EXTENSIONS OF ASSOCIATIVE ALGEBRAS

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ABSTRACT. In this paper, we translate the problem of extending an associative algebra by another associative algebra into the language of codifferentials. The authors have been constructing moduli spaces of algebras and studying their structure by constructing their versal deformations. The codifferential language is very useful for this purpose. The goal of this paper is to express the classical results about extensions in a form which leads to a simpler construction of moduli spaces of low dimensional algebras.

1. Introduction

The authors have been studying moduli spaces of low dimensional $\mathbb{Z}_2$-graded L_∞ and A_∞ algebras. In [2], we gave a constructive method for computing versal deformations of infinity algebras, and have been implementing this construction in a series of articles where we have computed moduli spaces of infinity algebras, and then used a codifferential approach to explore how this moduli space is glued together through versal deformations. Along the way, we noticed certain patterns in the moduli spaces.

First, we discovered that the moduli spaces of $1|n$-dimensional L_∞ algebras can be constructed as extensions of a $1|0$-dimensional algebra by a $0|n$-dimensional algebra. In fact, these algebras are given by an infinity algebra module structure; but one needs a more general notion of an infinity algebra module structure than was given in [18] and others.

Secondly, when studying the $3|0$-dimensional complex L_∞ algebras, which are just the ordinary 3-dimensional complex Lie algebras, we noticed that the moduli space had a stratification by complex orbifolds [4, 20], and we verified that the same type of stratification occurs for 4-dimensional complex Lie algebras in [5]. This led us to a conjecture

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that finite dimensional complex Lie algebras have a certain natural stratification by complex orbifolds, which we have since been exploring.

For very low dimensions, one can construct the moduli space by simply solving the associativity relations. But even for ordinary 3-dimensional associative algebras (see [8, 9]), we already found that it was too difficult to implement this approach using the computer algebra system Maple, because the associativity conditions give a relatively large system of quadratic equations, which is a hard problem to solve in general.

A better approach is to construct the moduli space of n-dimensional algebras by using extensions, and we have successfully implemented this strategy to construct moduli spaces of associative algebras up to total dimension 4, and Lie algebras up to dimension 5. The main purpose of this paper is to express the standard theory of the construction of extensions of associative algebras by extensions in the language of codifferentials, because that is the language in which we have developed extensive computational tools using Maple worksheets.

The real aim of the authors is to lay out a theory of extensions of infinity algebras, by generalizing the results about associative algebras to the A_∞ case. There has been recent interest in the idea of extensions of infinity algebras, so we decided to include some discussion of the idea here.

The authors have been constructing moduli spaces of low dimensional A_∞ algebras, along with some undergraduate researchers. One interesting complication is that no classification of simple algebras exists. If one defines an algebra to be simple if it has no (graded) ideals, then we have found that in the A_∞ algebra case, some simple algebras have deformations. Our work on the classification of low dimensional A_∞ algebras is in the early stages. In the case of L_∞ algebras, we have been more successful, in some cases, being able to give a complete description of the moduli space [3].

2. Preliminaries

Extensions of Lie and associative algebras by ideals is a classical subject [17, 19], which has been recast in many forms and generalized extensively [16, 15], in terms of diagrams of algebras. Deformation theory of associative algebras is still an active subject of research [1].

Our goal in this paper is to recast the classical ideas in the modern language of codifferentials of coalgebras introduced in [22]. (A *codifferential* is simply an odd coderivation whose square is zero.) The goal is to describe the theory of extensions of associative algebras in a more constructive approach, because our ultimate aim is to use the extensions as a tool to construct moduli spaces of low dimensional algebras.

The authors have been studying moduli spaces of algebras in several recent papers, from the point of view of algebras as codifferentials on certain coalgebras. The modern language of codifferentials makes it possible to express the ideas involved in extensions in a more explicit form, which makes it easier to apply the theory in practice. In this paper, we will illustrate how to use the presentation of the main results by giving examples of the construction of moduli spaces of extensions. In [8, 9], the ideas presented here were used to construct moduli spaces of three-dimensional complex associative algebras.

In some recent works, [4, 5, 20], moduli spaces of low dimensional Lie algebras have been constructed and interpreted using versal deformations of the algebras. These versal deformations were constructed by analyzing the space of coderivations of the symmetric algebra of the underlying vector space, so giving a description of the theory of extensions in terms of codifferentials, as we do in this paper, makes it possible to use the computational tools we have already developed to study the moduli spaces of algebras more effectively.

The extension problem has a cohomological interpretation in terms of certain codifferentials arising from the algebra structures. We also give a classification of infinitesimal deformations of extensions in terms of a certain triple cohomology. Finally, we study the problem of deformations of representations of associative algebras, also in terms of cohomology. We also give a partial description of the generalization of these ideas to the A_∞ case, but plan to give a detailed description elsewhere.

The results in this paper have immediate applications to the construction of moduli spaces of associative algebras using extensions. The authors have been using Maple worksheets developed by one of the authors and his students, which calculate cohomology and deformations of associative algebras. The authors have already been using these results in conjunction with the Maple software to construct moduli spaces, and we expect that this software will eventually be used by others for similar calculations.

In section 3, we recall the definition of an extension in terms of coderivations. In section 5 we recall the notion of equivalence of extensions, giving a definition of a restricted equivalence in terms of commutative diagrams. In section 6 we classify infinitesimal extensions, and then in section 7 we classify the extensions of an algebra by a fixed bimodule structure. In section 8 we classify the extensions of an associative algebra by an ideal in terms of the restricted notion of equivalence, and then we go on to classify the extensions in terms of a more general notion of equivalence in section 9. In section 12 we give some simple examples illustrating the application of the classification in constructing moduli spaces of extensions. In section 13, we classify infinitesimal deformations of extensions and in section 14 we classify infinitesimal deformations of representations.

The multiplication map $m : V \otimes V \to V$ of a $\mathbb{Z}_2$-graded associative algebra is an even map. It is a standard result that if we let $W = \Pi V$ be the parity reversion of V, that is, $W_0 = V_1$ and $W_1 = V_0$, where V_i is the subspace of V consisting of elements of parity i, then the induced map $d : W \otimes W \to W$, given by $d = \pi \circ m \circ (\pi^{-1} \otimes \pi^{-1})$, is an odd codifferential on the tensor coalgebra $T^c(W)$, in other words, it is an odd coderivation satisfying $[d, d] = 0$. When V is a completely even space, representing a non-graded algebra, then W is completely odd, and the map d is just $-m$, so the multiplication in V can be read from the codifferential d easily. More generally, $d(\pi a, \pi b) = (-1)^{a+1} \pi(m(a, b))$, so that, in general, d and m differ only by a sign change.

3. Extensions of Associative Algebras

The algebras we will consider are associative algebras, not necessarily unital, defined over a field $\mathbb{K}$, which we will assume has characteristic not equal to 2, although the results on extensions can be adapted to this case as well, by replacing the condition $[d, d] = 0$ with the condition $d^2 = 0$.

We refer to an exact sequence of algebras

$$0 \to M \to V \to W \to 0. \tag{1}$$

as an *extension of the algebra W by the algebra M*.

From our point of view, W and M are equipped with codifferentials, but the form of the exact sequence is unaltered in this point of view. In fact, one can avoid the codifferential language by considering the space of cochains to be equipped with the *Gerstenhaber bracket*.

We introduce the following notation for certain subspaces of the tensor coalgebra $T(V) = \sum_{n=0}^{\infty} T^n(V)$ of $V = M \oplus W$.

$$\begin{aligned} T^{k,0}(M, W) =& T^k(M) \\ T^{0,l}(M, W) =& T^l(W) \\ T^{k,l}(M, W) =& M \otimes T^{k-1,l}(M, W) \oplus W \otimes T^{k,l-1}(M, W). \end{aligned}$$

In other words, $T^{k,l}(M, W)$ is the subspace of $T^{k+1}(V)$ spanned by tensors of k elements from M and l elements from W. We also introduce a notation for certain spaces of cochains $C(V) = \mathrm{Hom}(T(V), V)$ on V.

$$C^k = \mathrm{Hom}(T^k(W), W)$$
$$C^{k,l} = \mathrm{Hom}(T^{k,l}(M, W), M).$$

Recall that $C(V)$ is identifiable with the space $\mathrm{Coder}(T(V)$ of coderivations of the tensor coalgebra $T(V)$, which means that $C(V)$ has a $\mathbb{Z}_2$-graded Lie bracket. We shall sometimes refer to cochains in $C(V)$ as coderivations.

In terms of the induced bracket of cochains, we have

$$[C^k, C^l] \subseteq C^{k+l-1}$$
$$[C^{k,l}, C^{r,s}] \subseteq C^{k+r-1,l+s}$$
$$[C^k, C^{r,s}] \subseteq C^{r,k+s-1}.$$

The algebra structure on V is determined by the following odd cochains:

$\delta \in C^2 = \mathrm{Hom}(W^2, W):$	the algebra structure on W
$\psi \in C^{0,2} = \mathrm{Hom}(W^2, M):$	the "cocycle" with values in M
$\lambda \in C^{1,1} = \mathrm{Hom}(WM \oplus MW, M):$	the "bimodule" structure on M
$\mu \in C^{2,0} = \mathrm{Hom}(M^2, M):$	the algebra structure on M

The fact that d has no terms from $\mathrm{Hom}(WM \oplus MW \oplus M^2, W)$ reflects the fact that M is an ideal in V. The associativity relation on V is that the odd coderivation

$$d = \delta + \lambda + \rho + \mu + \psi$$

is an odd *codifferential* on $T(V)$, which simply means that $[d, d] = 0$. Now, in general, we see that $[d, d] \in \mathrm{Hom}(V^3, V)$. By decomposing this space and considering which parts the brackets of the terms δ, λ, μ and ψ are defined on, we obtain

(2) $[\delta, \delta] = 0:$ The algebra structure δ on W is associative.

(3) $[\mu, \mu] = 0:$ The algebra structure μ on M is associative.

(4) $[\delta, \lambda] + 1/2[\lambda, \lambda] + [\mu, \psi] = 0:$ Module Relations.

(5) $[\mu, \lambda] = 0:$ Module-Algebra compatibility relations.

(6) $[\delta + \lambda, \psi] = 0:$ ψ is a "cocycle" with values in M.

We also have the relations $[\mu, \delta] = [\psi, \psi] = 0$, which follow automatically, and therefore are not conditions, *per se*, on the structure d.

When $[\mu, \psi] = 0$, condition (4) is called the Maurer-Cartan formula (MC formula), implying that $\delta + \lambda$ is a codifferential, and in that case, condition (6) is simply the condition that ψ is a cocycle with respect to this codifferential. In general, ψ is not really a cocycle, because $\delta+\lambda$ is not a codifferential. We shall refer to condition (4) as the MC condition, although this terminology is not precisely correct.

