

Deformations and contractions of Lie algebras

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Abstract

We discuss the mutually opposite procedures of deformations and contractions of Lie algebras. Our main purpose is to illustrate the fact that, with appropriate combinations of both procedures, we obtain new Lie algebras. Firstly, we discuss low-dimensional Lie algebras, and these simple examples illustrate that, whereas for every contraction there exists a reverse deformation, the converse is not true in general. We point out that otherwise ordinary members of parameterized families of Lie algebras are singled out by this irreversibility of deformations and contractions. Then, we remind that global deformations of the Witt, Virasoro, and affine Kac-Moody algebras allow one to retrieve Lie algebras of Krichever-Novikov type and show that, in turn, contractions of the latter lead to new infinite dimensional Lie algebras.

1 Introduction

When understood in a general and rather vague sense, deformations of Lie algebras are continuous modifications of the structure constants of these Lie algebras, and they occur in mathematics and in physics under various guises. There exist two main categories of such modifications of the structure constants: *contractions*, which typically transform a Lie algebra into a ‘more abelian’ Lie algebra, and *deformations*, which will lead to a Lie algebra with more intricate Lie brackets. In the literature, physicists have been mostly interested in contractions, whereas most articles about deformations have appeared in mathematics. Both concepts have been investigated by researchers with different approaches and different goals. Therefore, there exists a plethora of definitions for both contractions and deformations. In the next section, we recall general definitions of the two concepts within the framework of Lie algebras. Our main objective is to use various examples to illustrate that the combined use of deformations and contractions may unify a wider variety of algebraic structures and unveil new Lie algebras.

One-parameter deformations of arbitrary rings and associative algebras, and their related cohomology questions, were first investigated by Gerstenhaber [1] and then applied to Lie algebras by Nijenhuis and Richardson [2]. They turn out to be a particular case of ‘formal’ deformations, the latter being obtained when classical deformations are formal power series of one or several variables. Formal deformations are characterized by a complete local algebra base and they describe a local neighbourhood of the given object. This more general setup for Lie algebras was introduced by Fialowski [3, 4]. This framework is general enough to describe all non-equivalent formal deformations. Namely there is a versal object in this category [4, 5]. ‘Versal’ means that such a deformation induces all other deformations and is unique at the infinitesimal level. A more general deformation theory is obtained by considering an arbitrary commutative algebra with unity as the base of deformation. Such deformations are called ‘global’ and appear in the work of Fialowski and Schlichenmaier [6, 7].

A procedure opposite to deformations, contractions, is important in physics because it explains, in terms of Lie algebras, why some theories arise as a limit regime of more ‘exact’ theories. Motivated by the need to relate the symmetries underlying Einstein’s mechanics and Newtonian mechanics, Inönü and Wigner introduced the concept of contraction, which consists in multiplying the generators of the symmetry by ‘contraction parameters’, such that when

these parameters reach some singularity point, one obtains a different (i.e. non-isomorphic) Lie algebra with the same dimension [8]. A similar procedure had been mentioned previously by Segal [9]. The method has been generalized a few years later by Saletan [10]. Nice reviews can be found in [11]. Contractions were used by Lévy-Leblond to emphasize that the condition of largely timelike intervals is just as crucial as the infinite velocity of light, in order to contract the Poincaré algebra to the Galilei algebra [12]. Another physical example is the contraction of the de Sitter algebras to the Poincaré algebra, in the limit of large (universe) radius. These examples suggest that deformations are likely to be more useful than contractions in the investigation of fundamental theories [13].

In the mathematics literature, there are concepts similar to contractions, known as ‘degeneration’, ‘perturbation’, and ‘orbit closure’. Orbit closures arise in many areas of mathematics where algebraic or topological transformation groups are considered, such as invariant theory, representation theory, theory of singularities, etc. For algebraic structures on a fixed finite dimensional vector space, degeneration means that the orbits under the action of the general linear group are the isomorphism classes, and so orbit closure coincides with the closure of these classes.

This article is organized as follows. General definitions of deformations and contractions are given in the next section. In Section 3, we discuss the three-dimensional complex and real Lie algebras in order to get new insight about the two concepts with those known examples. We use the generalized Inönü-Wigner contraction method [14] and the classification of deformations in reference [15]. In Section 4 we turn to infinite dimensional Lie algebras. Our main result therein is that appropriate combinations of deformation and contraction procedures allow us to construct and classify new Lie algebras in a natural and elegant manner. We expect these new Lie algebras to be of interest, particularly in conformal field theories, just like non-semisimple finite dimensional Lie algebras are important in physics. Other articles address various aspects of both deformation and contraction methods [16]-[20].

2 Deformations and contractions of Lie algebras

A nice review of the concepts of deformations and contractions is given in reference [21]. Consider a Lie algebra \mathfrak{g} of dimension N over an arbitrary field k . Hereafter, we are interested in $k = \mathbb{R}$ and \mathbb{C} . Let us denote the basis elements of \mathfrak{g} by $\{x_1, \dots, x_N\}$, and write the Lie bracket as

$$[x_i, x_j] = C_{ij}^k x_k, \quad (1)$$

where the coefficients C_{ij}^k are the structure constants. We denote by \mathcal{L}_N , or $\mathcal{L}_N(k)$, the space of structural tensors of N -dimensional Lie algebras. Then a *one-parameter deformation* of a Lie algebra \mathfrak{g} , whose structure constants belong to $\mathcal{L}_N(k)$, is a continuous curve over $\mathcal{L}_N(k)$. The deformation is said to be (piecewise) smooth, analytic, etc. if the defining curve itself is (piecewise) smooth, analytic, respectively.

A *formal* one-parameter deformation is defined by the Lie brackets :

$$[a, b]_t = F_0(a, b) + tF_1(a, b) + \dots + t^m F_m(a, b) + \dots \quad (2)$$

where F_0 denotes the original Lie bracket $[\cdot, \cdot]$. Jacobi identity implies relations between the tensors F_m . The first such deformation relation is that F_1 must be a two-cocycle of \mathfrak{g} . We call $[\cdot, \cdot]_t$ a first-order, or *infinitesimal*, deformation if it satisfies the Jacobi identity up to t^2 . It follows that first-order deformations correspond to elements of the space of two-cocycles $Z^2(\mathfrak{g}, \mathfrak{g})$. A deformation is called of order n if it is defined modulo t^{n+1} .

Consider now a deformation $\mathfrak{g}_t = [\cdot, \cdot]_t$ not as a family of Lie algebras, but as a Lie algebra over the ring $k[[t]]$ of formal power series over k . A natural generalization is to allow more parameters, which amounts to consider $k[[t_1, \dots, t_k]]$ as the base, or, even more generally, to take an arbitrary commutative algebra A over k , with unit as the base. Assume that A admits an augmentation $\epsilon : A \rightarrow k$, such that ϵ is a k -algebra homomorphism and $\epsilon(1_A) = 1$. The ideal $m_\epsilon := \ker(\epsilon)$ is a maximal ideal of A , and, given a maximal ideal m of A with $A/m \cong k$, the natural quotient map defines an augmentation. If A has a unique maximal ideal, the deformation with base A is called *local*. If A is the projective limit of local algebras, the deformation is called *formal*.

