THE MODULI SPACE OF 4-DIMENSIONAL NON-NILPOTENT COMPLEX ASSOCIATIVE ALGEBRAS

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Abstract. In this paper, we study the moduli space of 4-dimensional complex associative algebras. We use extensions to compute the moduli space, and then give a decomposition of this moduli space into strata consisting of complex projective orbifolds, glued together through jump deformations. Because the space of 4-dimensional algebras is large, we only classify the non-nilpotent algebras in this paper.

1. Introduction

The classification of associative algebras was instituted by Benjamin Peirce in the 1870's [19], who gave a partial classification of the complex associative algebras of dimension up to 6, although in some sense, one can deduce the complete classification from his results, with some additional work. The classification method relied on the following remarkable fact:

Theorem 1.1. *Every finite dimensional algebra which is not nilpotent contains a nontrivial idempotent element.*

A nilpotent algebra A is one which satisfies $A^n = 0$ for some n, while an idempotent element a satisfies $a^2 = a$. This observation of Peirce eventually leads to two important theorems in the classification of finite dimensional associative algebras. Recall that an algebra is said to be simple if it has no nontrivial proper ideals, and it is not the trivial 1-dimensional nilpotent algebra over $\mathbb{K}$ which is given by the trivial product.

Theorem 1.2 (Fundamental Theorem of Finite Dimensional Associative Algebras)**.** *Suppose that A is a finite dimensional algebra over a field $\mathbb{K}$. Then A has a maximal nilpotent ideal N, called its radical. If A is not nilpotent, then A/N is a semisimple algebra, that is, a direct sum of simple algebras.*

In fact, in the literature, the definition of a semisimple algebra is often given as one whose radical is trivial, and then it is a theorem that semisimple algebras are direct sums of simple algebras. Moreover, when A/N satisfies a property called separability over $\mathbb{K}$, then A is a semidirect product of its radical and a semisimple algebra. Over the complex numbers, every semisimple algebra is separable. To apply this theorem to construct algebras by extension, one uses the following characterization of simple algebras.

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Theorem 1.3 (Wedderburn)**.** *If A is a finite dimensional algebra over $\mathbb{K}$, then A is simple iff A is isomorphic to a tensor product $M \otimes D$, where $M = \mathfrak{gl}(n, \mathbb{K})$ and D is a division algebra over $\mathbb{K}$.*

One can also say that A is a matrix algebra with coefficients in a division algebra over $\mathbb{K}$. An associative division algebra is a unital associative algebra where every nonzero element has a multiplicative inverse. (One has to modify this definition in the case of graded algebras, but we will not address this issue in this paper.) Over the complex numbers, the only division algebra is $\mathbb{C}$ itself, so Wedderburn's theorem says that the only simple algebras are the matrix algebras. In particular, there is exactly one simple 4-dimensional complex associative algebra, $\mathfrak{gl}(2, \mathbb{C})$, while there is one additional semisimple algebra, the direct sum of 4 copies of $\mathbb{C}$.

According to our investigations, there are two basic prior approaches to the classification. The first is the old paper by Peirce [19] which attempts to classify all the nilpotent algebras, including nonassociative ones. There are some evident mistakes in that paper, for example, it gives a classification of the commutative nilpotent associative algebras which contains nonassociative algebras as well. The second approach [18] classifies the unital algebras only. It turns out that classification of unital algebras is not sufficient.

Let us consider the unital algebra of one higher dimension which is obtained by adjoining a multiplicative identity as the unital enlargement of the algebra. Two nonisomorphic non-nilpotent algebras can have isomorphic unital enlargements, so they cannot be recovered so easily. Nevertheless, let us suppose that there were some efficient method of constructing all unital algebras of arbitrary dimension, and to determine their maximal nilpotent ideals. In that case, we could recover all nilpotent algebras of dimension n from their enlargements. Moreover, to recover all algebras of dimension n, one would only have to consider extensions of nilpotent algebras of dimension k by semisimple algebras of dimension $n-k$, where $0 \leq k \leq n$. Our method turns out to be efficient in constructing extensions of nilpotent algebras by semisimple ones.

Thus, even if the construction of unital algebras could be carried out simply, which is by no means obvious from the literature, one would still need our methodology to construct most of the algebras. So the role of our paper is to explore the construction method which leads to the description of all algebras.