When neither μ nor ψ vanish, then in general, λ is not a bimodule structure. However, we can still interpret the conditions in terms of a MC formula as follows. The sum of the two algebra structures $\delta+\mu$ is a codifferential on V, and with respect to this structure $\lambda+\psi$ satisfies an MC formula; *i.e.*,

$$[\delta+\mu,\lambda+\psi]+\tfrac{1}{2}[\lambda+\psi,\lambda+\psi]=0. \tag{7}$$

Thus the combined structure $\lambda+\psi$ plays the same role as the module structure plays in the simpler case. The moduli space of all extensions of the algebra structure δ on W by the algebra structure μ on W is given by the solutions to the MC formula (7). This point of view is useful if we consider extensions where μ is assumed to be a fixed algebra structure on M.

We can adopt a different point of view by noticing that $\mu+\lambda+\psi$ satisfies an MC formula with respect to the codifferential δ; *i.e.*, that

$$[\delta,\mu+\lambda+\psi]+\tfrac{1}{2}[\mu+\lambda+\psi,\mu+\lambda+\psi]=0. \tag{8}$$

This formulation is useful if we are interested in studying the moduli space of all extensions of W by M, where we don't assume any fixed multiplication structure μ on M.

All these facts are well-known, for example see [10, 11, 12, 13, 14, 19]. Our purpose in making them explicit here is to cast the ideas in the language of codifferentials. We summarize the main results in the theorem below.

Theorem 3.1. *Let δ be an associative algebra structure on W and μ be an associative algebra structure on M. Then $\lambda\in C^{1,1}$ and $\psi\in C^{0,2}$ determine an associative algebra structure on $M\oplus W$ precisely when the following conditions hold.*

$$\begin{aligned}[\delta,\lambda]+\tfrac{1}{2}[\lambda,\lambda]+[\mu,\psi]&=0\\ [\mu,\lambda]&=0\\ [\delta+\lambda,\psi]&=0\end{aligned}$$

4. Extensions of A_∞ algebras

We use the same notation as in the previous section. An A_∞ algebra structure on W is an odd element of $C^\bullet=\operatorname{Hom}(T(W),W)$, and can be expressed as a sum of an infinite number of terms,

$$\delta = \delta^1 + \delta^2 + \cdots ,$$

where $\delta^i \in C^i$. The condition for δ to determine an A_∞ algebra structure is still $[\delta, \delta] = 0$. Similarly, an odd $\mu \in C^{\bullet,0} = \mathrm{Hom}(T(M), M)$ determines an A_∞ structure on M when $[\mu, \mu] = 0$.

It is less useful to separate the extension into λ and ψ terms, because we cannot separate the conditions for an extension as in the previous section. If we let $\lambda \in C^{*,*+1}$, then an extension of the A_∞ algebra structure δ on W by the A_∞ structure μ on M is an odd codifferential $d = \delta + \mu + \lambda$, satisfying $[d, d] = 0$.

There is, however, a MC-equation which applies here. Note that $\delta + \mu$ is a codifferential, and that $[d, d] = 0$ is equivalent to

$$[\delta + \mu, \lambda] + \tfrac{1}{2}[\lambda, \lambda] = 0.$$

When $\mu = 0$, λ determines the structure of a *module* over the infinity algebra given by δ. It is probably more traditional in this case to assume $\lambda^{0,k} = 0$ for all k, to conform with the definition of a module in the associative case. In other words, if we let $\psi = \sum_{k=1}^{\infty} \lambda^{0,k}$, then we are assuming that $\psi = 0$ for λ to be a module structure.

Definition 4.1. Let δ be an A_∞-algebra sructure on W, and $\lambda \in C^{\bullet+1,\bullet+1}$ on the space $V = M \oplus W$. Then λ determines a *module* structure on M over δ if the Maurer-Cartan equation

$$[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] = 0$$

is satisfied.

The definition of a module structure above generalizes the definition given in [18] and is more natural in the context of the study of extensions.

Because of the separability of the condition for extensions of associative algebras into three separate conditions, it is much easier to classify the extensions of associative algebras than the corresponding problem for A_∞ algebras.

If we let $d^n = \delta^n + \mu^n + \sum_{k=1}^{n} \lambda^{n-k,k}$, then the condition $[d, d] = 0$ is equivalent to the infinite sequence of relations

$$\sum_{k+l=n+1} [d^k, d^l] = 0, \quad n = 1, \ldots .$$

Now we have

$$[\mu^i, \lambda^{j,n}] \in C^{i+j-1,n}, \quad [\lambda^{i,k}, \lambda^{j,l}] \in C^{i+j-1,k+l}, \quad [\delta^k, \lambda^{m,l}] \in C^{m,k+l-1}.$$

Therefore, we obtain that λ determines an extension precisely when the infinite set of relations

$$\sum_{i+j=m+1}[\mu^i,\lambda^{j,n}]+\sum_{i+j=m+1,k+l=n}[\lambda^{i,k},\lambda^{j,l}]+\sum_{k+l=n+1}[\delta^k,\lambda^{m,l}]=0,$$

In constructing the moduli space of A_∞ structures on a vector space V, the first step in the classification is to classify the codifferentials of the form $d^k \in C^k(V)$, in other words, those codifferentials of a fixed exterior degree. Then one constructs the equivalence classes of "extensions" of d^k to a codifferential $d = d^k + d^{k+1} + \cdots$. In fact, in any codifferential d, the lowest order nontrivial term d^k is a codifferential, and if we define $D(\varphi) = [d^k, \varphi]$, then $D^2 = 0$ and the condition $[d, d] = 0$ can be recast in the form

$$D(d^n)=\sum_{i=k+1}^{n-1}[d^i,d^{n+k-i}],\quad n=k+1,\dots. \tag{9}$$

Note that the sum on the right only involves terms of order lower than n. Suppose that $d^{k+1},\cdots,d^{n-1}$ have been constructed to satisfy the relations above. Then, as is usual in deformation theory, one can show that the right hand side of equation (9) can be shown to be a D-cocycle, and therefore the existence of a d^n extending the relations is equivalent to that cocycle being trivial. This is precisely the type of problem which is solved in deformation theory.

As a consequence, the extension problem for A_∞ algebras can be separated into two parts. The first part is to determine the equivalence classes of extensions of a codifferential δ^k on W by a codifferential μ^k on M. This problem is similar to the classical problem of extensions of associative algebras, gives an extension d^k of δ^k by μ^k. The second problem, to extend d^k to a codifferential $d = d^k + d^{k+1} + \cdots$, is more like a problem in deformation theory.

In [6], the problem of extending an L_∞ algebra given by a codifferential of exterior degree 2 to a general codifferential was studied.

In the examples below, we consider two cases of extensions of an infinity algebra of a fixed degree n by a infinity algebra of the same degree, to yield an infinity algebra of the same degree. In other words, we have $\delta \in C^n$, $\mu \in C^{n,0}$ and $\lambda \in \sum_{l=1}^{n} C^{n-l,l}$.

Example 1. If $n = 1$, then we have $\delta = \delta^1 \in C^1$, $\mu = \mu^1 \in C^{1,0}$ and $\lambda = \lambda^{0,1} \in C^{0,1}$. The condition for λ to determine an extension is just $[\delta + \mu, \lambda] = 0$. In other words, λ is a cocycle with respect to the codifferential $\delta + \mu$.

In some sense, the study of extensions of a degree 1 codifferential δ is not very interesting, because all codifferentials d on a space of total dimension at least 2 arise in this fashion in a more or less trivial manner. For example, one can choose $M = d(V)$, and W to be a complementary subspace. Since $d^2 = 0$, this yields a decomposition of the space V of the required sort.

The classification of codifferentials of degree 1 on a $\mathbb{Z}_2$-graded space V is just the same as the classifications of differentials on a $\mathbb{Z}_2$-graded vector space, so is very easy.

Example 2. If $n = 3$, then $\lambda = \lambda^{2,1} + \lambda^{1,2} + \lambda^{0,3}$. The condition for λ to determine an extension is equivalent to the five conditions

$$\begin{aligned}
&[\mu^3, \lambda^{2,1}] = 0\\
&\tfrac{1}{2}[\lambda^{2,1}, \lambda^{2,1}] + [\mu^3, \delta^3 + \lambda^{1,2}] = 0\\
&[\lambda^{2,1}, \delta^3 + \lambda^{1,2}] + [\mu^3, \lambda^{0,3}] = 0\\
&\tfrac{1}{2}[\delta^3 + \lambda^{1,2}, \delta^3 + \lambda^{1,2}] + [\lambda^{2,1}, \lambda^{0,3}] = 0\\
&[\delta^3 + \lambda^{1,2}, \lambda^{0,3}] = 0
\end{aligned}$$

In fact, it is not difficult to see that in general, that λ consists of n terms $\lambda^{n-1,1}, \cdots, \lambda^{0,n}$, and that there are always $2n-1$ conditions that need to be satisfied.

5. Restricted Equivalence of Extensions

A (restricted) equivalence of extensions is given by a commutative diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0\\
 & & \| & & \downarrow f & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

where we assume that in the bottom row, V is equipped with the codifferential $d = \delta + \mu + \lambda + \psi$, and in the top row, it is equipped with the codifferential $d' = \delta + \mu + \lambda' + \psi'$, and f is a morphism of associative algebras (which is necessarily an isomorphism). The condition that f is an isomorphism is simply $d' = f^*(d) = f^{-1}df$. In order for the diagram to commute, we must have $f(m, w) = (m + \beta(w), w)$, where $\beta \in C^{0,1} = \operatorname{Hom}(W, M)$. We can also express $f = \exp(\beta)$, which is convenient because then $f^* = \exp(-\operatorname{ad}_\beta)$, so that

$$f^*(d) = \exp(-\operatorname{ad}_\beta)(d) = d + [d, \beta] + \tfrac{1}{2}[[d, \beta], \beta] + \cdots . \tag{10}$$

This series is actually finite, because $[[[d, \beta], \beta], \beta] = 0$. Moreover, $[\mu, \beta] \in C^{1,1}$, while $[\delta, \beta]$, $[\lambda, \beta]$ and $[\rho, \beta]$ all lie in $C^{0,2}$. The only nonzero term in $[[d, \beta], \beta]$ is $[[\mu, \beta], \beta]$, which also lies in $C^{0,2}$. It follows that $\lambda' = \lambda + [\mu, \beta]$ and $\psi' = \psi + [\lambda + \frac{1}{2}[[\mu, \beta], \beta]$. Thus we have shown

Theorem 5.1. *If $d = \delta + \mu + \lambda + \psi$ and $d' = \delta + \mu + \lambda' + \psi'$ are two extensions of an associative algebra structure δ on W by an associative algebra structure μ on M, then they are equivalent (in the restricted sense) precisely when there is some $\beta \in \operatorname{Hom}(W, M)$ such that*

$$\lambda' = \lambda + [\mu, \beta] \tag{11}$$

$$\psi' = \psi + [[\delta + \lambda + \tfrac{1}{2}[\mu, \beta], \beta]. \tag{12}$$

We will denote the group of restricted equivalences by G_{rest}. Its elements consist of the exponentials of $\beta \in C^{0,1}$.