In general, let us consider a Lie algebra \mathfrak{g} over the field k , ϵ a fixed augmentation of the commutative algebra A , and $m_\epsilon := \ker(\epsilon)$, the associated

maximal ideal. We define a *global deformation* λ of \mathfrak{g} with base (A, m) as a Lie A -algebra structure on the tensor product $A \otimes_k \mathfrak{g}$ with bracket $[\cdot, \cdot]_\lambda$ such that

$$\epsilon \otimes \text{id} : A \otimes \mathfrak{g} \rightarrow k \otimes \mathfrak{g} = \mathfrak{g}$$

is a Lie algebra homomorphism [4, 5]. It means that, for all $a, b \in A$ and $x, y \in \mathfrak{g}$, we have the following conditions:

- (a) $[a \otimes x, b \otimes y]_\lambda = (ab \otimes \text{id})[1 \otimes x, 1 \otimes y]_\lambda$,
- (b) $[\cdot, \cdot]_\lambda$ is skew-symmetric and satisfies the Jacobi identity,
- (c) $\epsilon \otimes \text{id}([1 \otimes x, 1 \otimes y]_\lambda) = 1 \otimes [x, y]$.

Condition (a) means that, in order to describe a global deformation, it is sufficient to know the elements $[1 \otimes x, 1 \otimes y]_\lambda$, for all $x, y \in \mathfrak{g}$. From condition (c), it follows that the Lie bracket of these elements has the form

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y] + \sum_i a_i \otimes z_i,$$

with $a_i \in m$ and $z_i \in \mathfrak{g}$. The sum given in this expression is a finite sum.

Intuitively, *rigidity* of a Lie algebra \mathfrak{g} means that we cannot deform it. Or, given a family of Lie algebras containing \mathfrak{g} as the special element \mathfrak{g}_0 , any element \mathfrak{g}_t in the family ‘nearby’ will be isomorphic to \mathfrak{g}_0 . We call a Lie algebra *infinitesimally rigid* if every infinitesimal deformation is equivalent to the trivial one, and *formally rigid* if every formal deformation is trivial. For a finite dimensional Lie algebra \mathfrak{g} , if the cohomology space vanishes, i.e. $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, then \mathfrak{g} is rigid in any sense [1, 2]. Thus, for instance, any finite dimensional semisimple Lie algebra ($k = \mathbb{R}$ or \mathbb{C}) is rigid. In infinite dimension, the vanishing cohomology only implies that \mathfrak{g} is formally rigid [4].

A contraction is, in a sense, opposite to the operation of deformation. The commutation relations of a *contracted Lie algebra*, or *contraction*, \mathfrak{g}' of a Lie algebra \mathfrak{g} , are given by the limit :

$$[x, y]' \equiv \lim_{\epsilon \rightarrow \epsilon_0} \mathcal{U}_\epsilon^{-1}([\mathcal{U}_\epsilon(x), \mathcal{U}_\epsilon(y)]), \quad (3)$$

where $\mathcal{U}_\epsilon \in \text{GL}(N, k)$ is a non-singular linear transformation of \mathfrak{g} , with ϵ_0 being a singularity point of its inverse $\mathcal{U}_\epsilon^{-1}$. In mathematical terms, the orbits under the action of $\text{GL}(N, k)$ are the Lie algebra isomorphism classes, and a Lie bracket $[\cdot, \cdot]'$ is a contraction of $[\cdot, \cdot]$ if it is in the Zariski closure of the orbit of $[\cdot, \cdot]$ [19]. Since the fundamental definitions of deformations (given

in equation (2)) and contractions (equation (3)) are so different, it is not so clear *a priori* whether they are opposite concepts.

Throughout the paper, however, we shall utilize the *generalized Inönü-Wigner* (or *Weimar-Woods*) contractions, which are defined by splitting the Lie algebra \mathfrak{g} into an arbitrary number of subspaces:

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 + \cdots + \mathfrak{g}_p, \quad (4)$$

and by taking the matrix \mathcal{U}_ε of equation (3) as

$$\mathcal{U}_\varepsilon^{WW} = \oplus_j \varepsilon^{n_j} \text{id}_{\mathfrak{g}_j}, \quad \varepsilon > 0, \quad n_j \in \mathbb{R}, \quad j = 1, 2, \dots, p, \quad (5)$$

where $p \leq \dim \mathfrak{g}$. The Inönü-Wigner contractions correspond to the particular case $p = 1$, with $n_0 = 0$ and $n_1 = 1$ [8]. From equations (1) and (5), and if we denote by \mathfrak{g}_i the subspace in (4) to which the element x_i belongs, then equation (3) becomes

$$[x_i, x_j]' = \lim_{\varepsilon \rightarrow 0} \varepsilon^{n_i + n_j - n_k} C_{ij}^k x_k. \quad (6)$$

This shows that the exponents in equation (5) must be such that

$$n_i + n_j - n_k \geq 0, \quad \text{unless } C_{ij}^k = 0.$$

Then the structure constants of the contracted algebra \mathfrak{g}' are given by

$$(C')_{ij}^k = \begin{cases} C_{ij}^k, & \text{if } n_i + n_j = n_k, \\ 0, & \text{if } n_i + n_j > n_k. \end{cases}$$

Clearly, two trivial contractions always exist: first, the abelian Lie algebra, and second, the original Lie algebra itself, for which the commutation relations are left unchanged. Likewise, an abelian Lie algebra can be deformed to every Lie algebra of the same dimension.

3 Three-dimensional Lie algebras

In this section, we compare deformations and contractions of complex and real three-dimensional Lie algebras. Hereafter the deformations of real three-dimensional Lie algebras are classified for the first time by using the recent deformation classification of the corresponding complex Lie algebras [15].

Contractions of low-dimensional real Lie algebras are discussed in the literature: one, two and three dimensions in references [14, 22], and four dimensions in reference [23]. In references [22, 23], Inönü-Wigner contractions are utilized, whereas in reference [14], Weimar-Woods contractions are introduced, and applied to the three-dimensional real Lie algebras. We notice that to each contraction, there exists an opposite deformation, but the converse is not always true. This irreversibility occurs within families of Lie algebras.