The main goal of this paper is to give a complete description of the moduli space of nonnilpotent 4-dimensional associative algebras, including a computation of the miniversal deformation of every element. We get the description with the help of extensions, which is the novelty of our approach. The nilpotent cases will be classified in another paper. We also give a canonical stratification of the moduli space into projective orbifolds of a very simple type, so that the strata are connected only by deformations factoring through jump deformations, and the elements of a particular stratum are given by neighborhoods determined by smooth deformations.

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2. Construction of algebras by extensions

In [7], the theory of extensions of an algebra W by an algebra M is described. Consider the exact sequence

$$0 \to M \to V \to W \to 0$$

of associative $\mathbb{K}$-algebras, so that $V = M \oplus W$ as a $\mathbb{K}$-vector space, M is an ideal in the algebra V, and $W = V/M$ is the quotient algebra. Suppose that $\delta \in C^2(W)$ and $\mu \in C^2(M)$ represent the algebra structures on W and M respectively. We can view μ and δ as elements of $C^2(V)$. Let $T^{k,l}$ be the subspace of $T^{k+l}(V)$ given recursively by

$$T^{0,0} = \mathbb{K}$$
$$T^{k,l} = M \otimes T^{k-1,l} \oplus V \otimes T^{k,l-1}$$

Let $C^{k,l} = \operatorname{Hom}(T^{k,l}, M) \subseteq C^{k+l}(V)$. If we denote the algebra structure on V by d, we have

$$d = \delta + \mu + \lambda + \psi,$$

where $\lambda \in C^{1,1}$ and $\psi \in C^{0,2}$. Note that in this notation, $\mu \in C^{2,0}$. Then the condition that d is associative: $[d, d] = 0$ gives the following relations:

$$[\delta, \lambda] + \tfrac{1}{2}[\lambda, \lambda] + [\mu, \psi] = 0, \quad \text{The Maurer-Cartan equation}$$
$$[\mu, \lambda] = 0, \quad \text{The compatibility condition}$$
$$[\delta + \lambda, \psi] = 0, \quad \text{The cocycle condition}$$

Since μ is an algebra structure, $[\mu, \mu] = 0$. Then if we define D_μ by $D_\mu(\varphi) = [\mu, \varphi]$, then $D_\mu^2 = 0$. Thus D_μ is a differential on $C(V)$. Moreover $D_\mu : C^{k,l} \to C^{k+1,l}$. Let

$$Z^{k,l}_\mu = \ker(D_\mu : C^{k,l} \to C^{k+1,l}), \quad \text{the } (k,l)\text{-cocycles}$$
$$B^{k,l}_\mu = \operatorname{Im}(D_\mu : C^{k-1,l} \to C^{k,l}), \quad \text{the } (k,l)\text{-coboundaries}$$
$$H^{k,l}_\mu = Z^{k,l}_\mu / B^{k,l}_\mu, \quad \text{the } D_u \ (k,l)\text{-cohomology}$$

Then the compatibility condition means that $\lambda \in Z^{1,1}$. If we define $D_{\delta+\lambda}(\varphi) = [\delta + \lambda, \varphi]$, then it is not true that $D^2_{\delta+\lambda} = 0$, but $D_{\delta+\lambda} D_\mu = -D_\mu D_{\delta+\lambda}$, so that $D_{\delta+\lambda}$ descends to a map $D_{\delta+\lambda} : H^{k,l}_\mu \to H^{k,l+1}_\mu$, whose square is zero, giving rise to the $D_{\delta+\lambda}$-cohomology $H^{k,l}_{\mu,\delta+\lambda}$. Let the pair (λ, ψ) give rise to a codifferential d, and (λ, ψ') give rise to another codifferential d'. Then if we express $\psi' = \psi + \tau$, it is easy to see that $[\mu, \tau] = 0$, and $[\delta + \lambda, \tau] = 0$, so that the image $\bar{\tau}$ of τ in $H^{0,2}_\mu$ is a $D_{\delta+\lambda}$-cocycle, and thus τ determines an element $\{\bar{\tau}\} \in H^{0,2}_{\mu,\delta+\lambda}$.