6. Infinitesimal extensions and infinitesimal equivalence

An infinitesimal extension of δ by μ is one of the form

$$d = \delta + \mu + t(\lambda + \psi),$$

where t is an infinitesimal parameter (*i.e.*, $t^2 = 0$).

The conditions for d to be an infinitesimal extension are

$$[\delta, \lambda] + [\mu, \psi] = 0 \tag{13}$$

$$[\mu, \lambda] = 0 \tag{14}$$

$$[\delta, \psi] = 0 \tag{15}$$

If $\alpha \in C(V)$, then denote $D_\alpha = \operatorname{ad}_\alpha$. When α is odd and $[\alpha, \alpha] = 0$, then $D_\alpha^2 = 0$, so D_α is called a *coboundary operator* on $C(V)$, and $H_\alpha = \ker(D_\alpha)/\operatorname{Im}(D_\alpha)$ is the *cohomology* induced by α. The image of the cocycles from C^k in H_α will be denoted by H^k_α, and similarly, the image of the cocycles from $C^{k,l}$ in H_α will be denoted by $H^{k,l}_\alpha$. An element ϕ such that $D_\alpha(\phi) = 0$ is called a *D_α-cocycle*. The bracket on $C(V)$ descends to a bracket on H_α, so H_α inherits the structure of a Lie superalgebra. Since δ, μ and ψ are all codifferentials, they determine coboundary operators. In general, $\delta + \lambda$ is not a codifferential, so $D_{\delta+\lambda}$ is not a coboundary operator.

Note that in the conditions for d to be an infinitesimal extension, there is a certain symmetry in the roles played by the codifferentials δ and μ, in the sense that if we interchange δ with μ, and ψ with λ, then the conditions remain the same. We have

$$D_\delta : C^k \to C^{k+1} \qquad D_\delta : C^{k,l} \to C^{k,l+1}$$

$$D_\mu : C^k \to 0 \qquad D_\mu : C^{k,l} \to C^{k+1,l}.$$

Since $[\delta, \mu] = 0$, it follows that D_δ and D_μ anticommute. As a consequence,

$$D_\mu : \ker(D_\delta) \to \ker(D_\delta),$$

so we can define the cohomology $H_\mu(\ker\delta)$ determined by the restriction of D_μ to $\ker(D_\delta)$. For simplicity, let us denote the cohomology class of a D_μ-cocycle φ by $\bar{\varphi}$. Let us consider a D_μ-cocycle λ in $C^{1,1}$. Then the existence of a $\psi \in C^{0,2}$ such that $[\delta,\lambda]+[\mu,\psi]=0$ and $[\delta,\psi]=0$ is equivalent to the assertion that $\overline{[\delta,\lambda]}=0$ in $H^{1,2}_\mu(\ker(\delta))$.

Note that even though the condition for the existence of a ψ depends explicitly on λ, rather than the cohomology class $\overline{\lambda}$, if such a ψ exists for a particular λ in $\overline{\lambda}$, then one exists for any element in $\overline{\lambda}$. This follows because if λ is replaced by $\lambda'=\lambda+[\mu,\beta]$ and ψ by $\psi'=\psi+[\delta,\beta]$, where $\beta\in C^{0,1}$, then we obtain a new codifferential $d'=\delta+\mu+t(\lambda'+\psi')$, which is *infinitesimally equivalent* to d. By infinitesimal equivalence, we mean an equivalence determined by an *infinitesimal automorphism* $f=\exp(t\beta)$, where $\beta\in C^{0,1}$. (Actually, this is a restricted version of infinitesimal equivalence. We will introduce a more general notion later.) Since $d'=f^*(d)$, it follows that d' satisfies the conditions for an infinitesimal extension.

Now consider a fixed D_μ-cocycle λ such that $\overline{[\delta,\lambda]}=0$ in $H^{1,2}_\mu(\ker(\delta))$, and choose some ψ such that $[\delta,\lambda]+[\mu,\psi]=0$. If $\psi'=\psi+\tau$ is another solution, then $[\mu,\tau]=0$ and $[\delta,\tau]=0$. Now $[\mu,\delta]=0$, so the D_μ-cohomology class $\bar{\delta}$ is defined. In the Lie superalgebra structure on H_μ, we have

$$[\bar{\alpha},\bar{\beta}]=\overline{[\alpha,\beta]}.$$

Since $[\bar{\delta},\bar{\delta}]=\overline{[\delta,\delta]}=0$, $\bar{\delta}$ determines a coboundary operator $D_{\bar{\delta}}$ on H_μ. Denote the cohomology of $D_{\bar{\delta}}$ by $H_{\mu,\delta}$, and the cohomology class of a $D_{\bar{\delta}}$-cocycle $\bar{\varphi}$ by $[\bar{\varphi}]$. Then $[\bar{\delta},\bar{\tau}]=0$, so τ determines a cohomology class $[\bar{\tau}]$.

On the other hand, suppose that $\bar{\tau}$ is any $D_{\bar{\delta}}$-cocycle. Then $[\bar{\delta},\bar{\tau}]=0$ implies that $[\delta,\tau]$ is a D_μ-coboundary. Since $[\delta,\tau]\in C^{0,3}$, this forces $[\delta,\tau]=0$. Thus, every $D_{\bar{\delta}}$-cocycle $\bar{\tau}$ determines an extension. In other words, $\tau\in C^{0,2}$ determines an extension precisely when $[\mu,\tau]=0$ and $[\delta,\tau]=0$.

We wish to determine when two extensions $d=\delta+\mu+t(\lambda+\psi+\tau)$ and $d'=\delta+\mu+t(\lambda+\psi+\tau')$ are infinitesimally equivalent. First, let us suppose that $[\overline{\tau'}]=[\bar{\tau}]$. Then $\overline{\tau'}=\bar{\tau}+[\bar{\delta},\bar{\alpha}]$, for some $\alpha\in C^{0,1}$. Since $\tau',\tau\in C^{0,2}$, which contains no D_μ-coboundaries, it follows that $\tau'=\tau+[\delta,\alpha]$. It is easy to see that this implies that $d'=\exp(t\alpha)^*(d)$. Thus elements of $[\bar{\tau}]$ give rise to infinitesimally equivalent extensions. The converse is also easy to see, so the equivalence classes of infinitesimal extensions determined by λ are classified by the $H^{0,2}_{\mu,\delta}$ cohomology classes $[\bar{\tau}]$.

We summarize these results in the following theorem.

Theorem 6.1. *The infinitesimal extensions of an associative algebra structure δ on W by an associative algebra structure μ on M are completely classified by the cohomology classes $\bar{\lambda} \in H^{1,1}_{\mu}$ which satisfy the formula*

$$\overline{[\delta, \lambda]} = 0 \in H^{1,2}_{\mu,\delta}(\ker(D_\delta))$$

together with the cohomology classes $[\bar{\tau}] \in H^{0,2}_{\mu,\delta}$.

7. Classification of extensions of an associative algebra by a bimodule

In this section we consider the special case of an extension of W by a bimodule structure λ on M. This means that $\mu = 0$, so the MC formula (4) reduces to the usual MC formula

$$[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] = 0.$$

Let us relate the definition of bimodule given here with the notion of left and right module structures. Since

$$C^{1,1} = \mathrm{Hom}(WM, M) \oplus \mathrm{Hom}(MW, M),$$

we can express $\lambda = \lambda_L + \lambda_R$, where $\lambda_L \in \mathrm{Hom}(WM, M)$ and $\lambda_R \in \mathrm{Hom}(MW, M)$. Then the MC formula above is equivalent to the three conditions on λ_L and λ_R below.

$[\delta, \lambda_L] + \frac{1}{2}[\lambda_L, \lambda_L] = 0$	λ_L is a left-module structure.
$[\delta, \lambda_R] + \frac{1}{2}[\lambda_R, \lambda_R] = 0$	λ_R is a right-module structure.
$[\lambda_L, \lambda_R] = 0$	The two module structures are compatible.

Thus our definition of a bimodule structure is equivalent to the usual notion of a bimodule.

Now, λ determines a bimodule structure precisely when $[\mu, \psi] = 0$, owing to (4) in the conditions for an extension. Thus $\bar{\psi}$ is well defined in H_μ. Since $[\mu, \delta + \lambda] = 0$, we can define $D_{\bar{\delta}+\bar{\lambda}}$ on H_μ. Moreover $D^2_{\bar{\delta}+\bar{\lambda}} = 0$, so we can define its cohomology $H_{\mu,\delta+\lambda}$ on H_μ. Now for a D_μ-cocycle $\psi \in C^{0,2}$, $[\delta+\lambda, \psi] = 0$ precisely when $[\bar{\delta}+\bar{\lambda}, \bar{\psi}] = 0$, because $[\delta+\lambda, \psi] \in C^{0,3}$, which contains no D_μ-coboundaries. Therefore, a D_μ-cocycle ψ determines an extension iff $\bar{\psi}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle.

On the other hand, $\overline{\psi'} \in [\bar{\psi}]$ iff $\psi' = \psi + [\delta + \lambda, \beta]$ for some $\beta \in C^{0,1}$. But this happens precisely in the case when the extensions determined by ψ and ψ' are equivalent (in the restricted sense). Thus we have shown the following theorem.

Theorem 7.1. *The extensions of δ by μ determined by a fixed bimodule structure λ are classified by the cohomology classes $[\bar{\psi}] \in H^{0,2}_{\mu}$.*

8. Classification of restricted equivalence classes of extensions

In this section, we assume that δ and μ are fixed associative algebra structures on M and W, respectively. We want to classify the equivalence classes of extensions under the action of the group G_{rest} of restricted equivalences given by exponentials of maps $\beta \in \mathrm{Hom}(W, M)$. First, note that δ is a D_μ-cocycle, and λ must be a D_μ-cocycle by condition (5), so they determine D_μ-cohomology classes $\bar{\delta}$ and $\bar{\lambda}$ in H_μ. If λ, ψ determine an extension, and $\lambda' \in \bar{\lambda}$, then λ', ψ' determine an equivalent extension, where λ' and ψ' are given by the formulas (11) and (12). Moreover, condition (4) yields the MC formula

$$[\bar{\delta}, \bar{\lambda}] + \tfrac{1}{2}[\bar{\lambda}, \bar{\lambda}] = 0, \tag{16}$$

which means that given a representative λ of a cohomology class $\bar{\lambda}$, there is a ψ satisfying (4) precisely when $\bar{\lambda}$ satisfies the MC-equation for $\bar{\delta}$, which is a codifferential in H_μ.

