

Classification of Graded Lie Algebras with Two Generators

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This article considers infinite-dimensional Lie algebras over a field of characteristic 0 with basis e_1, e_2, \dots which satisfy the condition

$$[e_i, e_j] = c_{ij}e_{i+j}.$$

A complete description is given of such algebras with two generators. In particular, it follows from the proposed classification that if the number of independent relations between the generators of a Lie algebra of this type is finite, then it is equal to 2.

In this paper we classify the graded Lie algebras $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ over a field \mathbb{K} of characteristic 0, for which $\dim \mathfrak{g}_i = 1$, with minimum possible number of generators. Obviously this number is equal to 2.

There are three well-known Lie algebras of the above type: L_1, n_1, n_2 . The Lie algebra L_1 consists of vector fields on the real line with polynomial coefficients which vanish together with their first derivative at the coordinate origin [1]. The algebras n_1 and n_2 are maximal nilpotent subalgebras in Kac-Moody algebras $A_1^{(1)}$ and $A_2^{(2)}$, respectively [2].

The problem of classification of algebras such as L_1, n_1, n_2 is naturally related to the problem formulated by V. Kac in [3], which involves the classification of all simple graded Lie algebras $\mathcal{L} = \bigoplus_{i \in \mathbb{Z}} \mathcal{L}_i$, where $\dim \mathcal{L}_i = 1$.

In the algebra $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ we choose a basis of homogeneous elements $e_i \in \mathfrak{g}_i$. The generators of \mathfrak{g} are e_1 and e_2 . Note that $[e_1, e_2] \neq 0$ and $[e_1, [e_1, e_2]] \neq 0$. We specify the explicit form of the commutator in the algebras L_1, n_1, n_2 as follows.

$$L_1 : [e_i, e_j] = (j - i)e_{i+j},$$

$$n_1 : [e_i, e_j] = a_{ij}e_{i+j}, \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if } j - i \equiv 1 \pmod{3} \\ 0 & \text{if } j - i \equiv 0 \pmod{3} \\ -1 & \text{if } j - i \equiv -1 \pmod{3} \end{cases},$$

$$n_2 : [e_i, e_j] = b_{ij}e_{i+j}, \quad \text{where the numbers } b_{ij} \text{ depend only on the residue obtained when dividing } i \text{ and } j \text{ by 8 according to the rule } b_{ij} + b_{i',j'} = 0 \text{ if the numbers } i + i' \text{ and } j + j' \text{ are divisible by 8. The accompanying table gives the numbers } b_{ij} \text{ (the remaining } b_{ij} \text{ are determined from the relations } b_{ij} = -b_{ji} \text{ and } b_{ij} + b_{8-i,8-j} = 0.$$

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		$j \bmod 8$						
		1	2	3	4	5	6	7
$i \bmod 8$	0	1	-2	-1	0	1	2	-1
	1	0	1	-1	3	-2	0	1
	2	-1	0	0	0	1	-1	0
	3	1	0	0	-3	1	1	-2

In addition to these three algebras, we will need two particular algebras and also a special family of algebras. These are:

\mathfrak{m}_1 : The algebra in which $[e_1, e_i] = e_{i+1}$ for $i > 1$ and $[e_i, e_j] = 0$ for $i, j > 1$,

\mathfrak{m}_2 : The algebra in which the commutator is set up as follows: $[e_i, e_j] = 0$ for $i, j > 2$, while $[e_1, e_j] = e_{j+1}$ for $j \geq 2$ and $[e_2, e_j] = e_{j+2}$ for $j > 2$.

$\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \dots)$: A family of Lie algebras with countably many parameters $\lambda_{4k} \in \mathbb{K}\mathbb{P}^1$. The commutator is defined as follows: $[e_1, e_4] = 0$, $[e_3, e_4] = 0$, $[e_i, e_j] = 0$ if i is even but not 2 and j is any positive integer. Furthermore,

$$[e_1, e_{4k-1}] = \alpha_{4k} e_{4k}, \quad \text{and} \quad [e_2, e_{4k-2}] = \beta_{4k} e_{4k}, \quad k = 2, 3, 4, \dots,$$

where the α_{4k} and β_{4k} are the homogeneous coordinates of the point $\lambda_{4k} \in \mathbb{K}\mathbb{P}^1$. The remaining commutators can be uniquely reconstructed from the above formulas. Their structural constants are homogeneous polynomials of α_{4k} and β_{4k} . See the Appendix for some explicit formulas for the commutators.