3.1 Complex Lie algebras

In Table 1, we list the Lie brackets of the three-dimensional complex Lie algebras. Note that $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$ coincides with $\mathfrak{r}_{3,\lambda=0}(\mathbb{C})$, which explains why $\lambda = 0$ is excluded from the family $\mathfrak{r}_{3,\lambda}(\mathbb{C})$ in Table 1. We do so *a priori* just to follow the accepted nomenclature. However, we shall see that, in the sense of contractions and deformations, $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$ behaves just like any member of the family, so it is more appropriate to include it into the family. The results agree with the calculations (based on the method of orbit closures) in reference [24].

To illustrate the procedures, let us briefly discuss the contraction from $\mathfrak{sl}_2(\mathbb{C})$ to $\mathfrak{r}_{3,-1}(\mathbb{C})$. We express the Lie brackets of $\mathfrak{sl}_2(\mathbb{C})$ in the Cartan basis:

$$[h, e] = e, \quad [h, f] = -f, \quad [e, f] = 2h.$$

Then, we may introduce the contraction parameters as follows:

$$e \rightarrow \varepsilon e, \quad f \rightarrow \varepsilon f, \quad h \rightarrow h,$$

before taking the limit $\varepsilon \rightarrow 0$. This results in $[e, f] \rightarrow 0$, with $[h, e]$ and $[h, f]$ unchanged, i.e. the Lie brackets for $\mathfrak{r}_{3,-1}(\mathbb{C})$. Now, let us illustrate the reverse deformation with this simple example. The original Lie brackets of $\mathfrak{r}_{3,-1}(\mathbb{C})$ are such that, in equation (2), the non-zero F_0 's are $F_0(h, e) = e$ and $F_0(h, f) = -f$. Then, in order to deform it to $\mathfrak{sl}_2(\mathbb{C})$, we may write equation (2) as

$$\begin{aligned} [h, e]_t &= e + tF_1(h, e), \\ [h, f]_t &= -f + tF_1(h, f), \\ [e, f]_t &= tF_1(e, f), \end{aligned}$$

where

$$F_1(h, e) = 0, \quad F_1(h, f) = 0, \quad F_1(e, f) = 2h,$$

\mathbb{C}^3 :	$[x_i, x_j] = 0, \quad i, j = 1, 2, 3$
$\mathfrak{n}_3(\mathbb{C})$:	$[x_1, x_2] = x_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$:	$[x_1, x_2] = x_2$
$\mathfrak{r}_3(\mathbb{C})$:	$[x_1, x_2] = x_2, [x_1, x_3] = x_2 + x_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}), (\lambda \in \mathbb{C}^*, \lambda \leq 1)$:	$[x_1, x_2] = x_2, [x_1, x_3] = \lambda x_3$
$\mathfrak{sl}_2(\mathbb{C})$:	$[x_1, x_2] = x_3, [x_2, x_3] = x_1, [x_3, x_1] = x_2$

Table 1: Three-dimensional complex Lie algebras

as suggested clearly by the contraction. The resulting Lie algebra is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, for any non-zero t .

The results of contractions and deformations of three-dimensional complex Lie algebras are displayed on Fig. 1. The lines and arrows should be interpreted as follows: an arrow points toward the deformation, whereas a simple line connect Lie algebras related by both deformation and contraction, with the deformed Lie algebra lying upward. The left-pointing arrow symbol over $\mathfrak{r}_{3,\lambda \neq \pm 1}(\mathbb{C})$ means that it deforms inside the family.

Let us note that a non-trivial contraction always induces a non-trivial (inverse) deformation. However, the converse is not true: there are deformations which do not admit a inverse contraction. For instance, one can never have a contraction inside a parameterized family of Lie algebras, but there is always a deformation inside a family. Also, nothing can be contracted to the parameterized family, whereas there are many non-trivial deformations in dimension three to the family $\mathfrak{r}_{3,\lambda \neq \pm 1}(\mathbb{C})$. Let us emphasize that the irreversibility occurs only when the family is involved.

The family of Lie algebras $\mathfrak{r}_{3,\lambda \neq \pm 1}(\mathbb{C})$ has a non-trivial deformation into itself. Note that the two Lie algebras $\mathfrak{r}_{3,1}(\mathbb{C})$ and $\mathfrak{r}_{3,-1}(\mathbb{C})$ are singled out for two reasons. First, $\mathfrak{r}_{3,1}(\mathbb{C})$ can be deformed into $\mathfrak{r}_3(\mathbb{C})$ whereas $\mathfrak{r}_{3,\lambda \neq 1}(\mathbb{C})$ cannot, and $\mathfrak{r}_{3,-1}(\mathbb{C})$ can deform into $\mathfrak{sl}_2(\mathbb{C})$, whereas $\mathfrak{r}_{3,\lambda \neq -1}(\mathbb{C})$ cannot. Second, $\mathfrak{r}_{3,1}(\mathbb{C})$ is special because it cannot be contracted to $\mathfrak{n}_3(\mathbb{C})$, unlike $\mathfrak{r}_{3,\lambda \neq 1}(\mathbb{C})$.

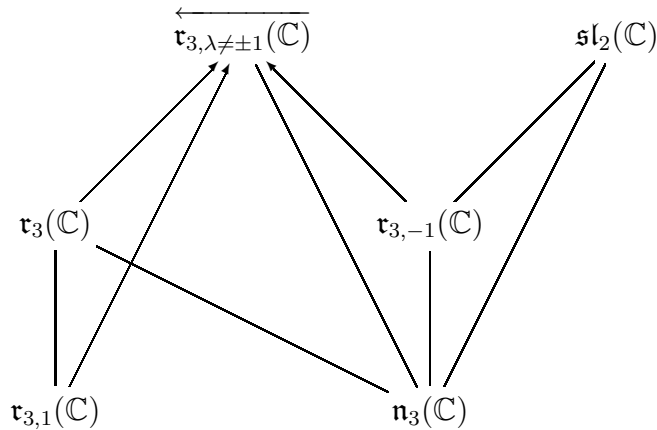


Fig.1 Contractions and deformations of the three-dimensional complex Lie algebras

3.2 Real Lie algebras

The Lie brackets of three-dimensional real Lie algebras are given in Table 2. Also, we list in Table 3 different notations found in the literature, because we shall present just the results for the contractions taken from references [14, 22]. Note that the Lie algebra $\mathfrak{t}_2(\mathbb{R}) \oplus \mathbb{R}$ may be included within the family $\mathfrak{t}_{3,\lambda}(\mathbb{R})$ as the particular case $\lambda = 0$. However, although it might be tempting to view $\mathfrak{n}_2(\mathbb{R})$ likewise, as $\mathfrak{t}'_{3,\lambda=0}(\mathbb{R})$, we shall see that it is not appropriate, because the deformation pattern of $\mathfrak{n}_2(\mathbb{R})$ is different from $\mathfrak{t}'_{3,\lambda}(\mathbb{R})$.