We also need ψ to satisfy condition (6); *i.e.*, $\psi \in \ker(D_{\delta+\lambda})$, which is not automatic. However, note that since $[\mu, \delta + \lambda] = 0$, $D_{\delta+\lambda}$ anticommutes with D_μ, which implies that D_μ induces a coboundary operator on $\ker(D_{\delta+\lambda})$. Because the triple bracket of any coderivation vanishes, $[\delta + \lambda, \delta + \lambda] \in \ker(D_{\delta+\lambda})$. As a consequence, we obtain that the existence of an extension with module structure λ is equivalent to the condition that $[\delta + \lambda, \delta + \lambda]$ is a D_μ-coboundary in the restricted complex $\ker(D_{\delta+\lambda})$. In other words, there is an extension with module structure λ precisely when $\overline{[\delta + \lambda, \delta + \lambda]} = 0$ in the restricted cohomology $H_\mu(\ker(D_{\delta+\lambda}))$.

Even though the complex $\ker(D_{\delta+\lambda})$ depends on λ, the existence of an extension with module structure λ depends only on the D_μ-cohomology class of λ. Thus the assertion that $\overline{[\delta + \lambda, \delta + \lambda]} = 0$ in $H_\mu(\ker(D_{\delta+\lambda}))$ depends only on $\bar{\lambda}$, and not on the choice of a representative. Of course, the ψ satisfying equation (4) does depend on λ. We encountered a similar situation when analyzing infinitesimal extensions, except that there, one had to consider only $H_\mu(\ker(\delta))$, instead of $H_\mu(\ker(D_{\delta+\lambda}))$.

If $\bar{\lambda}$ satisfies equation (16) in H_μ, then $D^2_{\bar{\delta}+\bar{\lambda}} = 0$, so we can define an associated cohomology, which we denote by $H_{\mu,\delta+\lambda}$. If $\bar{\varphi}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle, then denote its cohomology class in $H_{\mu,\delta+\lambda}$ by $[\bar{\varphi}]$. Note that equation (16) is satisfied whenever $\overline{[\delta + \lambda, \delta + \lambda]} = 0$ in $H_\mu(\ker(D_{\delta+\lambda}))$.

Now fix λ and ψ determining an extension. Suppose λ and ψ' also determines an extension, and let $\tau = \psi' - \psi$. Then it follows that $[\mu, \tau] = 0$ and $[\delta + \lambda, \tau] = 0$. Thus $\bar{\tau}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle. Since $\tau \in C^{0,2}$, the condition $D_{\bar{\delta}+\bar{\lambda}}(\bar{\tau}) = 0$ is equivalent to the conditions $[\mu, \tau] = 0$

and $[\delta+\lambda,\tau]=0$. Clearly, if $D_{\bar\delta+\bar\lambda}(\bar\tau)=0$, then $\psi'=\psi+\tau$ determines an extension. Thus the set of extensions with a fixed λ are determined by the $D_{\bar\delta+\bar\lambda}$-cocycles $\bar\tau$.

We wish to determine when two extensions $d=\delta+\mu+\lambda+\psi+\tau$ and $d=\delta+\mu+\lambda+\psi+\tau'$ are equivalent. If $\overline{\tau'}\in[\bar\tau]$, then $\overline{\tau'}=\bar\tau+[\bar\delta+\bar\lambda,\bar\beta]$, for some $\beta\in C^{0,1}$, and since $\tau\in C^{0,1}$ which contains no D_μ-coboundaries, $\tau'=\bar\tau+[\delta+\lambda,\beta]$. It follows that $d'=\exp(\beta)^*(d)$, so the extensions are equivalent. Conversely, if $d'=\exp(\beta)^*(d)$, then $[\mu,\beta]=0$ and $\tau'=\tau+[\delta+\lambda,\beta]$, so $\overline{\tau'}\in[\bar\tau]$.

Theorem 8.1. *The equivalence classes of extensions of the associative algebra structure δ on W by an associative algebra structure μ on M under the action of the group G_{rest} of restricted equivalences are completely classified by cohomology classes $\bar\lambda\in H^{1,1}_\mu$ which satisfy the condition*

$$\overline{[\delta+\lambda,\delta+\lambda]}=0\in H^{1,2}_\mu(\ker(D_{\delta+\lambda}))$$

together with the cohomology classes $[\bar\tau]\in H^{0,2}_{\mu,\delta+\lambda}$.

9. General Equivalence Classes of Extensions

In the standard construction of equivalence of extensions, we have assumed that the homomorphism $f:V\to V$ acts as the identity on M and W. We could consider a more general commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

where η and γ are isomorphisms. It is easy to see that under this circumstance, if d' is the codifferential on the top line, and d is the one below, then $\eta^*(\mu)=\mu'$ and $\gamma^*(\delta)=\delta'$. Therefore, if one is interested in studying the most general moduli space of all possible extensions of all codifferentials on M and W, where equivalence of elements is given by diagrams above, then for two extensions to be equivalent, μ' must be equivalent to μ as a codifferential on M, and δ' must be equivalent to δ as a codifferential on W, with respect to the action of the automorphism group $\mathbf{GL}(M)$ on M and $\mathbf{GL}(W)$ on W.

Thus, in classifying the elements of the moduli space, we first have to consider equivalence classes of codifferentials on M and W. As a consequence, after making such a choice, we need only consider diagrams which preserve μ and δ; in other words, we can assume that $\eta^*(\mu)=\mu$ and that $\gamma^*(\delta)=\delta$.

Next note that we can always decompose a general extension diagram into one of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \Vert & & \downarrow{\scriptstyle f} & & \Vert & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle g=(\eta,\gamma)} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & M & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0
\end{array}$$

where $f = \exp(\beta)$, and $g = (\eta, \gamma)$ is an element of the group G_Δ consisting of block diagonal matrices. The group G_{gen} of general equivalences is just the group of block upper triangular matrices, and is the semidirect product of G_{rest} with G_Δ; that is, $G_{\text{gen}} = G_{\text{rest}} \rtimes G_\Delta$. In fact, if $g \in G_\Delta$, then $g^{-1}\exp(\beta)g = \exp(g^*(\beta))$.

The group G_Δ acts in a simple manner on cochains. If $g \in G_\Delta$, then $g^*(C^{k,l}) \subseteq C^{k,l}$ and $g^*(C^k) \subseteq C^k$. Since $g^*D_\mu = D_{g^*(\mu)}g^*$, the action induces a map

$$g^* : H_\mu \to H_{g^*(\mu)},$$

given by $g^*(\bar{\varphi}) = \overline{g^*(\varphi)}$. Similarly, $g^*D_{\delta+\lambda} = D_{g^*(\delta)+g^*(\lambda)}g^*$, so we obtain a map

$$g^* : H_{\mu,\delta+\lambda} \to H_{g^*(\mu),g^*(\delta)+g^*(\lambda)},$$

given by $g([\bar{\varphi}]) = [\overline{g^*(\varphi)}]$.

Let $G_\Delta(\mu, \delta)$ be the subgroup of G_Δ consisting of those elements g satisfying $g^*(\mu) = \mu$ and $g^*(\delta) = \delta$. Then $G_\Delta(\mu, \delta)$ acts on H_μ, and induces a map $H_{\mu,\delta+\lambda} \to H_{\mu,\delta+g^*(\lambda)}$. Let $G_\Delta(\mu, \delta, \lambda)$ be the subgroup of g in $G_\Delta(\mu, \delta)$ such that $g^*(\lambda) = \lambda$. Thus $G_\Delta(\mu, \delta, \lambda)$ acts on both on H_μ and $H_{\mu,\delta+\lambda}$.

It is easy to study the behaviour of elements in $G_\Delta(\mu, \delta)$ on extensions. If λ gives an extension and $g \in G_\Delta(\mu, \delta)$, then any element $\lambda' \in g^*(\bar{\lambda})$ will determine an equivalent extension, and thus equivalence classes of $\bar{\lambda}$ under the action of the group $G_\Delta(\mu, \delta)$ correspond to equivalent extensions.

Now suppose that λ, ψ gives an extension, and $\bar{\tau}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle. If $g \in G_\Delta(\mu, \delta, \lambda)$, then

$$g^*(\psi + \tau) = \psi + g^*(\psi) - \psi + g^*(\tau),$$

so that

$$[\bar{\tau}] \mapsto [\overline{g^*(\psi) - \psi + g^*(\tau)}]$$

determines an action of $G_\Delta(\mu, \delta, \lambda)$ on $H_{\mu,\delta+\lambda}$ whose equivalence classes determine equivalent representations.

To understand the action of G_{gen} on extensions, first note that any element $h \in G_{\mathrm{gen}}$ can be expressed uniquely in the form $h = g\exp(\beta)$ where $g \in G_\Delta$. If $d' = h^*(d)$, for an extension d, then we compute the components of the extension d' as follows.

$$\begin{aligned}
\delta' &= g^*(\delta)\\
\mu' &= g^*(\mu)\\
\lambda' &= g^*(\lambda) + [\mu', \beta]\\
\psi' &= g^*(\psi) + [\delta' + \lambda' - \tfrac{1}{2}[\mu', \beta], \beta].
\end{aligned}$$

Clearly, $\delta' = \delta$ and $\mu' = \mu$ precisely when $g \in G_\Delta(\mu, \delta)$. Define the group $G_{\mathrm{gen}}(\mu, \delta)$ to be the subgroup of G_{gen} consisting of those $h = g\exp(\beta)$ such that $g^*(\delta) = \delta$ and $g^*(\mu) = \mu$. We have a simple decomposition $G_{\mathrm{gen}}(\mu, \delta) = G_\Delta(\mu, \delta) \rtimes G_{\mathrm{rest}}$.

Define $G_{\mathrm{gen}}(\mu, \delta, \lambda)$ to be the subgroup of $G_{\mathrm{gen}}(\mu, \delta)$ consisting of those h such that $\lambda = g^*(\lambda) + [\mu, \beta]$, then $G_{\mathrm{gen}}(\mu, \delta, \lambda)$ does not have a simple decomposition in terms of $G_\Delta(\mu, \delta, \lambda)$, because the condition $\lambda' = \lambda$ does not force $g \in G_\Delta(\mu, \delta, \lambda)$. However, we can still define an action of $G_{\mathrm{gen}}(\mu, \delta, \lambda)$ on $H^{0,2}_{\mu,\delta+\lambda}$ by

$$[\bar\tau] \to [\overline{g^*(\psi) - \psi + g^*(\tau) + [\delta + \lambda - \tfrac{1}{2}[\mu, \beta], \beta]}],$$

whose equivalence classes determine equivalent representations. Note that for any element φ in $C^{0,2}$, $g^*(\varphi) = h^*(\varphi)$, so we can use h in place of g in the formula above.