If $\beta \in C^{0,1}$, then $g = \exp(\beta) : \mathcal{T}(V) \to \mathcal{T}(V)$ is given by $g(m, w) = (m + \beta(w), w)$. Furthermore $g^* = \exp(-\operatorname{ad}_\beta) : C(V) \to C(V)$ satisfies $g^*(d) = d'$, where $d' = \delta + \mu + \lambda' + \psi'$ with $\lambda' = \lambda + [\mu, \beta]$ and $\psi' = \psi + [\delta + \lambda + \frac{1}{2}[\mu, \beta], \beta]$. In this case, we say that d and d' are equivalent extensions in the restricted sense. Such equivalent extensions are also equivalent as codifferentials on $\mathcal{T}(V)$. Note that λ and λ' differ by a D_μ-coboundary, so $\bar{\lambda} = \bar{\lambda}'$ in $H^{1,1}_\mu$. If λ satisfies the MC equation for some ψ, then any element λ' in $\bar{\lambda}$ also gives a solution of the MC equation, for the ψ' given above. The cohomology classes of those λ for which a solution of the MC equation exists determine distinct restricted equivalence classes of extensions.

Let $G_{M,W} = \mathbf{GL}(M) \times \mathbf{GL}(W) \subseteq \mathbf{GL}(V)$. If $g \in G_{M,W}$ then $g^* : C^{k,l} \to C^{k,l}$, and $g^* : C^k(W) \to C^k(W)$, so $\delta' = g^*(\delta)$ and $\mu' = g^*(\mu)$ are codifferentials on $\mathcal{T}(M)$ and $\mathcal{T}(W)$ respectively. The group $G_{\delta,\mu}$ is the subgroup of $G_{M,W}$ consisting of those elements g such that $g^*(\delta) = \delta$ and $g^*(\mu) = \mu$. Then $G_{\delta,\mu}$ acts on the restricted equivalence classes of extensions, giving the equivalence classes of general extensions. Also $G_{\delta,\mu}$ acts on $H^{k,l}_\mu$, and induces an action on the classes $\bar\lambda$ of λ giving a solution to the MC equation.

Next, consider the group $G_{\delta,\mu,\lambda}$ consisting of the automorphisms h of V of the form $h = g\exp(\beta)$, where $g \in G_{\delta,\mu}$, $\beta \in C^{0,1}$ and $\lambda = g^*(\lambda) + [\mu,\beta]$. If $d = \delta+\mu+\lambda+\psi+\tau$, then $h^*(d) = \delta+\mu+\lambda+\psi+\tau'$ where

$$\tau' = g^*(\psi) - \psi + [\delta+\lambda-\tfrac{1}{2}[\mu,\beta],\beta] + g^*(\tau).$$

Thus the group $G_{\delta,\mu,\lambda}$ induces an action on $H^{0,2}_{\mu,\delta+\lambda}$ given by $\{\bar\tau\}\to\{\overline{\tau'}\}$.

The general group of equivalences of extensions of the algebra structure δ on W by the algebra structure μ on M is given by the group of automorphisms of V of the form $h = \exp(\beta)g$, where $\beta \in C^{0,1}$ and $g \in G_{\delta,\mu}$. Then we have the following classification of such extensions up to equivalence.

Theorem 2.1 ([7])**.** *The equivalence classes of extensions of δ on W by μ on M is classified by the following:*

(1) *Equivalence classes of $\bar\lambda \in H^{1,1}_\mu$ which satisfy the MC equation*

$$[\delta,\lambda] + \tfrac{1}{2}[\lambda,\lambda] + [\mu,\psi] = 0$$

for some $\psi \in C^{0,2}$, under the action of the group $G_{\delta,\mu}$.

(2) *Equivalence classes of $\{\bar\tau\} \in H^{0,2}_{\mu,\delta+\lambda}$ under the action of the group $G_{\delta,\mu,\lambda}$.*

Equivalent extensions will give rise to equivalent algebras on V, but it may happen that two algebras arising from nonequivalent extensions are equivalent. This is because the group of equivalences of extensions is the group of invertible block upper triangular matrices on the space $V = M\oplus W$, whereas the the equivalence classes of algebras on V are given by the group of all invertible matrices, which is larger.