The results obtained by Weimar-Woods [14], which encompass those of Conatser [22], are summarized in Fig. 2. The only difference between the two classifications is that Weimar-Woods obtains the direct contraction from $\mathfrak{sl}_2(\mathbb{R})$ into $\mathfrak{n}_3(\mathbb{R})$, whereas Conatser obtains this relation with a sequence of contractions, the intermediary step going through $\mathfrak{n}_2(\mathbb{R})$. This is not at all surprising, since Weimar-Woods utilizes a more general contraction method than Conatser's.

We have classified the deformations following the same lines as in reference [15], using the corresponding real forms of the complex Lie algebras. They are displayed in Fig. 2. Let us make some general comments. Different real forms of a given complex Lie algebra do not interact via deformations or contractions because, if they did, they would induce a non-trivial deformation or contraction at the complex level which is the identity mapping. Also, a subset of the real forms is formally identical to the corresponding complex

\mathbb{R}^3 :	$[x_i, x_j] = 0, \quad i, j = 1, 2, 3$
$\mathfrak{n}_3(\mathbb{R})$:	$[x_1, x_2] = x_3$
$\mathfrak{t}_3(\mathbb{R})$:	$[x_1, x_2] = x_2, [x_1, x_3] = x_2 + x_3$
$\mathfrak{t}_{3,\lambda}(\mathbb{R}), (\lambda \in \mathbb{R}, \lambda < 1)$:	$[x_1, x_2] = x_2, [x_1, x_3] = \lambda x_3$
$\mathfrak{t}_{3,1}(\mathbb{R})$:	$[x_1, x_2] = x_2, [x_1, x_3] = x_3$
$\mathfrak{t}_{3,-1}(\mathbb{R})$:	$[x_1, x_2] = x_2, [x_1, x_3] = -x_3$
$\mathfrak{t}'_{3,\lambda}(\mathbb{R}), (\lambda \in \mathbb{R}^*)$:	$[x_1, x_2] = \lambda x_2 + x_3, [x_1, x_3] = -x_2 + \lambda x_3$
$\mathfrak{n}_2(\mathbb{R}) = \mathfrak{t}'_{3,0}(\mathbb{R})$:	$[x_1, x_2] = x_3, [x_1, x_3] = -x_2$
$\mathfrak{sl}_2(\mathbb{R})$:	$[x_1, x_3] = -2x_2, [x_1, x_2] = x_1, [x_2, x_3] = x_3$
\mathfrak{su}_2 :	$[x_1, x_2] = x_3, [x_2, x_3] = x_1, [x_3, x_1] = x_2$
$\mathfrak{t}_2(\mathbb{R}) \oplus \mathbb{R}$:	$[x_1, x_2] = x_2$

Table 2: Three-dimensional real Lie algebras

Onishchik, Vinberg ^[21]	Patera et al ^[25]	Lévy-Nahas ^[17]	Conatser ^[22]	Weimar-Woods ^[14]
$\mathfrak{n}_3(\mathbb{R})$	$A_{3,1}$	A_2	C_2	L_S
$\mathfrak{t}_3(\mathbb{R})$	$A_{3,2}$	A_4	C_5	L_{SA}
$\mathfrak{t}_{3,\lambda}(\mathbb{R}), (\lambda \in \mathbb{R}^*, \lambda < 1)$	$A_{3,5}^a$	$A_5(\lambda)$	$C_{4,\lambda}$	L_{SSA-}^λ
$\mathfrak{t}_{3,1}(\mathbb{R})$	$A_{3,3} = A_{3,5}^{a=1}$	$A_3 = A_5(1)$	$C_{4,\lambda=+1}$	L_A
$\mathfrak{t}_{3,-1}(\mathbb{R})$	$A_{3,4} = A_{3,5}^{a=-1}$	$A_6 = A_5(-1)$	$C_{4,\lambda=-1}$	L_{SS-}
$\mathfrak{n}_2(\mathbb{R})$	$A_{3,6} = A_{3,7}^{a=0}$	A_7	$C_{6,\lambda=0}$	L_{SS+}
$\mathfrak{t}'_{3,\lambda}(\mathbb{R}), (\lambda \in \mathbb{R}^*)$	$A_{3,7}^a$		$C_{6,\lambda}$	L_{SSA+}^λ
$\mathfrak{sl}_2(\mathbb{R})$	$A_{3,8}$	A_8	C_8	L_{SSS-}
\mathfrak{su}_2	$A_{3,9}$	A_9	C_7	L_{SSS+}
$\mathfrak{t}_2(\mathbb{R}) \oplus \mathbb{R}$	$A_{3,5}^{a=0}$	$A_5(0)$	C_3	$L_{SSA-}^{\lambda=0}$
\mathbb{R}^3	$3A_{1,1}$	A_1	C_1	

Table 3: Three-dimensional real Lie algebras

forms, so they inherit a similar pattern for contractions and deformations. As a consequence, we can restrict ourselves to the contractions and deformations involving the new objects only.

As mentioned above, the Lie algebra $\mathfrak{n}_2(\mathbb{R})$ acquires a special status, compared to the other members of the family $\mathfrak{r}'_{3,\lambda}(\mathbb{R})$, with respect to deformations. This is because $\mathfrak{n}_2(\mathbb{R})$ can be deformed into \mathfrak{su}_2 and $\mathfrak{sl}_2(\mathbb{R})$ and no other members of the family can. Therefore, the value $\lambda = 0$ stands out, in a way parallel to the values $\lambda = \pm 1$ in the complex case. Nevertheless, there exists a natural deformation of $\mathfrak{n}_2(\mathbb{R})$ into the family. The non-trivial deformed Lie brackets are the following, expressed as in equation (2):

$$\begin{aligned} [x_1, x_2]_t &= [x_1, x_2]_0 + tF_1(x_1, x_2), & \text{where } F_1(x_1, x_2) &= x_2, \\ [x_1, x_3]_t &= [x_1, x_3]_0 + tF_1(x_1, x_3), & \text{where } F_1(x_1, x_3) &= x_3. \end{aligned}$$

It is true in general that any particular member of a family can be deformed into the family.