Theorem 9.1. *The equivalence classes of extensions of W by M under the action of the group G_{gen} are classified by the following data:*

1. *Equivalence classes of codifferentials δ on W under the action $\mathbf{GL}(W)$.*
2. *Equivalence classes of codifferentials μ on M under the action of the group $\mathbf{GL}(M)$.*
3. *Equivalence classes of D_μ-cohomology classes $\bar\lambda \in H^{1,1}_\mu$ which satisfy the MC-equation*
$$\overline{[\delta + \lambda, \delta + \lambda]} = 0 \in H^{1,2}_\mu(\ker(D_{\delta+\lambda}))$$
under the action of the group $G_\Delta(\mu, \delta)$ on H_μ.
4. *Equivalence classes of $D_{\bar\delta+\bar\lambda}$-cohomology classes $[\bar\tau] \in H^{0,2}_{\mu,\delta+\lambda}$ under the action of the group $G_{gen}(\mu, \delta, \lambda)$.*

We are more interested in the moduli space of extensions of W by M preserving fixed codifferentials on these spaces.

Theorem 9.2. *The equivalence classes of extensions of a codifferential δ on W by a codifferential μ on M under the action of the group $G_{gen}(\mu,\delta)$ are classified by the following data:*

(1) *Equivalence classes of D_μ-cohomology classes $\bar\lambda \in H^{1,1}_\mu$ which satisfy the MC-equation*

$$\overline{[\delta+\lambda,\delta+\lambda]} = 0 \in H^{1,2}_\mu(\ker(D_{\delta+\lambda}))$$

under the action of the group $G_\Delta(\mu,\delta)$ on H_μ.

(2) *Equivalence classes of $D_{\bar\delta+\bar\lambda}$-cohomology classes $[\bar\tau] \in H^{0,2}_{\mu,\delta+\lambda}$ under the action of the group $G_{gen}(\mu,\delta,\lambda)$.*

To illustrate why this more general notion of equivalence is useful, we give some simple examples of extensions of associative algebras. For simplicity, we assume that the base field is $\mathbb{C}$ in all our examples.

10. Fundamental Theorem of Finite Dimensional Associative Algebras

In this section, we give $\mathbb{Z}_2$-graded versions of Wedderburn's Theorem and the Fundamental Theorem of Finite Dimensional Associative Algebras. There are a few modifications of the classical theory which need to be made in the $\mathbb{Z}_2$-graded case.

Definition 10.1. A $\mathbb{Z}_2$-graded division algebra is a unital, associative algebra for which every nonzero *homogeneous* element is invertible.

This definition reduces to the standard definition when the algebra is not graded, because in that case, all elements are even. Over $\mathbb{C}$, the only two division algebras are $\mathbb{C}$ and its *double* $\mathbb{C}_1 = \mathbb{C}\oplus\Pi\mathbb{C}$, which has multiplication given by $(a,b)*(c,d) = (ac+bd, ad+bc)$.

Theorem 10.2 (Wedderburn)**.** *Suppose that A is a finite dimensional $\mathbb{Z}_2$-graded simple associative algebra. Then $A = M\otimes D$ where D is a division algebra and M is a matrix algebra. Moreover, any algebra of this form is simple.*

Let $M = \mathfrak{gl}(V)$ where $\dim(V) = k|l$. then $\dim M = (k^2+l^2)|2kl$. Moreover, $\dim(M\otimes\mathbb{C}_1) = (k+l)^2|(k+l)^2$. Thus, we have a simple manner of calculating the dimensions of the simple complex associative algebras.

The (graded) *center* $Z(A)$ of a $\mathbb{Z}_2$-graded algebra A is the graded subalgebra generated by all homogeneous $c\in A$ such that $ca = (-1)^{ac}ac$ for all homogeneous $a\in A$. A unital $\mathbb{K}$-algebra is said to be *connected* over $\mathbb{K}$ if its center is $\mathbb{K}$.

Theorem 10.3. *Suppose that* $\mathrm{Char}(\mathbb{K}) \neq 2$*. Then the center of a* $\mathbb{Z}_2$*-graded division algebra over* $\mathbb{K}$ *is a field.*

Example 3. Let $\mathbb{K} = \mathbb{Z}_2$, and $A = \mathbb{K} \oplus \Pi\mathbb{K}$ be the double $\mathbb{K}_1$. Then A is a division algebra, and it is graded commutative. However $(1+\theta)(1-\theta) = 0$, so A is not a field.

Because of the example above, we will assume from now on that the field $\mathbb{K}$ does not have characteristic 2.

Definition 10.4. The *radical* $\mathrm{Rad}(A)$ of an associative algebra is the set of all *properly nilpotent* elements of A, which are the elements $a \in A$ such that the principal ideal (a) is nilpotent.

In the $\mathbb{Z}_2$-graded case, we have to modify the definition of the principal ideal (a), because we want this ideal to be the smallest graded ideal containing a. The ideal (a) is the ideal generated by the homogeneous parts of a. It is a straightforward to see that the radical of A is just the union of all graded nilpotent ideals in A, so is the largest graded nilpotent ideal.

The simplest version of the fundamental theorem of associative algebras is

Theorem 10.5. *Let A be a finite dimensional non-nilpotent $\mathbb{Z}_2$-graded algebra. Then* $A/\mathrm{Rad}(A)$ *is a semisimple algebra. As a consequence, there is an exact sequence*

$$0 \to N \to A \to S \to 0,$$

where N is a nilpotent algebra and S is a semisimple algebra.

There is a stronger version of this theorem, which requires the notion of a separable algebra.

Definition 10.6. A semisimple algebra S is separable over $\mathbb{K}$ if $S \otimes F$ remains semisimple for any finite extension of $\mathbb{K}$.

Actually, if S is simple, then S is separable over $\mathbb{K}$ precisely when its center, which always a field, is a separable field extension of $\mathbb{K}$.

Theorem 10.7. *Suppose that* $\mathrm{Char}(\mathbb{K}) \neq 2$ *and A is a nonnilpotent* $\mathbb{K}$*-algebra and that* $S = A/\mathrm{Rad}(A)$ *is separable. Then* $A = \mathrm{Rad}(A) \rtimes S$*.*

The proof of this theorem which we gave in [**?**] required that the characteristic not be equal to 2. Thus there are two reasons for excluding characteristic 2, which is probably an issue of 2 not being a good prime for $\mathbb{Z}_2$-graded algebras.

The main advantage in the formulation above, is that when we have a semidirect product, this means that in the decomposition of the extension in terms of λ and ψ, that ψ can be taken to be equal to zero. This simplifies the construction of extensions immensely, because this means we only have to consider D_μ-cocycles which satisfy the MC-equation, up to coboundaries.

Unfortunately, we do not have a similar result for nilpotent algebras. However, we do have the following.

Theorem 10.8. *Let A be a nonzero nilpotent finite dimensional algebra. Then A has a codimension 1 ideal M. If $\dim(A) = n|1$, then then A has an ideal of dimension $n|0$.*

The above theorem means, in the codifferential framework, that a nilpotent algebra defined by a codifferential on a $1|n$-dimensional space is given by an extension of a $1|0$-dimensional trivial codifferential δ by a nilpotent $0|n$-dimensional algebra W. Because of this, it follows that $C^{1,1}$ is an entirely even space, so that there are no nontrivial odd elements $\lambda \in C^{1,1}$. As a consequence, the construction of extensions is greatly simplified, because $\lambda = 0$.

11. Further Analysis of the MC equation

Here we analyze the form of λ, in an extension of W by M. Let us suppose that $V = \langle v_1, \cdots, v_n\rangle$. By convention, we always assume the basis is homogeneous and that the even elements are listed first. We also assume that W and M have bases given by subsets of this basis of V, so that $W = \langle v_{w(1)}, \cdots, v_{w(p)}\rangle$ and $M = \langle v_{m(1)}, \cdots, v_{m(q)}\rangle$.

Then an arbitrary $\lambda \in C^{1,1}$ is of the form

$$\lambda = \psi^{w(k)m(j)}_{m(i)}(L_k)^i_j + \psi^{m(j)w(k)}_{m(i)}(R_k)^i_j,$$

where L_k and R_k are matrices of linear maps $M \to M$, which have the opposite parity to $v_{w(k)}$. Then we have

$$\begin{aligned}\tfrac{1}{2}[\lambda,\lambda] = {}& (-1)^{w(k)}\psi^{w(k)w(l)m(j)}_{m(i)}(L_kL_l)^i_j + \psi^{m(j)w(k)w(l)}_{m(i)}(R_lR_k)^i_j \\ &+ \psi^{w(k)m(j)w(l)}_{m(i)}(R_lL_k + (-1)^{v_{w(k)}}L_kR_l)^i_j.\end{aligned}$$

It is important to note that the formula above is given in terms of matrix multiplication. This is significant from the computational view as we shall illustrate below. It is interesting to note that matrices in $G_{M,W}$, which are block diagonal maps $\operatorname{diag}(G_1, G_2)$, where $G_1 \in \mathbf{GL}(M)$ and $G_2 \in \mathbf{GL}(W)$, act on λ in a manner which can be described in terms of the matrices above. First, G_1 acts by conjugating all the matrices L_k and R_k simultaneously. Secondly, the matrix G_2 acts on the k indices.

Now suppose that $\delta = \psi^{w(k)w(k)}_{w(k)}$. Let $J : M \to M$ be the diagonal operator given by $Jm = (-1)^m m$. We have

$$[\delta, \lambda] = \psi^{w(k)w(k)m(j)}_{m(i)}(L_k)^i_j + \psi^{m(j)w(k)w(k)}_{m(i)}(R_k J)^i_j.$$

As a consequence, if W is completely odd, and $\delta = \sum_k \psi^{w(k)w(k)}_{w(k)}$ represents the algebra structure $\mathbb{C}^q$, the direct sum of q copies of the simple algebra $\mathbb{C}$, then we can completely solve the MC equation $[\delta, \lambda] + \frac{[}{\lambda}, \lambda] = 0$ in terms of matrix multiplication.