The fundamental theorem of finite dimensional algebras allows us to restrict our consideration of extensions to two cases. First, we can consider those extensions where δ is a semisimple algebra structure on W, and μ is a nilpotent algebra structure on M. In this case, because we are working over $\mathbb{C}$, we can also assume that $\psi = \tau = 0$. Thus the classification of the extension reduces to considering equivalence classes of λ.

Secondly, we can consider extensions of the trivial algebra structure $\delta = 0$ on a 1-dimensional space W by a nilpotent algebra μ. This is because a nilpotent algebra has a codimension 1 ideal M, and the restriction of the algebra structure to M is nilpotent. However, in this case, we cannot assume that ψ or τ vanish, so we need to use the classification theorem above to determine the equivalence classes of extensions. In many cases, in solving the MC equation for a particular λ, if there is any ψ yielding a solution, then $\psi = 0$ also gives a solution, so the action of $G_{\delta,\mu,\lambda}$ on $H^{0,2}_\mu$ takes on a simpler form than the general action we described above.

In addition to the complexity which arises because we cannot take the cocycle term ψ in the extension to be zero, there is another issue that complicates the construction of the extensions. If an algebra is not nilpotent, then it has a maximal nilpotent ideal which is unique, and it can be constructed as an extension of a semisimple algebra by this unique ideal. Both the semisimple and nilpotent parts in this construction are completely determined by the algebra. Therefore, a classification of extensions up to equivalence of extensions will be sufficient to classify the algebras. This means that the equivalence classes of the module structure λ determine the algebras up to isomorphism.

For nilpotent algebras, we don't have this assurance. The same algebra structure may arise by extensions of the trivial algebra structure on a 1-dimensional space by two different nilpotent algebra structures on the same $n-1$-dimensional space.

In addition, the deformation theory of the nilpotent algebras is far more involved than the deformation theory of the nonnilpotent algebras. Thus, we decided to discuss the nilpotent 4-dimensional complex algebras in a separate paper. In this paper, we only look at extensions of semisimple algebras by nilpotent algebras, which is precisely what is necessary to classify all non-nilpotent algebras.

3. Associative algebra structures on a 4-dimensional vector space

Denote the basis elements of a 4-dimensional associative algebra by f_1, f_2, f_3, f_4 and let ψ_k^{ij} denote the product $f_i f_j = f_k$. We will recall the classification of algebras on a 2-dimensional space given in [1], and the classification of algebras on a 3-dimensional space given in [8].

Codifferential	H^0	H^2	H^1	H^3	H^4
$d_1 = \psi_1^{11} + \psi_2^{22}$	2	0	0	0	0
$d_2 = \psi_2^{22} + \psi_1^{12}$	0	0	0	0	0
$d_3 = \psi_2^{22} + \psi_1^{21}$	0	0	0	0	0
$d_4 = \psi_2^{22} + \psi_1^{12} + \psi_1^{21}$	2	1	1	1	1
$d_5 = \psi_2^{22}$	2	1	1	1	1
$d_6 = \psi_1^{22}$	2	2	2	2	2

Table 1. Two dimensional complex associative algebras and their cohomology

Actually, we only need to know the nilpotent algebras from lower dimensions as well as the semisimple algebras. In dimension 1, there is one nontrivial algebra structure $d_1 = \psi_1^{11}$, which is just complex numbers $\mathbb{C}$.

Thus, in dimension 2, the algebra $d_1 = \psi_1^{11} + \psi_2^{22}$ is semisimple, while the algebra $d_6 = \psi_1^{22}$ is nilpotent. These are the only algebras of dimension 2 (other than the trivial algebra) which play a role in the construction of 4-dimensional algebras by extensions. The algebra d_1 is just the direct sum $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$.

In the case of 3-dimensional algebras, only $d_1 = \psi_1^{11} + \psi_2^{22} + \psi_3^{33}$ is semisimple, and only the algebras $d_{19} = \psi_2^{13} + \psi_2^{31} + \psi_1^{33}$, $d_{20}(p : q) = \psi_2^{13} p + \psi_2^{31} q + \psi_2^{33}$, and $d_{21} = \psi_2^{13} - \psi_2^{31}$ are nilpotent. The algebra d_1 is just the direct sum of three copies of $\mathbb{C}$.

