta))$ . The Lie algebra  $L(0,1)$  can be embedded in the topological Lie algebra of all vector fields on the line. That means we can define the continuous cohomology

$$H_c^*(L(0,1), M(-\alpha, -\beta)).$$

**Proposition 1.** *The space  $H^*(L(0,1), M(-\alpha, -\beta))$  is different from zero only in the case where there exist two nonnegative integers  $k, l$  such that*

$$\alpha = \frac{3k^2 \pm k}{2}, \quad \beta = \frac{3l^2 \pm l}{2}.$$

The space

$$H^* \left( L(0,1), M \left( -\frac{3k^2 \pm k}{2}, -\frac{3l^2 \pm l}{2} \right) \right)$$

is a free module over  $H^*(L(0,1))$ , with one generator of degree  $k+l$ .

The proof is a standard exercise in continuous cohomology theory. The generator can be obtained as follows. Let  $L(p)$  be the Lie algebra of vector fields on the line vanishing at  $p \in \mathbf{R}$ , and let  $\overline{M}(\alpha)$  be the module on which  $f(z)d/dz$  acts as multiplication by  $f'(p)$ . The cohomology space  $H^*(L(p), \overline{M}(\alpha))$  is known,

see e.g. [6]. It is nontrivial only if  $\alpha = (3k^2 \pm k)/2$  where  $k$  is a nonnegative integer, and in that case  $H^*(L(p), \overline{M}(\alpha))$  is a free module over  $H_c^*(L(p))$  with one generator of degree  $k$ . Further,  $L(0, 1)$  can be embedded in  $L(0)$  and in  $L(1)$ . The restrictions of  $\overline{M}(\alpha)$  and  $\overline{M}(\beta)$  give the modules  $M(\alpha, 0)$  and  $M(0, \beta)$ . The product of the restricted classes of  $H_c^k$  and  $H_c^l$  ( $\alpha = (3k^2 \pm k)/2$ ,  $\beta = (3l^2 \pm l)/2$ ) gives the generator of  $H_c^{k+l}$ . Finally, one can show that  $H^*(L(0, 1), M(-\alpha, -\beta))$  is isomorphic to  $H_c^*(L(0, 1), M(-\alpha, -\beta))$ .

The standard complex

$$C_*(L_1, \mathcal{F}_{\lambda, \mu}) = \Lambda^* L_1 \otimes \mathcal{F}_{\lambda, \mu},$$

which yields  $H_*(L_1, \mathcal{F}_{\lambda, \mu})$ , is naturally graded, as  $L_1$  is a graded algebra and  $\mathcal{F}_{\lambda, \mu}$  is a graded module over  $L_1$  (e.g. the chain  $e_1 \otimes f_1$  has degree zero). Let  $C_*^0(L_1, \mathcal{F}_{\lambda, \mu})$  be the subcomplex of the elements of degree zero. It is clear that  $C_*(L_1, \mathcal{F}_{\lambda, \mu}) \cong \bigoplus_k C_*^0(L_1, \mathcal{F}_{\lambda, \mu+k})$ , where the sum is taken over all natural numbers  $k$ . That means that it is enough to compute the cohomology of  $C_*^0(L_1, \mathcal{F}_{\lambda, \mu})$ .

**Proposition 2.** *The complex  $C_*^0(L_1, \mathcal{F}_{\lambda, \mu})$  is isomorphic to the complex  $M(\alpha, \beta) \otimes \Lambda^* L(0, 1)$ , which yields the homology of  $L(0, 1)$  with coefficients in  $M(\alpha, \beta)$ , where  $\alpha - \mu - \lambda$ ,  $\beta = \lambda - 1$ .*

*Proof.* Observe that  $L(0, 1)$  can be realized as the subalgebra of  $\mathcal{L}$  with basis  $\bar{e}_i = e_i - e_{i-1}$ ,  $i = 1, 2, \dots$ . Put  $e'_i = \bar{e}_i - e_0$ ; on  $M(\alpha, \beta)$ ,  $e'_i$  induces multiplication by  $i\beta - \alpha$ . Let  $z$  be a generator in  $M(\alpha, \beta)$ . The differential  $M(\alpha, \beta) \otimes \Lambda^* L(0, 1)$  acts in the following way:

$$\begin{aligned} & d(z \otimes e'_{i_1} \wedge \dots \wedge e'_{i_k}) \\ &= \sum_{r,s} (-1)^{r+s} (i_r - i_s) z \otimes e'_{i_r+i_s} \wedge \dots \wedge \hat{e}'_{i_s} \wedge \dots \wedge \hat{e}'_{i_r} \wedge \dots \wedge e'_{i_k} \\ &+ \sum_s (-1)^s (i_1 + \dots + i_k - i_s) \cdot z \otimes e'_{i_1} \wedge \dots \wedge \hat{e}'_{i_s} \wedge \dots \wedge e'_{i_k} \\ &+ \sum_s (-1)^{s+1} (i_s \beta - \alpha) \cdot z \otimes e'_{i_1} \wedge \dots \wedge \hat{e}'_{i_s} \wedge \dots \wedge e'_{i_k}. \end{aligned} \quad (1)$$

The first two sums correspond to the bracket with  $e'_{i_r}$  and  $e'_{i_s}$  (we remark that  $[e'_i, e'_j] = (j-i)e'_{i+j} + ie'_i - je'_j$ ), while the last one corresponds to the action of  $e'_j$  on  $z$ . Now we determine the differential in  $C_*^0(L_1, \mathcal{F}_{\lambda, \mu})$ . The elements  $f_{-j} \otimes e_{i_1} \wedge \dots \wedge e_{i_k}$ ,  $j = i_1 + i_2 + \dots + i_k$ , form a basis of  $C_*^0(L_1, \mathcal{F}_{\lambda, \mu})$ . Then

$$\begin{aligned} & d(f_{-j} \otimes e_{i_1} \wedge \dots \wedge e_{i_k}) \\ &= \sum (-1)^{r+s} f_{-j} \otimes (i_r - i_s) e_{i_r+i_s} \wedge \dots \wedge \hat{e}_{i_s} \wedge \dots \wedge \hat{e}_{i_r} \wedge \dots \wedge e_{i_k} \\ &+ \sum (-1)^{s+1} (\lambda(i_s + 1) + \mu - j) f_{-j+i_s} \otimes e_{i_1} \wedge \dots \wedge \hat{e}_{i_s} \wedge \dots \wedge e_{i_k}. \end{aligned} \quad (2)$$

Observe that (1) becomes (2) if  $\alpha = -\lambda - \mu$ ,  $\beta = \lambda - 1$ .

Combining Propositions 1 and 2, we get a new method for computing  $H^*(L_1, \mathcal{F}_{\lambda, \mu})$ . Finally, we have the following result.

**Theorem.**  $H_*(L_1, \mathcal{F}_{\lambda, \mu}) \cong \bigoplus_k H_*(L(0, 1), M(-\lambda - \mu + k, \lambda - 1)), k \in \mathbf{Z}$ .

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INSTITUTE OF SOLID-STATE PHYSICS OF THE SOVIET ACADEMY OF SCIENCES,  
142432 CHERNOGOLOWKA (MOSCOW DISTRICT), USSR (ADDRESS OF B. L. FEIGIN)

INSTITUTE OF MATHEMATICS OF THE HUNGARIAN ACADEMY OF SCIENCES, H-  
1053 BUDAPEST, REÁLTANODA U. 13–15, HUNGARY

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY,  
CAMBRIDGE, MASSACHUSETTS 02139 USA