When W is completely odd, the matrices L_k and R_k represent even maps $M \to M$. In particular, we can express R_k in block diagonal form $R_k = \operatorname{diag}(T_k, B_k)$, where T_k and B_k represent the induced maps on the even and odd parts of M. In this case $R_k J = \operatorname{diag}(T_k, -B_k)$, so that for δ representing the algebra structure C^q, we can express the solution to the MC equation in the form

$$(L^k)^2 = L_k, \quad L_k L_l = 0, k \neq l, \quad L_k R_l = R_l L_k$$
$$(T_k)^2 = -T_k, \quad (B^k)^2 = B_k, \quad R_k R_l = 0, k \neq l$$

As a consequence, the set of matrices L_k and R_l are all commuting, diagonalizable matrices with L_k and B_k having eigenvalues 0 or 1, and T_k having eigenvalues 0 or -1.

The action of an element of G_μ on λ is by simultaneous conjugation of the matrices L_k and R_k. Moreover, if $\delta = C^q$, then G_δ is just the permutation matrices, so G_δ acts by permuting the indices k. When $\mu = 0$, $G_\mu = G_M = \mathfrak{gl}(M)$. As a consequence, in this case, we have a discrete set of solutions to the MC equation. Moreover, if the total dimension of M is p, then since $L_k L_l = 0$ when $k \neq l$, there can be at most p nonzero matrices L_k, and similarly, at most p nonzero matrices R_k. This means that if $q \geq 2p$, the set of solutions to the MC equation is the same. It can be shown that the deformations of the codifferentials $d = \delta + \lambda$ are also essentially the same. Thus, if we fix a space M of dimension $k|l$, then whenever W is a $r|s$-dimensional space where $r \geq 2k$ and $s \geq 2l$, the number and even the form of the codifferentials is independent of $r|s$. We call this the *stable* case.

When $\mu \neq 0$, then two factors modify the analysis above. First, the compatibility relation $[\mu, \lambda] = 0$ restricts the set of allowable λ, and the set of equivalence classes of such λ under the D_μ-cohomology restricts the solutions even further (up to equivalence). On the other hand, the group G_μ is smaller than the group $\mathfrak{gl}(M)$, so two elements which might be equivalent under the full group of symmetries may no longer be equivalent. It still turns out that there is a stable dimension $r|s$, as in the previous case, but it is harder to determine what that dimension will be. In examples which we have computed, it often turns out that the dimension much lower than $2k|2l$.

When δ is not C^n, the situation is more interesting. In [?, ?, ?, ?], we studied extensions where δ has a term which is isomorphic to the double of $\mathbb{C}$, and the analysis of these extensions is more subtle than the extensions of $\mathbb{C}^n$. We have not studied extensions where δ contains a matrix algebra, as the smallest such example occurs in dimension 5.

12. Simple examples of extensions of associative algebras

The notion of a Lie superalgebra is expressible in terms of coderivations on the symmetric coalgebra of a $\mathbb{Z}_2$-graded vector space. These superalgebras have been well studied. The corresponding notion of an associative algebra on a $\mathbb{Z}_2$-graded vector space is not well known. One reason for this might be that the definition of an associative algebra on a graded vector space is the same as for a non-graded space; the associativity relation does not pick up any signs as happens with the graded Jacobi identity on a superspace. At first glance, it does not appear that there is any reason to study such super associative algebras. However, the notion of an A_∞ algebra, which generalizes the idea of an associative algebra, naturally arises in the $\mathbb{Z}_2$-graded setting. The study of associative algebra structures on $\mathbb{Z}_2$-graded spaces is really the first step in the study of A_∞ algebras.

The manner in which the $\mathbb{Z}_2$-grading appears in the classification of associative algebra structures on a $\mathbb{Z}_2$-graded space is in terms of the parity of the multiplication, which is always required to be even, and in the parity of automorphisms of the vector space, which are also required to be even maps. Thus, for a $\mathbb{Z}_2$-graded space, not every multiplication on the underlying ungraded space is allowed, and only certain automorphisms of the space are allowed. This effects the moduli space in two ways. First, there are fewer codifferentials, and secondly, the set of equivalences is restricted, so it is not obvious whether the moduli space of $\mathbb{Z}_2$-graded associative algebras is larger or smaller than the moduli space on the associated ungraded space. In fact, there is a natural map between the moduli space of $\mathbb{Z}_2$-graded associative algebras and the moduli space of ungraded algebras. In general, this map may be neither surjective, nor injective.

Because it is convenient to work in the parity reversed model, an ungraded vector space will correspond to a completely odd space in our setting. Moreover, the associativity relation picks up signs in the parity reversed model. In fact, if d is an odd codifferential in $C^2(W) =$

$\mathrm{Hom}(T^2(W), W)$, then the associativity relation becomes

$$d(d(a,b),c) + (-1)^a d(a, d(b,c)) \tag{17}$$

Note that when V is completely odd, then $(-1)^a = -1$ for all a, which gives the usual associativity relation. Nevertheless, the relation above gives the usual associativity relation on the parity reversion $V = \Pi(W)$, because the induced multiplication is given by

$$m(x,y) = (-1)^x \pi d(\pi^{-1}(x), \pi^{-1}(y)).$$

If $V = \langle e_1, \cdots, e_m\rangle$ is a $\mathbb{Z}_2$-graded space, then a basis for the n-cochain space $C^n(V) = \mathrm{Hom}(T^n(V), V)$ is given by the coderivations φ_i^I, where $I = (i_1, \cdots, i_n)$ is a multi-index, and

$$\varphi_i^I(e_J) = \delta_J^I e_i.$$

Here $e_J = e_{j_1} \otimes \cdots \otimes e_{j_n}$ is a basis element of $T^n(V)$, determined by the multi-index J and δ_J^I is the Kronecker delta. When φ_i^I is odd, we denote it by ψ_i^I to emphasize this fact.

If $V = \langle e_1\rangle$ is a 1-dimensional odd vector space, (we denote its dimension by $0|1$) then it has only one nontrivial odd codifferential of degree 2, namely $d = \psi_1^{1,1}$. In other words $d(e_1, e_1) = e_1$. On the other hand, an even 1-dimensional vector space ($1|0$-dimensional) has no nontrivial odd codifferentials. As the first case of examples of the theory of extensions, we will study extensions of 1-dimensional spaces by 1-dimensional spaces, as the construction is very easy.

12.1. **Extensions where** $\dim(V) = 0|2$**.** The classification of complex 2-dimensional associative algebras dates back at least to [21]. These algebras form the moduli space of $0|2$-dimensional associative algebras. Let $V = \langle v_1, v_2\rangle$, where v_1 and v_2 are both odd.

The semisimple 2-dimensional complex associative algebra is given by the $d_1 = \psi_1^{11} + \psi_2^{22}$. Next, we classify the extensions of the $0|1$-dimensional simple algebra $\delta = \psi_1^{1,1}$ by the trivial nilpotent $0|1$-dimensional algebra $\mu = 0$. Using the matrices L_1 and R_1 introduced in the section above, we find they are 1×1 matrices (numbers) with eigenvalues 1 or 0. There are evidently 4 solutions $L_1 = 0, R_1 = 1$, $L_1 = 1, R_1 = 0$, $L_1 = 1, R_1 = 1$, and $L_1 = 0, R_1 = 0$, which give the codifferential $d_2, \cdots, d_5$, which are the left-unital, the right unital, the unital and commutative, and the direct sum extensions. Note that this is not the stable case of extensions of a $0|q$ dimensional algebra by a $0|1$-dimensional algebra. The stable case occurs when $q = 2$, which we will discuss below.

Finally, to study the nilpotent algebras, we need to consider extensions of the $0|1$-dimensional trivial algebra by the $0|1$-dimensional algebra. In this case, the MC equation forces $(L_1)^2 = (R_1)^2 = 0$, which means that $\lambda = 0$. Since $\psi = \psi_2^{11} a$ for some coefficient a, we obtain only one nontrivial nilpotent algebra $d_6 = \psi_2^{11}$.

Although we have tacitly assumed that the base field is $\mathbb{C}$, the construction above does not use any of the properties such as algebraic closure of $\mathbb{C}$. Moreover, the moduli space of $0|1$-dimensional algebras is independent of the base field. Thus, the construction above is valid in any field whose characteristic is not 2.

12.2. **Extensions where** $\dim V = 1|1$**.** The moduli space of complex associative algebras on V of dimension $1|1$ contains exactly 6 nonequivalent codifferentials, just as in the $0|2$-dimensional case. Let $V = \langle v_1, v_2\rangle$, where v_1 is even and v_2 is odd. The simple algebra $d_1 = \psi_2^{2,2} - \psi_2^{1,1} - \psi_1^{1,2} + \psi_1^{2,1}$ is the double of $\mathbb{C}$. Note, if the base field is $\mathbb{R}$, we actually have 2 nonequivalent simple $1|1$-dimensional algebras. There are infinitely many simple algebras over $\mathbb{Q}$. Thus, we will only consider the complex case here.

The extensions of the simple $0|1$-dimensional algebra $\delta = \psi_1^{1,1}$ are given by numbers L_1 and R_1 which satisfy $L_1^2 = L_1$ and $R_1^2 = -R_1$. Thus, we obtain 4 cases $L_1 = 1, R_1 = 0$, $L_1 = 0, R_1 = -1$, $L_1 = 1, R_1 = -1$, and $L_1 = 0, R_1 = 0$, corresponding to the codifferentials $d_2, \cdots, d_5$. These extensions are left-unital, right-unital, unital and graded commutative, and the direct sum.

Finally, we have to consider the nilpotent algebras. If we extend the trivial $0|1$-dimensional algebra by the trivial $1|0$-dimensional algebra, then there are no odd elements in $C^{0,2}$, so $\psi = 0$. Since $\lambda = 0$ as well, this only gives the zero codifferential. On the other hand, if we extend the trivial $1|0$-dimensional algebra by the $0|1$-dimensional algebra, $\psi = \psi_2^{1,1}$ is odd, and we obtain, up to equivalence, exactly 1 nontrivial $1|1$-dimensional nilpotent algebra.

12.3. **Extensions where** $\dim V = 0|3$**.** Let $V = \langle v_1, v_2, v_3\rangle$. The codifferential $d = \psi_1^{1,1} + \psi_2^{2,2} + \psi_3^{3,3}$, representing the algebra structure $\mathbb{C}^3$, is the only semisimple $0|3$-dimensional algebra. All other algebras are extensions of the semisimple $0|2$-dimensional algebra $\delta = \psi_2^{2,2} + \psi_3^{3,3}$, the simple $0|1$-dimensional algebra $\delta = \psi_3^{3,3}$ or the trivial $0|1$-dimensional algebra $\delta = 0$.