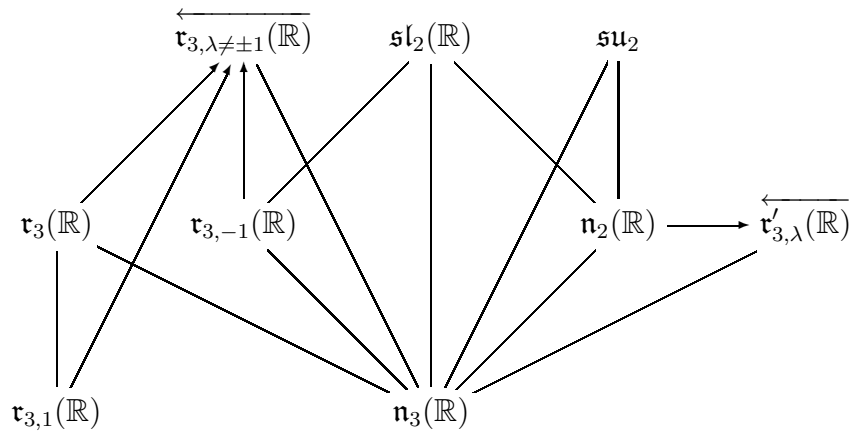


Fig.2 Contractions and deformations of the three-dimensional real Lie algebras

4 Infinite dimensional Lie algebras

In this section, we discuss deformations and contractions of some infinite dimensional Lie algebras. The physics literature about applications of infinite dimensional Lie algebras, namely to conformal field theory is enormous. Their interest stems from critical phenomena in two dimensions [26].

Whereas the Witt and Virasoro algebras describe local invariance of conformal field theories on the (zero genus) Riemann sphere, the Lie algebras of Krichever-Novikov type discussed hereafter corresponds to higher genus. The physical interpretation of the contraction parameters introduced within these Lie algebras remain to be further explored. The difficulty encountered when deforming the known Lie algebras is that formal deformations are no longer sufficient to describe general deformations. The examples discussed below are formally rigid, so that they admit no non-trivial formal deformations. Nevertheless, there exist very interesting non-trivial global deformations. In the global deformation theory, we no longer have the tool of computing cohomology in order to get deformations, so the picture is much more difficult, and there are very few results so far [6, 7]. This is where a combination of the contractions and deformations proves really fruitful, since it leads to new infinite dimensional objects, as we will show hereafter. In a domain where so few objects are known explicitly, each new object should be of interest both in mathematics and in physics, particularly, in conformal field theory. The deformations we consider here are over affine varieties, which are very special global deformations.

4.1 Witt, Virasoro, and Krichever-Novikov algebras

First, let us consider the Witt algebra \mathfrak{W}

$$[l_n, l_m] = (m - n)l_{n+m}, \quad n, m \in \mathbb{Z}.$$

Its only one-dimensional central extension is the Virasoro algebra \mathfrak{V} , with bracket operation:

$$[l_n, l_m] = (m - n)l_{n+m} + \frac{1}{12}m(m^2 - 1)\delta_{n+m,0} c, \quad [l_n, c] = 0,$$

where c denotes the central charge.

Krichever and Novikov invented the algebras of Virasoro type in reference [27]. It was shown recently that these infinite-dimensional Lie algebras can be interpreted as global deformations of the Witt or Virasoro algebra [6]. For simplicity, we will consider deformations of the Witt algebra, but this can be generalized in a natural way to the Virasoro algebra [6]. Despite its infinitesimal and formal rigidity, which prevents any non-trivial *formal* deformation,

the Witt algebra \mathfrak{W} can be non-trivially *globally* deformed into Krichever-Novikov type algebras \mathfrak{KN} [6]. Such a phenomenon does not appear with Lie algebras of finite dimension.

An example of Krichever-Novikov algebras \mathfrak{KN} is a two-dimensional family of Lie algebras parameterized over \mathbb{C}^2 . The generators are given by the fields:

$$\begin{aligned} V_{2n+1} &= (X - e_1)^n Y \frac{d}{dX}, \\ V_{2n} &= 2(X - e_1)^{n-1} (X - e_2)(X + e_1 + e_2) \frac{d}{dX}, \end{aligned}$$

which satisfy the following Lie brackets:

$$[V_n, V_m] = \begin{cases} (m - n)V_{n+m}, & n, m \text{ odd}, \\ (m - n)(V_{n+m} + 3e_1 V_{n+m-2} + (e_1 - e_2)(2e_1 + e_2)V_{n+m-4}), & n, m \text{ even}, \\ (m - n)V_{n+m} + (m - n - 1)3e_1 V_{n+m-2} + (n - m - 2)(e_1 - e_2)(2e_1 + e_2)V_{n+m-4}, & n \text{ odd}, m \text{ even}. \end{cases}$$

Note that for $e_1 = 0$, $e_2 = 0$, and $Y = X$, we recover the Witt algebra, and the fields reduce to

$$l_n = X^{n+1} \frac{d}{dX}.$$

A different \mathfrak{KN} algebra can be obtained as a one-parameter global deformation, by taking the following field basis:

$$V_{2n} \equiv X(X - \alpha)^n (X + \alpha)^n \frac{d}{dX}, \quad V_{2n+1} \equiv (X - \alpha)^{n+1} (X + \alpha)^{n+1} \frac{d}{dX}.$$

One calculates the Lie brackets:

$$[V_n, V_m] = \begin{cases} (m - n)V_{n+m}, & n, m \text{ odd}, \\ (m - n)(V_{n+m} + \alpha^2 V_{n+m-2}), & n, m \text{ even}, \\ (m - n)V_{n+m} + (m - n - 1)\alpha^2 V_{n+m-2}, & n \text{ odd}, m \text{ even}. \end{cases} \quad (7)$$

There are many other ways as well, if we specify the base of the deformation being different affine lines in \mathbb{C}^2 .

Clearly, they can be contracted back to the Witt algebra. Let us show it on the second type, equation (7), by utilizing a very elegant and simple Weimar-Woods contraction: if we define \mathcal{U}_ε in equations (4) and (5) such that

$$l_n \equiv \varepsilon^n V_n, \quad \text{for all } n \in \mathbb{Z}, \quad (8)$$

then equation (7) becomes

$$[l_n, l_m]_\varepsilon = \varepsilon^{n+m}[V_n, V_m] = \begin{cases} (m-n)l_{n+m}, & n, m \text{ odd,} \\ (m-n)(l_{n+m} + \varepsilon^2 \alpha^2 l_{n+m-2}), & n, m \text{ even,} \\ (m-n)l_{n+m} + (m-n-1)\varepsilon^2 \alpha^2 l_{n+m-2}, & n \text{ odd, } m \text{ even.} \end{cases}$$

Then, it is clear that, in the limit where ε approaches zero, we retrieve the commutation relations of \mathfrak{W} . Therefore, the operations of deformation and contraction are mutually reversible in this case.