EXAMPLE: For the algebra $\mathfrak{g}(1, 1, 1, \dots)$ the commutators are

$$\begin{aligned} [e_1, e_2] &= e_3, & [e_1, e_3] &= e_4 \\ [e_1, e_{2k+1}] &= [e_2, e_{2k}] = e_{2k+2} & \text{if } k \geq 2 \\ [e_2, e_{2k-1}] &= e_{2k+1} & \text{if } k \geq 2 \\ \text{the other commutators} & \text{are } 0. \end{aligned}$$

Theorem Let $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ be an \mathbb{N} -graded Lie algebra, where $\dim \mathfrak{g}_i = 1$, with basis e_1, e_2, e_3, \dots , generated by e_1 and e_2 . Then \mathfrak{g} is one of the following.

- Assume $[e_1, e_4] \neq 0$ and $[e_2, e_3] \neq 0$. If $[e_3, e_4] \neq 0$, then $\mathfrak{g} \cong L_1$ while if $[e_3, e_4] = 0$, then $\mathfrak{g} \cong \mathfrak{m}_2$.
- Assume $[e_2, e_3] = 0$. If $[e_3, e_4] \neq 0$, then $\mathfrak{g} \cong \mathfrak{n}_2$ while if $[e_3, e_4] = 0$, then $\mathfrak{g} \cong \mathfrak{m}_1$.
- Assume $[e_1, e_4] = 0$. If $[e_3, e_4] \neq 0$, then $\mathfrak{g} \cong \mathfrak{n}_1$ while if $[e_3, e_4] = 0$, then $\mathfrak{g} \cong \mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \dots)$ for some choice of the $\lambda_8, \lambda_{12}, \lambda_{16}, \dots$ ■

PROOF: (sketch) The Jacobi identity yields a system of linear equations for the usual structural constants c_{ij} ($[e_i, e_j] = c_{ij}e_{i+j}$). The numbers c_{12} and c_{13} can be regarded as arbitrary but non-zero. To be specific, assume that $c_{12} = 1$. Then the coefficients c_{14} and c_{23} can no longer be chosen arbitrarily since they must satisfy a linear equation. We fix some solution of this equation after which c_{15} and c_{24} are determined uniquely. At the next step we obtain two equations for c_{16} , c_{25} and c_{34} , etc.

As an example we consider the case in which $c_{14} \neq 0$, $c_{23} \neq 0$, and $c_{34} \neq 0$. We can assume that $c_{13} = 2$, $c_{23} = c_{34} = 1$. Solving this system step-by-step we find that if $c_{14}c_{25} \neq 9$, then the system of equations for c_{ij} with $i + j = 16$ does not have a non-zero solution. If, however, $c_{14}c_{25} = 9$, then all the equations can be solved non-trivially and uniquely so we obtain an algebra that is isomorphic to L_1 .

We can similarly consider the cases in which some of the numbers c_{14}, c_{23} , and c_{34} are zero.

Of interest are the relations that link the generators e_1 and e_2 . It is easy to show that these generators should satisfy at least two independent relations of weights 5 and 7 of the form

$$\begin{aligned} \lambda[e_1, [e_1, [e_1, e_2]]] + \mu[e_2, [e_2, e_1]] &= 0, \\ \alpha[e_1, [e_1, [e_1, [e_1, [e_1, e_2]]]]] + \beta[e_2, [e_2, [e_2, e_1]]] + \gamma[e_1, [e_1, [e_2, [e_2, e_1]]]] &= 0. \end{aligned} \quad (*)$$

In the algebras L_1 , \mathfrak{n}_1 , \mathfrak{n}_2 , and \mathfrak{m}_2 , the relations (*) make up a complete system of defining relations while for the algebras \mathfrak{m}_1 and $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \dots)$ this system is infinite. Specifically, in the algebra \mathfrak{m}_1 we should add relations of weights 13, 17, 21, ... to (*); the relations between generators in the algebras $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \dots)$ are not yet computed.