Note that $d_{20}(p : q)$ is a family of algebras parameterized by the projective orbifold $\mathbb{P}^1/\Sigma_2$. By this we mean that the algebras $d_{20}(p : q)$ and $d_{20}(tp : tq)$ are isomorphic if $t \neq 0$, which gives the projective parameterization, and that the algebras $d_{20}(p : q)$ and $d_{20}(q : p)$ are also isomorphic, which gives the action of the group Σ_2 on $\mathbb{P}^1$.

In constructing the elements of the moduli space by extensions, we need to consider three possibilities, extensions of the semisimple algebra structure on a 3-dimensional space W by the trivial algebra structure on a 1-dimensional space M, extensions of the semisimple algebra structure on a 2-dimensional space by a

Codifferential	H^0	H^2	H^1	H^3	H^4
$d_1=\psi^{33}_3+\psi^{22}_2+\psi^{11}_1$	3	0	0	0	0
$d_2=\psi^{22}_2+\psi^{33}_3+\psi^{21}_1+\psi^{13}_1$	1	0	0	0	0
$d_3=\psi^{22}_2+\psi^{33}_3+\psi^{12}_1$	1	0	0	0	0
$d_4=\psi^{22}_2+\psi^{33}_3+\psi^{21}_1$	1	0	0	0	0
$d_5=\psi^{22}_2+\psi^{33}_3+\psi^{21}_1+\psi^{12}_1$	3	1	1	1	1
$d_6=\psi^{22}_2+\psi^{33}_3$	3	1	1	1	1
$d_7=\psi^{33}_3+\psi^{22}_1+\psi^{31}_1+\psi^{32}_2+\psi^{13}_1+\psi^{23}_2$	3	2	2	2	2
$d_8=\psi^{33}_3+\psi^{22}_1$	3	2	2	2	2
$d_9=\psi^{33}_3+\psi^{31}_1+\psi^{32}_2$	0	3	0	0	0
$d_{10}=\psi^{33}_3+\psi^{13}_1+\psi^{23}_2$	0	3	0	0	0
$d_{11}=\psi^{33}_3+\psi^{31}_1+\psi^{23}_2$	0	1	0	1	0
$d_{12}=\psi^{33}_3+\psi^{13}_1+\psi^{32}_2+\psi^{23}_2$	1	1	1	1	1
$d_{13}=\psi^{33}_3+\psi^{31}_1+\psi^{32}_2+\psi^{23}_2$	1	1	1	1	1
$d_{14}=\psi^{33}_3+\psi^{32}_2$	1	1	2	2	2
$d_{15}=\psi^{33}_3+\psi^{23}_2$	1	1	2	2	2
$d_{16}=\psi^{33}_3+\psi^{32}_2+\psi^{23}_2$	3	2	2	2	2
$d_{17}=\psi^{33}_3+\psi^{31}_1+\psi^{13}_1+\psi^{32}_2+\psi^{23}_2$	3	4	6	12	24
$d_{18}=\psi^{33}_3$	3	4	8	16	32
$d_{19}=\psi^{13}_2+\psi^{31}_2+\psi^{33}_1$	3	3	3	3	3
$d_{20}(0:0)=\psi^{33}_2$	3	5	9	17	33
$d_{20}(1:0)=\psi^{13}_2+\psi^{33}_2$	1	2	5	8	11
$d_{20}(1:1)=\psi^{13}_2+\psi^{31}_2+\psi^{33}_2$	3	4	5	7	8
$d_{20}(1:-1)=\psi^{13}_2-\psi^{31}_2+\psi^{33}_2$	1	2	3	4	5
$d_{20}(p:q)=\psi^{13}_2 p+\psi^{31}_2 q+\psi^{33}_2$	1	2	3	3	4
$d_{21}=\psi^{13}_2-\psi^{31}_2$	1	4	5	8	9

TABLE 2. Three dimensional complex associative algebras and their cohomology

nilpotent algebra on a 2-dimensional space, and extensions of either the simple or the trivial 1-dimensional algebras by a nilpotent 3-dimensional algebra.