The extensions of $\delta = \psi_2^{2,2} + \psi_3^{3,3}$ by the trivial $0|1$-dimensional algebra $M = \langle v_1\rangle$ give the stable case for extensions of a $0|1$-dimensional algebra by C^q, since $q = 2$ is the stable dimension. There are 5 such extensions, one more than the $q = 1$ case. These extensions are given by numbers L_1, L_2, R_1, R_2 which are either 1 or 0, subject to the requirements that $L_1L_2 = R_1R_2 = 0$, up to permutations of the indices 1 and 2. We can assume that $L_2 = 0$, by permuting the indices if necessary. When $R_2 = 0$, these solutions correspond to the 4 cases arising from extending M by $\mathbb{C}^1$. There is one new case arising, which is given by $L_1 = 1$, $R_1 = 0$, and $R_2 = 1$. This is a unital but not commutative extension. Thus, the stable case gives 5 extensions. In [8], these codifferentials were given as $d_2, \cdots, d6$.

The extensions of $\delta = \psi_3^{3,3}$ by $M = \langle v_1, v_2\rangle$ are far from the stable case, which occurs when $\dim W = 4$. There are 12 such extensions. Of these, 10 occur by extending the trivial algebra structure by a pair of 2×2 diagonal matrices with eigenvalues 0 or 1. These pairs of L_1, R_1 are

$$\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

These extensions are given in [8] as the codifferentials $d_9, , \cdots, d_{18}$. We also obtain extensions of the nontrivial nilpotent algebra $\mu = \psi_1^{2,2}$. There are two obvious such extensions, the unital extension d_7 and the direct sum extension d_8, and these are the only such extensions. Moreover, this is the stable case for extensions of μ, so in this case, the stable dimension is $q = 1$, smaller than the stable dimension for extending the trivial codifferential. In fact, in all cases we have studied thus far, the stable dimension for extensions of a nontrivial μ is less than the stable dimension for the trivial μ.

Finally, one can extend the trivial δ on a $0|1$-dimensional space by either the nontrivial or trivial nilpotent μ. There are 3 nonequivalent extensions of the trivial μ, one of which is a family. The extensions of the nontrivial μ does not give anything which does not already arise as an extension of the trivial algebra. Details of this construction are already given in [8], so we do not reproduce it here.

13. Infinitesimal deformations of extensions of associative algebras

A natural question that arises when studying the moduli spaces arising from extensions is how to fit the moduli together as a space, and to answer that question, one needs to have a notion of how to move around in the moduli space. This notion is precisely the idea of deformations, in this case, deformations of the extensions. We will classify the infinitesimal deformations of an extension.

Let $d = \delta + \mu + \lambda + \psi$ be an extension, and consider the infinitesimal deformation

$$d_t = d + t(\eta + \zeta)$$

of this extension, where $\eta \in C^{1,1}$ represents a deformation of the λ structure, and $\zeta \in C^{0,2}$ gives a deformation of the ψ structure. In this section, we don't consider deformations which involve deforming the δ or μ structure. The infinitesimal condition is that $t^2 = 0$, in which case, as usual, the condition for η, ζ to determine a deformation is, infinitesimally, that $[d, \eta + \zeta] = 0$. We split this one condition up into the four conditions below.

$$[\delta + \lambda, \eta] + [\mu, \zeta] = 0 \tag{18}$$

$$[\delta + \lambda, \zeta] + [\psi, \eta] = 0 \tag{19}$$

$$[\mu, \eta] = 0 \tag{20}$$

$$[\psi, \zeta] = 0 \tag{21}$$

These conditions are symmetric in the roles of ψ and μ, but this symmetry is a bit misleading. For example the condition (21) is automatic for $\zeta \in C^{0,2}$, but condition (20) is not automatic for $\eta \in C^{1,1}$.

Note that since η is a D_μ-cocycle, $\bar{\eta}$ is well defined, and condition (18) implies that $\bar{\eta}$ is a $D_{\bar{\delta}+\bar{\lambda}}$-cocycle. Since $[\psi, \psi] = 0$, it determines a coboundary operator D_ψ as well. Denote the D_ψ-cohomology class of a D_ψ-cocycle φ by $\overline{\overline{\varphi}}$ and the set of cohomology classes by H_ψ. Note that H_ψ inherits the structure of a Lie algebra.

Since $[\delta + \lambda, \psi] = 0$, it follows that $\overline{\overline{\delta + \lambda}}$ is well defined. Moreover, we have

$$\begin{aligned}[\overline{\overline{\delta + \lambda}}, [\overline{\overline{\delta + \lambda}}, \overline{\overline{\varphi}}]] &= \overline{\overline{[\delta + \lambda, [\delta + \lambda, \varphi]]}} = \overline{\overline{[\tfrac{1}{2}[\delta + \lambda, \delta + \lambda], \varphi]}} \\ &= -\overline{\overline{[[\mu, \psi], \varphi]}} = -\overline{\overline{[\mu, [\psi, \varphi]]}} = 0,\end{aligned}$$

so $D_{\overline{\overline{\delta+\lambda}}}$ is a differential on H_ψ. Denote the cohomology class of a $D_{\overline{\overline{\delta+\lambda}}}$-cocycle $\overline{\overline{\varphi}}$ by $\left[\overline{\overline{\varphi}}\right]$ and the set of cohomology classes by $H_{\psi,\delta+\lambda}$. Note that $H_{\psi,\delta+\lambda}$ inherits the structure of a Lie algebra.

We first remark that conditions (18) and (20) imply that $[\bar{\eta}]$ is well defined, and (19) and (21) imply that $\left[\overline{\overline{\zeta}}\right]$ is well defined.

Next we introduce an action of D_ψ on $H_{\mu,\delta+\lambda}$. It is not possible to extend the operation of bracketing with ψ to the D_μ-cohomology, because $[\mu,\psi]\neq 0$. Moreover, even if $[\mu,\varphi]=0$, it does not follow that $[\mu,[\psi,\varphi]]=0$. However, we can extend the bracket to $H_{\mu,\delta+\lambda}$ as follows. A cohomology class $[\bar{\varphi}]$, is given by a φ such that $[\mu,\varphi]=0$ and $[\delta+\lambda,\varphi]=[\mu,\beta]$ for some β. Note that

$$\begin{aligned}[\mu,[\psi,\varphi]]&=[[\mu,\psi],\varphi]=-[\delta+\lambda,[\delta+\lambda,\varphi]]\\&=-[\delta+\lambda,[\mu,\beta]]=[\mu,[\delta+\lambda,\beta]].\end{aligned}$$

In [7], it was shown that the operator D_ψ on $H_{\mu,\delta+\lambda}$ given by

$$D_\psi([\bar{\varphi}])=\left[\overline{[\psi,\varphi]-[\delta+\lambda,\beta]}\right],$$

where β is any solution to $[\delta+\lambda,\varphi]=[\mu,\beta]$, is well defined, and that $D_\psi^2=0$. Moreover, if $H_{\mu,\delta+\lambda,\psi}$ denotes the associated cohomology, then the bracket on $H_{\mu,\delta+\lambda}$ descends to a bracket on $H_{\mu,\delta+\lambda,\psi}$, equipping it with the structure of a Lie superalgebra. Let us denote the D_ψ-cohomology class of a D_ψ-cocycle $[\overline{\varphi}]$ by $\{[\overline{\varphi}]\}$.

In a very similar manner, one can show that one can define D_μ on $H_{\psi,\delta+\lambda}$ by

$$D_\mu\left(\left[\overline{\overline{\varphi}}\right]\right)=\left[\overline{\overline{[\mu,\varphi]-[\delta+\lambda,\beta]}}\right],$$

where β is any coderivation satisfying $[\delta+\lambda,\varphi]=[\psi,\beta]$. Then D_μ is a Lie algebra morphism on $H_{\psi,\delta+\lambda}$ whose square is zero, and we denote the resulting cohomology by $H_{\psi,\delta+\lambda,\mu}$ and the cohomology class of a D_μ-cocycle $\left[\overline{\overline{\varphi}}\right]$ by $\left\{\left[\overline{\overline{\varphi}}\right]\right\}$.

We will call $H_{\mu,\delta+\lambda,\psi}$ and $H_{\psi,\delta+\lambda,\mu}$ *triple* cohomology groups. It turns out that the first one will play a more important role in the classification of infinitesimal deformations of extensions. The following lemma was proved in [7].

Lemma 13.1. *Let $d=\delta+\mu+\lambda+\psi$ be an extension of the codifferentials δ on W by μ on M, that $\eta\in\operatorname{Hom}(MW,M)$ and $\zeta\in\operatorname{Hom}(W^2,M)$. If*

$$d_t=d+t(\eta+\zeta)$$

determines an infinitesimal deformation of d then

(1) $\{[\overline{\eta}]\}$ *is well defined.*
(2) $\left\{\left[\overline{\overline{\zeta}}\right]\right\}$ *is well defined.*

Infinitesimal deformations can be characterized in terms of the triple cohomology $H_{\mu,\delta+\lambda,\psi}$ alone. The following theorem, proved in [7], gives a condition for an infinitesimal deformation to exist, depending on η alone, and classifies all such deformations.

Theorem 13.2. *Let $d = \delta + \mu + \lambda + \psi$ be an extension of the codifferentials δ on W by μ on M.*

An element $\eta \in C^{1,1}$ gives rise to an infinitesimal deformation for some $\zeta \in C^{0,2}$ if and only if the triple cohomology class $\{[\bar{\eta}]\}$ in $H^{1,1}_{\mu,\delta+\lambda,\psi}$ is well defined. In this case, if $\zeta \in C^{0,2}$ is any coderivation such that η, ζ determine an infinitesimal deformation, then $\zeta' = \zeta + \tau$ determines another infinitesimal deformation if and only if the double cohomology class $[\bar{\tau}]$ is well defined in $H^{0,2}_{\mu,dl,\psi}$.

Moreover the infinitesimal equivalence classes of infinitesimal deformations are classified by the triple cohomology classes $\{[\bar{\eta}]\} \in H^{0,1}_{\mu,\delta+\lambda,\psi}$ and $\{[\bar{\tau}]\} \in H^{0,2}_{\mu,\delta+\lambda,\psi}$.

The triple cohomology introduced in this section has a generalization in the case of the study of an A_∞-algebra given by an extension of a A_∞-algebra given by a degree k term by another A_∞-algebra given by a degree k term. In this case, we explained above, there are $2k-1$ nontrivial relations. There is also a sequence of $k+1$ -cohomology spaces H_i, each given by a coboundary operator D_{i+1} defined on the space H_i, determined by the element λ_i. The first of these spaces is $H_0 = H_\mu$, the second is $H_1 = H_{\mu,\delta+\lambda}$, and the third is $H_3 = H_{\mu,\delta+\lambda,\psi}$. We will give details of this construction elsewhere, but point out that the more general construction explains the somewhat mysterious third cohomology space given above.