In addition to retrieving the Witt algebra \mathfrak{W} , one may contract \mathfrak{KN} to other, so far unknown, Lie algebras. Let us discuss an example of such a contraction of \mathfrak{KN} which turns out to be a deformation of the respective contraction of \mathfrak{W} . Moreover, this exotic contraction of \mathfrak{KN} may be contracted back to the corresponding contraction of \mathfrak{W} by utilizing equation (8). In order to do so, let us define \mathcal{U}_ε à la Weimar-Woods:

$$\mathcal{U}_\varepsilon \equiv \varepsilon^{n_0} \text{id}_{\mathfrak{g}_0} + \varepsilon^{n_1} \text{id}_{\mathfrak{g}_1}, \quad (9)$$

where 0 and 1 denote the even and odd sectors of the powers of \mathfrak{KN} , respectively. Then, if we take the Lie brackets (7) as a specific example, we obtain the modified brackets:

$$[V_n, V_m]_\varepsilon = \begin{cases} \varepsilon^{2n_1-n_0}(m-n)V_{n+m}, & n, m \text{ odd,} \\ \varepsilon^{n_0}(m-n)(V_{n+m} + \alpha^2 V_{n+m-2}), & n, m \text{ even,} \\ \varepsilon^{n_0}[(m-n)V_{n+m} + (m-n-1)\alpha^2 V_{n+m-2}], & n \text{ odd, } m \text{ even.} \end{cases} \quad (10)$$

Clearly, we must have non-negative values of n_0 and $2n_1 - n_0$. We obtain the trivial abelian Lie algebra when these expressions take on positive values. Another trivial contraction is given by $n_0 = n_1 = 0$; then it leaves the commutators of equation (7) unchanged.

Two more contractions may be obtained with the splitting of equation (9). One is the Inönü-Wigner contraction, given by $n_0 = 0$ and $n_1 = 1$ (or any n_1 positive); then the contracted commutation relations read:

$$[V_n, V_m] = \begin{cases} 0, & n, m \text{ odd,} \\ (m-n)(V_{n+m} + \alpha^2 V_{n+m-2}), & n, m \text{ even,} \\ (m-n)V_{n+m} + (m-n-1)\alpha^2 V_{n+m-2}, & n \text{ odd, } m \text{ even.} \end{cases} \quad (11)$$

This defines a new family of infinite dimensional Lie algebras. In the spirit of sequences of contractions, if we further contract these Lie algebras with

\mathcal{U}_ε such as defined in equation (8), then we obtain

$$[l_n, l_m] = \begin{cases} 0, & n, m \text{ odd,} \\ (m-n)l_{n+m}, & n \text{ or } m \text{ even.} \end{cases}$$

This algebra is clearly non-isomorphic to \mathfrak{W} . Indeed, it is a contraction of \mathfrak{W} , obtained with equation (9), utilized this time with 0 being the even sector, and 1 the odd sector of \mathfrak{W} .

The second contraction of the \mathfrak{KN} algebra of equation (7) is constructed by choosing $n_0 > 0$ and $2n_1 - n_0 = 0$ in equation (10):

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd,} \\ 0, & n, m \text{ even,} \\ 0, & n \text{ odd, } m \text{ even.} \end{cases}$$

Again, this algebra is not isomorphic to the Witt algebra \mathfrak{W} .

4.2 Affine Kac-Moody and Krichever-Novikov algebras

In this section, we turn to deformations and contractions which involve Kac-Moody algebras. More specifically, we shall consider affine, or current, untwisted Kac-Moody algebras, $\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c$, which are defined in terms of a finite simple complex Lie algebra \mathfrak{g} , together with $\mathbb{C}[t, t^{-1}]$, the associative algebra of the Laurent polynomials, and the central extension c :

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + mcB(a, b)\delta_{m+n,0}. \quad (12)$$

Either one may contract the finite Lie algebra \mathfrak{g} first and then affinize the contracted algebra \mathfrak{g} , as done in Ref. [28], or one may contract the affine algebra. In the next subsection, we shall discuss the contractions of affine Kac-Moody algebras. Then, in the following subsection, we further construct new infinite dimensional Lie algebras by contracting Krichever-Novikov type deformations of affine Kac-Moody algebras.

Contractions of affine Kac-Moody algebras

Let us review the results of reference [28]. To do so, we use the notation

$$T_i^m \equiv x_i \otimes t^m,$$

together with the Killing form $B(x_i, x_j) = \frac{1}{2}\delta_{ij}$, so that the Lie brackets (12) reads

$$[T_i^m, T_j^n] = C_{ij}^k T_k^{m+n} + \frac{1}{2}cm\delta_{ij}\delta_{m+n,0}. \quad (13)$$

The structure constants are as in equation (1). In order to contract $\hat{\mathfrak{g}}$ à la Inönü-Wigner, first we split the vector space underlying \mathfrak{g} as in equation (4):

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1,$$

and we denote the basis elements as

$$\begin{aligned} x_\alpha &\in \mathfrak{g}_0, & \alpha &= 1, \dots, \dim \mathfrak{g}_0, \\ x_i &\in \mathfrak{g}_1, & i &= 1, \dots, \dim \mathfrak{g}_1. \end{aligned}$$

Then equation (6) leads to the following contraction of the corresponding loop algebra:

$$\begin{aligned} [x_\alpha \otimes t^m, x_\beta \otimes t^n] &= C_{\alpha\beta}^\gamma x_\gamma \otimes t^{m+n}, \\ [x_\alpha \otimes t^m, x_i \otimes t^n] &= C_{\alpha i}^j x_j \otimes t^{m+n}, \\ [x_i \otimes t^m, x_j \otimes t^n] &= 0. \end{aligned}$$

If the splitting is rather taken over to the affine Lie algebra as follows:

$$\hat{\mathfrak{g}}_0 = \{T_\alpha^m, c\}, \quad \hat{\mathfrak{g}}_1 = \{T_i^m\},$$

then we find [28]

$$\begin{aligned} [T_\alpha^m, T_\beta^n] &= C_{\alpha\beta}^\gamma T_\gamma^{m+n} + \frac{1}{2}cm\delta_{\alpha\beta}\delta_{m+n,0}, \\ [T_\alpha^m, T_i^n] &= C_{\alpha i}^j T_j^{m+n}, \\ [T_i^m, T_j^n] &= 0. \end{aligned}$$

Clearly, these contractions generate infinite dimensional Lie algebras which are not of Kac-Moody type. Physical applications, particularly in conformal field theory, of such exotic Lie algebras, certainly deserve further examination.