Corollary 1 *If the number of relations between the generators of the Lie algebra \mathfrak{g} is finite, then \mathfrak{g} is isomorphic to one of the four algebras L_1 , \mathfrak{n}_1 , \mathfrak{n}_2 , and \mathfrak{m}_2 , and the number of relations is 2.*

Let us consider a Lie algebra with generators e_1 and e_2 and relations (*). It turns out that for most points $(\lambda, \mu, \alpha, \beta, \gamma) \in \mathbb{K}^5$ this algebra is finite-dimensional. One can compute the following. If $6\lambda = \mu$, $120\alpha = 3\beta + 20\gamma$, then the Lie algebra is isomorphic to L_1 ; if $\mu = 0$, $\lambda = 0$, then it is isomorphic to \mathfrak{n}_1 ; if $\lambda = 0$, $\gamma = 0$, it is isomorphic to \mathfrak{n}_2 ; and if $\lambda = \mu$, $\alpha = \beta + \gamma$, it is isomorphic to \mathfrak{m}_2 . If, however, $\lambda = 0$, $\alpha = 0$ or $\mu = 0$, $\beta = 0$, then the dimension of the space of weight i in the Lie algebra increases exponentially with i , and adding additional relations converts this algebra into one of the algebras of the family $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \dots)$.

In conclusion, we should note that the cohomology of the Lie algebras L_1 , \mathfrak{n}_1 and \mathfrak{n}_2 with trivial coefficients are known [4], [5]. In all cases, for $i > 0$ the i -th cohomology space is two-dimensional. It would be interesting to calculate the cohomology of the other algebras considered in this article.

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Appendix

(added June 1997)

Here are some details for the structural constants of the algebras $\mathfrak{g}(\lambda_8, \lambda_{12}, \lambda_{16}, \dots)$. We compute $[e_i, e_j]$ for all levels $i + j \leq 17$. For these levels, we list only the (generically) non-zero brackets. Using a computer one can extend these much further.

$$\begin{aligned}
i + j = 3: & \quad [e_1, e_2] = e_3 \\
i + j = 4: & \quad [e_1, e_3] = e_4 \\
i + j = 5: & \quad [e_2, e_3] = e_5 \\
i + j = 6: & \quad [e_1, e_5] = [e_2, e_4] = e_6 \\
i + j = 7: & \quad [e_2, e_5] = e_7 \\
i + j = 8: & \quad [e_1, e_7] = \alpha_8 e_8, \quad [e_2, e_6] = \beta_8 e_8, \quad [e_3, e_5] = (\alpha_8 - \beta_8) e_8 \\
i + j = 9: & \quad [e_2, e_7] = e_9 \\
i + j = 10: & \quad [e_2, e_8] = e_{10}, \quad [e_1, e_9] = (2\alpha_8 - \beta_8) e_{10}, \quad [e_3, e_7] = (\alpha_8 - \beta_8) e_{10} \\
i + j = 11: & \quad [e_2, e_9] = e_{11} \\
i + j = 12: & \quad [e_1, e_{11}] = \alpha_{12} e_{12}, \quad [e_2, e_{10}] = \beta_{12} e_{12} \\
& \quad [e_3, e_9] = (\alpha_{12} + (\beta_8 - 2\alpha_8)\beta_{12}) e_{12}, \\
& \quad [e_5, e_7] = ((3\alpha_8 - 2\beta_8)\beta_{12} - \alpha_{12}) e_{12} \\
i + j = 13: & \quad [e_2, e_{11}] = e_{13} \\
i + j = 14: & \quad [e_2, e_{12}] = e_{14}, \\
& \quad [e_1, e_{13}] = (3\alpha_{12} + (3\beta_8 - 5\alpha_8)\beta_{12}) e_{14}, \\
& \quad [e_3, e_{11}] = (2\alpha_{12} + (3\beta_8 - 5\alpha_8)\beta_{12}) e_{14}, \\
& \quad [e_5, e_9] = ((3\alpha_8 - 2\beta_8)\beta_{12} - \alpha_{12}) e_{14} \\
i + j = 15: & \quad [e_2, e_{13}] = e_{15} \\
i + j = 16: & \quad [e_1, e_{15}] = \alpha_{16} e_{16}, \quad [e_2, e_{14}] = \beta_{16} e_{16} \\
& \quad [e_3, e_{13}] = (\alpha_{16} - 3\beta_{16}\alpha_{12} - \beta_{16}\beta_{12}(3\beta_8 - 5\alpha_8)) e_{16} \\
& \quad [e_5, e_{11}] = (-\alpha_{16} + 5\beta_{16}\alpha_{12} + 2\beta_{16}\beta_{12}(3\beta_8 - 5\alpha_8)) e_{16} \\
& \quad [e_7, e_9] = (\alpha_{16} - 6\beta_{16}\alpha_{12} + \beta_{16}\beta_{12}(13\alpha_8 - 8\beta_8)) e_{16} \\
i + j = 17: & \quad [e_2, e_{15}] = e_{17}
\end{aligned}$$