14. Infinitesimal Deformations of Representations

In this section, we give a complete classification of infinitesimal deformations of representations of associative algebras

Let M be an associative algebra with multiplication μ, which is also a module over W. In other words, we are studying an extension of W by M for which the cocycle ψ vanishes. There are two interesting problems we could study.

(1) Allow the module structure λ and the algebra structure δ to vary, but keep μ fixed. This case includes the study of deformations of a module structure where the module does not have an algebra structure.
(2) Allow the module structure λ and the multiplication μ to vary, but keep the algebra structure δ fixed.

In both of these scenarios, we think of the structures on M and W as being distinct, with interaction only through λ, so when considering automorphisms of the structures, it is reasonable to restrict to automorphisms of V which do not mix the W and M terms, in other words we allow only elements of G_Δ.

Then we have the following maps:

$$\begin{aligned} D_\delta &: C^n \to C^{n+1} \\ D_\lambda &: C^n \to C^{1,n} \\ D_{\delta+\lambda} &: C^{k,l} \to C^{k,l+1} \\ D_\mu &: C^{k,l} \to C^{k+1,l}. \end{aligned}$$

In the setup of this problem, we only are interested in $C^{k,l}$ for $k \geq 1$, so we shall restrict our space of cochains in this manner. Because of this restriction, we note that an element in $C^{1,1}$ can be a D_μ-cocycle, but never a D_μ- coboundary. Moreover $C^n \subseteq \ker(D_\mu)$, so an element in C^2 is always a D_μ-cocycle, and never a D_μ-coboundary.

Because $\psi = 0$, the MC-equation $[\delta, \lambda] + \frac{1}{2}[\lambda\lambda] = 0$ is satisfied, so that $D^2_{\delta+\lambda} = 0$. Since

$$\begin{aligned} (D_\lambda D_\delta + D_{\delta+\lambda} D_\lambda)(\varphi) =& [\lambda, [\delta, \varphi]] + [\delta + \lambda, [\lambda, \varphi]] \\ =& [[\lambda, \delta], \varphi] - [\delta, [\lambda, \varphi]] + [\delta, [\lambda, \varphi]] + [\lambda, [\lambda, \varphi]] \\ =& [[\delta, \lambda], \varphi] + [\tfrac{1}{2}[\lambda, \lambda], \varphi] = 0. \end{aligned}$$

we have

$$D_\lambda D_\delta + D_{\delta+\lambda} D_\lambda = 0.$$

If we denote the D_μ-cohomology class of a D_μ-cocycle φ by $\bar\varphi$ as usual, then since $\bar\lambda$ and $\bar\delta$ are defined, we get the following version of this equation, applicable to the cohomology space H_μ.

$$D_{\bar\lambda} D_{\bar\delta} + D_{\bar\delta+\bar\lambda} D_{\bar\lambda} = 0.$$

As usual, let us denote the $D_{\bar\delta+\bar\lambda}$-cohomology class of a $D_{\bar\delta+\bar\lambda}$-cocycle $\bar\varphi$ by $[\bar\varphi]$.

Let us study the first scenario, where we allow λ and δ to vary, in other words, we consider

$$d_t = d + t(\delta_1 + \lambda_1),$$

where $\delta_1 \in C^2$ and $\lambda_1 \in C^{1,1}$ represent the variations in δ and λ. The infinitesimal condition $[d_t, d_t] = 0$ is equivalent to the three conditions for a deformation of a module structure:

$$\begin{aligned} [\delta, \delta_1] &= 0 \\ [\lambda, \delta_1] + [\delta + \lambda, \lambda_1] &= 0 \\ [\mu, \lambda_1] &= 0. \end{aligned}$$

By the third condition above $\bar{\lambda}_1$ is well defined, and $\bar{\delta}_1$ is defined. We claim that if $D_{\bar{\delta}}(\bar{\delta}_1) = 0$, which is the first condition, then the $D_{\bar{\delta}+\bar{\lambda}}$-cohomology class $[D_{\bar{\lambda}}(\delta_1)]$ is well defined and depends only on the D_δ-cohomology class of δ_1. It is well defined because

$$D_{\bar{\delta}+\bar{\lambda}}D_{\bar{\lambda}}(\bar{\delta}_1) = -D_{\bar{\lambda}}D_{\bar{\delta}}(\bar{\delta}_1) = 0.$$

To see that it depends only on the D_δ-cohomology class of $\bar{\delta}$, we apply $D_{\bar{\lambda}}$ to a $D_{\bar{\delta}}$-coboundary $D_{\bar{\lambda}}(\bar{\varphi})$ to obtain

$$D_{\bar{\lambda}}D_{\bar{\delta}}(\bar{\varphi}) = D_{\bar{\delta}+\bar{\lambda}}D_{\bar{\lambda}}(-\bar{\varphi}),$$

which is a $D_{\bar{\delta}+\bar{\lambda}}$-coboundary. The second condition for a deformation of the module structure implies that $[D_{\bar{\lambda}}(\bar{\delta})] = 0$. Moreover, if this statement holds, then there is some λ_1 such that δ_1 and λ_1 determine a deformation of the module structure. We see that $\lambda_1' = \lambda + \tau$ is another solution precisely when $\bar{\tau}$ exists and $D_{\bar{\delta}+\bar{\lambda}}(\bar{\tau}) = 0$. Thus, given one solution λ_1, the set of solutions is determined by the $D_{\bar{\delta}+\bar{\lambda}}$-cocycles $\tau \in C^{1,1}$.

Now let us consider infinitesimal equivalence. We suppose that $\alpha \in C^{1,0}$ and $\gamma \in C^1$, and $g = \exp(t(\alpha + \beta)$. If $d_t' = g^*(d_t)$ is given by the cochains δ_1' and λ_1', then we have

$$\begin{aligned} \delta_1' &= \delta_1 + D_\delta(\gamma) \\ \lambda_1' &= \lambda_1 + D_\lambda(\alpha + \gamma) \\ D_\mu(\alpha + \gamma) &= 0. \end{aligned}$$

It follows that the set of equivalence classes of deformations are determined by D_δ cohomology classes of $\delta_1 \in C^2$. If we fix δ_1 such that $D_\delta(\delta_1) = 0$ and λ_1 satisfying the rest of the conditions of a deformation, then expressing $\tau' = \tau + D_\lambda(\alpha + \gamma)$. But, since $D_\delta(\alpha + \gamma) = 0$, this means we can express $\tau' = \tau + D_{\delta+\lambda}(\alpha + \gamma)$ and $D_\mu(\alpha + \gamma)$, which means that $\bar{\tau}' = \bar{\tau} + D_{\bar{\delta}+\bar{\lambda}}(\bar{\alpha} + \bar{\gamma})$, and the solutions for τ are given by $D_{\bar{\delta}+\bar{\lambda}}$-cohomology classes of D_μ-cocycles $\tau \in C^{1,1}$. Thus we obtain

Theorem 14.1. *The infinitesimal deformations of a module M with multiplication μ over an associative algebra δ, allowing the algebra structure δ on W and module structure λ to vary are classified by*

(1) *D_δ-cohomology classes of D_δ-cocycles $\delta_1 \in C^2$ satisfying the condition*

$$[D_{\bar{\lambda}}(\bar{\delta}_1)] = 0$$

.

(2) *$D_{\bar{\delta}+\bar{\lambda}}$-cohomology classes $[\bar{\tau}]$ of $D_{\bar{\delta}+\bar{\lambda}}$-cocycles $\bar{\tau}$ of D_μ-cocycles $\tau \in C^{1,1}$.*

Finally, let us study the second scenario, where we allow λ and μ, but not δ, to vary. We write $d_t = d + t(\lambda_1 + \mu_1)$, where $\lambda_1 \in C^{1,1}$ is the variation of λ and $\mu_1 \in C^{2,0}$ is the variation in μ. The Jacobi identity $[d_t, d_t] = 0$ gives three conditions for a deformation of the module structure.

$$
\begin{aligned}
&D_\mu(\mu_1) = 0\\
&D_{\delta+\lambda}(\mu_1) + D_\mu(\lambda_1) = 0\\
&D_{\delta+\lambda}(\lambda_1) = 0.
\end{aligned}
$$

Recall that D_μ maps $\ker(D_{\delta+\lambda})$ to itself, so $H_\mu(\ker(\delta+\lambda))$ is well defined. The first condition on a deformation says that $\bar{\mu}_1$ is well defined. We claim that in that case, $\overline{D_{\delta+\lambda}(\mu_1)}$ is a well defined element of $H_\mu(\ker(\delta+\lambda)$ which depends only on $\bar{\mu}_1$. This is clear, since $D_{\delta+\lambda}(\mu_1) \in \ker(D_{\delta+\lambda})$, and $D_\mu D_{\delta+\lambda}(\mu_1) = -D_{\delta+\lambda} D_u(\mu_1) = 0$. The second condition on a deformation says simply that $\overline{D_{\delta+\lambda}(\mu_1)} = 0$, and the fact that this statement is true in $H_\mu(\ker(\delta+\lambda))$ is the third condition. Therefore, assuming that $\overline{D_{\delta+\lambda}(\mu_1)} = 0$, we can find a λ_1 so that all of the conditions for a deformation are satisfied.

If $\lambda_1 + \tau$ gives another solution, then $D_\mu(\tau) = 0$ and $D_{\delta+\lambda}(\tau) = 0$. Because we do not allow elements of $C^{0,1}$ as cochains, these two equalities are equivalent to $[\bar{\tau}]$ being well defined.

If $d'_t = \exp(t(\alpha+\gamma))$, then we obtain the following.

$$
\begin{aligned}
[\delta, \alpha+\gamma] &= 0\\
\lambda'_1 &= \lambda + [\lambda, \alpha+\gamma]\\
\mu'_1 &= \mu_1 + [\mu, \alpha+\gamma].
\end{aligned}
$$

Thus, up to equivalence, a deformation is given by a D_μ-cohomology class $\bar{\mu}_1$. If we fix μ_1, and then look at the variation in τ, one also obtains that up to equivalence, the deformation is determined by the $D_{\bar{\delta}+\bar{\lambda}}$-cohomology class $[\bar{\tau}]$ of the D_μ-cohomology class $\bar{\tau}$. Thus we have shown

Theorem 14.2. *The infinitesimal deformations of a module M with associative algebra structure μ over an associative algebra δ allowing the algebra structure μ on W and module structure λ to vary are classified by*

1. *D_μ cohomology classes $\bar{\mu}_1$ of D_μ-cocycles μ_1 lying in $C^{2,0}$.*
2. *$D_{\bar{\delta}+\bar{\lambda}}$-cohomology classes $[\bar{\tau}]$ of $D_{\bar{\delta}+\bar{\lambda}}$-cocycles $\bar{\tau}$ of D_μ-cocycles τ lying in $C^{1,1}$.*

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