There is another class of contractions, not discussed in reference [28]. It is investigated, in the context of ‘graded contractions’ [29], to contract affine Kac-Moody algebras [30]. We may begin with a splitting defined by the powers of the Laurent polynomials $\mathbb{C}[t, t^{-1}]$. Let us consider the two blocks:

$$\hat{\mathfrak{g}}_0 = \{T_i^{2m}, c\}, \quad \hat{\mathfrak{g}}_1 = \{T_i^{2m+1}\},$$

which admits a natural Z_2 grading. Then the equation (6) leads to the contracted Lie brackets:

$$\begin{aligned} [T_i^{2m}, T_j^{2n}] &= C_{ij}^k T_k^{2(m+n)} + cm\delta_{ij}\delta_{m+n,0}, \\ [T_i^{2m}, T_j^{2n+1}] &= C_{ij}^k T_k^{2(m+n)+1}, \\ [T_i^{2m+1}, T_j^{2n+1}] &= 0. \end{aligned}$$

If we define

$$T_i^n \rightarrow \varepsilon^n T_i^n, \quad c \rightarrow c,$$

then equation (13) contracts to the non-extended affine Lie algebra:

$$[T_i^m, T_j^n] = C_{ij}^k T_k^{m+n}.$$

Contractions of Krichever-Novikov type deformations of affine Kac-Moody algebras

Examples of global deformations of affine Kac-Moody algebras are also given by \mathfrak{KN} type algebras. However, they do not have such a nice algebraic description as for the Witt and Virasoro algebra. In reference [7], it is shown that the trivially extended affine algebras, that is, equation (12) with $k = 0$, may be deformed to the following \mathfrak{KN} type algebra, parameterized over the affine plane \mathbb{C}^2 , or, described algebraically, over the polynomial algebra $\mathbb{C}[e_1, e_2]$:

$$[a \otimes A^n, b \otimes A^m] = \begin{cases} [a, b] \otimes A^{n+m}, & n \text{ or } m \text{ even,} \\ [a, b] \otimes A^{n+m} + 3e_1 [a, b] \otimes A^{n+m-2} + \\ \quad + (e_1 - e_2)(2e_1 + e_2)[a, b] \otimes A^{n+m-4}, & n \text{ and } m \text{ odd,} \end{cases} \quad (14)$$

where a and b belong to a finite dimensional complex Lie algebra \mathfrak{g} . For $(e_1, e_2) = (0, 0)$, we simply obtain the original affine algebra.

If we take, as base variety, the affine line $e_1 = 0$, then we get the one-parameter \mathfrak{KN} Lie algebras [7]:

$$[a \otimes A^n, b \otimes A^m] = \begin{cases} [a, b] \otimes A^{n+m}, & n \text{ or } m \text{ even,} \\ [a, b] \otimes A^{n+m} - e [a, b] \otimes A^{n+m-4}, & n \text{ and } m \text{ odd.} \end{cases} \quad (15)$$

First, as we have done in the previous section, let us see that both deformations may be contracted back to the original Kac-Moody algebra by defining the transformation \mathcal{U}_ε in analogy with equation (8):

$$a \otimes t^n \equiv \varepsilon^n a \otimes A^n, \quad \text{for all } n \in \mathbb{Z}, \quad (16)$$

so that the Lie brackets (14) become

$$[a \otimes A^n, b \otimes A^m]_\varepsilon = \begin{cases} [a, b] \otimes A^{n+m}, & n \text{ or } m \text{ even,} \\ [a, b] \otimes A^{n+m} + 3e_1 \varepsilon^2 [a, b] \otimes A^{n+m-2} + \\ \quad + (e_1 - e_2)(2e_1 + e_2) \varepsilon^4 [a, b] \otimes A^{n+m-4}, & n \text{ and } m \text{ odd.} \end{cases}$$

As the contraction parameter ε approaches zero, this leads to equation (12) with $k = 0$. The situation is absolutely similar for equation (15).

Now, let us obtain other Lie algebras by following procedures similar to what we have done for the Witt algebra. First, we split the \mathfrak{RM} algebra described by equations (14) or (15) as we have done in equation (9). Then, just to illustrate this contraction with equation (14), we find

$$[a \otimes A^n, b \otimes A^m]_\varepsilon = \begin{cases} \varepsilon^{n_0} [a, b] \otimes A^{n+m}, & n \text{ or } m \text{ even,} \\ \varepsilon^{2n_1 - n_0} ([a, b] \otimes A^{n+m} + 3e_1 [a, b] \otimes A^{n+m-2} + \\ \quad + (e_1 - e_2)(2e_1 + e_2)[a, b] \otimes A^{n+m-4}), & n \text{ and } m \text{ odd.} \end{cases} \quad (17)$$

The Inönü-Wigner contraction discussed after equation (10), for which $n_0 = 0$ and $n_1 = 1$, leads to commutation relations analogous to equation (11):

$$[a \otimes A^n, b \otimes A^m]' = \begin{cases} [a, b] \otimes A^{n+m}, & n \text{ or } m \text{ even,} \\ 0 & n \text{ and } m \text{ odd.} \end{cases}$$

If we take n_0 positive, and $2n_1 - n_0 = 0$, then equation (17) gives

$$[a \otimes A^n, b \otimes A^m]_\varepsilon = \begin{cases} 0 & n \text{ or } m \text{ even,} \\ [a, b] \otimes A^{n+m} + 3e_1 [a, b] \otimes A^{n+m-2} + \\ \quad + (e_1 - e_2)(2e_1 + e_2)[a, b] \otimes A^{n+m-4}, & n \text{ and } m \text{ odd.} \end{cases}$$

Evidently, there are countless possibilities, if we replace equation (16) with a different splitting of the Lie algebras. We obtain similar results for the Lie algebras of equation (15).

Let us now turn to contractions where the splitting is not done only with respect to the degrees of the Laurent polynomials, but within the underlying finite Lie algebra \mathfrak{g} . For the sake of illustration, let us split the finite underlying Lie algebra \mathfrak{g} according to the following \mathbb{Z}_2 -graded structure:

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1, \quad (18)$$

such that

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_0. \quad (19)$$

With $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$, for instance, this is satisfied if we choose $\mathfrak{g}_0 = \{h_\alpha, h_\beta, e_{\pm\alpha}\}$ and $\mathfrak{g}_1 = \{e_{\pm\beta}, e_{\pm(\alpha+\beta)}\}$. Next, let us write equation (5) as follows:

$$\mathcal{U}_\varepsilon \equiv \varepsilon^{n_{00}} \text{id}_{\mathfrak{g}_{00}} + \varepsilon^{n_{01}} \text{id}_{\mathfrak{g}_{01}} + \varepsilon^{n_{10}} \text{id}_{\mathfrak{g}_{10}} + \varepsilon^{n_{11}} \text{id}_{\mathfrak{g}_{11}},$$

where

$$\begin{aligned}\mathfrak{g}_{00} &= \mathfrak{g}_0 \otimes A^{2n}, \\ \mathfrak{g}_{01} &= \mathfrak{g}_0 \otimes A^{2n+1}, \\ \mathfrak{g}_{10} &= \mathfrak{g}_1 \otimes A^{2n}, \\ \mathfrak{g}_{11} &= \mathfrak{g}_1 \otimes A^{2n+1},\end{aligned}$$

where n is an integer. Such a decomposition carries a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ -graded structure [29]. Since both indices add up modulo 2, the commutator $[\mathfrak{g}_\mu, \mathfrak{g}_\nu]$ becomes

$$[\mathfrak{g}_\mu, \mathfrak{g}_\nu]_\varepsilon = \varepsilon^{n_\mu + n_\nu - n_{\mu+\nu}} \mathfrak{g}_{\mu+\nu},$$

where μ and ν are double indices. Then, from equation (6), we see that the exponents $n_{00}, n_{01}, n_{10}, n_{11}$, must be such that the following expressions

$$\begin{aligned}n_{00}, \\ n_{01} + n_{10} - n_{11}, \quad n_{10} + n_{11} - n_{01}, \quad n_{11} + n_{01} - n_{10}, \\ 2n_{01} - n_{00}, \quad 2n_{10} - n_{00}, \quad 2n_{11} - n_{00},\end{aligned} \tag{20}$$

are *non-negative*. The second array shows that the exponents n_{01}, n_{10}, n_{11} are either all equal to zero, or all non-zero. In the latter situation, we may choose $n_{11} = n_{01} + n_{10}$ (for instance, $n_{01} = 1 = n_{10}, n_{11} = 2$), which implies that the other two terms in the second line (equation above) are positive. Obviously, there are two other possibilities, i.e. with n_{01} and n_{10} respectively replacing n_{11} . When $n_{00} \neq 0$, then the third line of equation (20) implies that n_{01}, n_{10} and n_{11} must be positive also. To summarize, we have the following possibilities:

1. All the exponents n_μ are equal to zero.
2. $n_{00} = 0$ and the three others different from zero, with $n_{11} = n_{01} + n_{10}$ (so that $n_{01} > 0$ and $n_{10} > 0$). There is a total of three such possibilities.
3. $n_{00} \neq 0$, then the others are also different from zero, with a pattern similar to the previous case, and for which there are three possibilities.

For the sake of illustration, let us display the effect of such contraction on the \mathfrak{KN} type algebra (15). We use the notation a_0 to mean that $a \in \mathfrak{g}_0$ of equation (18), and a_1 if $a \in \mathfrak{g}_1$. Before taking the limit, the Lie brackets

(15) become:

$$\begin{aligned}
[a_0 \otimes A^{2n}, b_0 \otimes A^{2m}]_\varepsilon &= \varepsilon^{n_{00}} [a_0, b_0] \otimes A^{2(n+m)}, \\
[a_0 \otimes A^{2n}, b_0 \otimes A^{2m+1}]_\varepsilon &= \varepsilon^{n_{00}} [a_0, b_0] \otimes A^{2(n+m)+1}, \\
[a_0 \otimes A^{2n}, b_1 \otimes A^{2m}]_\varepsilon &= \varepsilon^{n_{00}} [a_0, b_1] \otimes A^{2(n+m)}, \\
[a_0 \otimes A^{2n}, b_1 \otimes A^{2m+1}]_\varepsilon &= \varepsilon^{n_{00}} [a_0, b_1] \otimes A^{2(n+m)+1}, \\
[a_0 \otimes A^{2n+1}, b_0 \otimes A^{2m+1}]_\varepsilon &= \varepsilon^{2n_{01}-n_{00}} ([a_0, b_0] \otimes A^{2(n+m)+2} - e[a_0, b_0] \otimes A^{2(n+m)-2}), \\
[a_0 \otimes A^{2n+1}, b_1 \otimes A^{2m}]_\varepsilon &= \varepsilon^{n_{01}+n_{10}-n_{11}} ([a_0, b_1] \otimes A^{2(n+m)+1}, \\
[a_0 \otimes A^{2n+1}, b_1 \otimes A^{2m+1}]_\varepsilon &= \varepsilon^{n_{01}+n_{11}-n_{10}} ([a_0, b_1] \otimes A^{2(n+m)+2} - e[a_0, b_1] \otimes A^{2(n+m)-2}), \\
[a_1 \otimes A^{2n}, b_1 \otimes A^{2m}]_\varepsilon &= \varepsilon^{2n_{10}-n_{00}} [a_1, b_1] \otimes A^{2(n+m)}, \\
[a_1 \otimes A^{2n}, b_1 \otimes A^{2m+1}]_\varepsilon &= \varepsilon^{n_{01}+n_{11}-n_{01}} [a_1, b_1] \otimes A^{2(n+m)+1}, \\
[a_1 \otimes A^{2n+1}, b_1 \otimes A^{2m+1}]_\varepsilon &= \varepsilon^{2n_{11}-n_{00}} ([a_1, b_1] \otimes A^{2(n+m)+2} - e[a_1, b_1] \otimes A^{2(n+m)-2}).
\end{aligned}$$

If we choose $n_{00} = 0$, $n_{01} = 1 = n_{10}$ and $n_{11} = 2$, then these relations become, in the limit where ε approaches zero:

$$\begin{aligned}
[a_0 \otimes A^{2n}, b_0 \otimes A^{2m}]' &= [a_0, b_0] \otimes A^{2(n+m)}, \\
[a_0 \otimes A^{2n}, b_0 \otimes A^{2m+1}]' &= [a_0, b_0] \otimes A^{2(n+m)+1}, \\
[a_0 \otimes A^{2n}, b_1 \otimes A^{2m}]' &= [a_0, b_1] \otimes A^{2(n+m)}, \\
[a_0 \otimes A^{2n}, b_1 \otimes A^{2m+1}]' &= [a_0, b_1] \otimes A^{2(n+m)+1}, \\
[a_0 \otimes A^{2n+1}, b_0 \otimes A^{2m+1}]' &= 0, \\
[a_0 \otimes A^{2n+1}, b_1 \otimes A^{2m}]' &= [a_0, b_1] \otimes A^{2(n+m)+1}, \\
[a_0 \otimes A^{2n+1}, b_1 \otimes A^{2m+1}]' &= 0, \\
[a_1 \otimes A^{2n}, b_1 \otimes A^{2m}]' &= 0, \\
[a_1 \otimes A^{2n}, b_1 \otimes A^{2m+1}]' &= 0, \\
[a_1 \otimes A^{2n+1}, b_1 \otimes A^{2m+1}]' &= 0.
\end{aligned}$$

If we consider $n_{00} = 1$, $n_{01} = 1 = n_{11}$ and $n_{10} = 2$, then all the terms above become equal to zero, except

$$[a_0 \otimes A^{2n+1}, b_1 \otimes A^{2m+1}]' = [a_0, b_1] \otimes A^{2(n+m)+2} - e[a_0, b_1] \otimes A^{2(n+m)-2}.$$

Clearly, these relations describe Lie algebras non-isomorphic to the original Krichever-Novikov algebra (15).

Many more infinite dimensional Lie algebras may be obtained if we take other values of the n_μ 's. Evidently, one may begin also with decompositions different from equation (18). It is interesting to note that when the Laurent polynomials are graded by another group than \mathbb{Z}_2 , then the behaviour of different terms in the second line of equations (14) and (15) will be different, and thus the contracted Lie algebras will have a richer structure. A systematic investigation of such contractions deserves further study.

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