Consider the general setup, where an n-dimensional space $W=\langle f_{m+1},\dots f_{m+n}\rangle$ is extended by an m-dimensional space $M=\langle f_1,\cdots,f_m\rangle$. Then the module structure is of the form

$$\lambda=\psi^{kj}_i(L_k)^i_j+\psi^{jk}_i(R_k)^i_j,\quad i,j=1,\dots m,k=m+1\dots m+n,$$

and we can consider L_k and R_k to be $m\times m$ matrices. Then we can express the bracket $\frac{1}{2}[\lambda,\lambda]$, which appears in the MC equation in terms of matrix multiplication.

$$\tfrac{1}{2}[\lambda,\lambda]=\psi^{jkl}_i(R_lR_k)^i_j+\psi^{kjl}_i(L_kR_l-R_kL_l)^i_j-\psi^{klj}_i(L_kL_l)^i_j, \tag{1}$$

where $i,j=1,\dots m$, and $k,l=m+1,\dots m+n$.

Next, suppose that $\delta=\psi^{m,m}_m+\cdots+\psi^{m+n,m+n}_{m+n}$ is the semisimple algebra structure $\mathbb{C}^n$ on W. Then we can also express $[\delta,\lambda]$ in terms of matrix multiplication.

$$[\delta,\lambda]=\psi^{kkj}_i(L_k)^i_j-\psi^{jkk}_i(R_k)^i_j. \tag{2}$$

Since δ is semisimple, one can ignore the cocycle ψ in constructing an extension, so the MC equation is completely determined by the equations (2) and (1), so we obtain the conditions. Therefore, the MC equation holds precisely when

$$L_k^2 = L_k, \quad R_k^2 = R_k, \quad L_kL_l = R_kR_l = 0 \text{ if } k \neq l, \quad L_kR_l = R_lL_k.$$

As a consequence, both L_k and R_k must be commuting nondefective matrices whose eigenvalues are either 0 or 1, which limits the possibilities. Moreover, it can be shown that G_δ, the group of automorphisms of W preserving δ is just the group of permutation matrices. Thus if $G = \operatorname{diag}(G_1, G_2)$ is a block diagonal element of $G_{\delta,\mu}$, the matrix G_2 is a permutation matrix. The action of G on λ is given by simultaneous conjugation of the matrices L_k and R_k by G_1, and a simultaneous permutation of the k-indices determined by the permutation associated to G_2.

When μ is zero, this is the entire story. When $\mu \neq 0$, the matrices G_1 are required to preserve μ, and the compatibility condition $[\mu, \lambda]$ also complicates the picture.

It is important to note that given an m and a nilpotent element μ on an m-dimensional space M, there is an n beyond which the extensions of the semisimple codifferential on an N dimensional space with N greater than n are simply direct sums of the extensions of the n-dimensional semisimple algebra $\mathbb{C}^n$ and the semisimple algebra $\mathbb{C}^{N-n}$. We say that the extension theory becomes stable at n. Moreover, the deformation picture stabilizes as well.

In higher dimensions, there are semisimple algebras which are not of the form $\mathbb{C}^n$. Also, as m increases, the complexity of the nontrivial nilpotent elements μ increases as well. In dimension 4, there is a simple algebra, $\mathfrak{gl}(2,\mathbb{C})$, represented by the codifferential d_1, and a semisimple algebra $\mathbb{C}^4$, represented by the algebra d_2. All other 4-dimensional nonnilpotent algebras are extensions of a semisimple algebra of the type $\mathbb{C}^n$, for $n = 1, 2, 3$.

4. Extensions of the 3-dimensional semisimple algebra $\mathbb{C}^3$ by the 1-dimensional trivial algebra $\mathbb{C}_0$

Let $W = \langle f_2, f_3, f_4\rangle$ and $M = \langle f_1\rangle$. The matrices L_k and R_k determining λ are 1×1 matrices, in other words, just numbers; in fact, they are either 0 or 1. By applying a permutation to the indices $2, 3, 4$, we can assume that either all the L_k vanish, or $L_2 = 1$ and both L_3 and L_4 vanish. In the first case, either $R_2 = 1$ or $R_2 = 0$ and $R_3 = R_4 = 0$. In the second case, we can either have $R_2 = 1$ and $R_3 = 0$, $R_2 = 0$ and $R_3 = 1$, or both R_2 and R_3 vanish. In all three cases, $R_4 = 0$. Note that in all of these solutions, we can assume that $L_4 = R_4 = 0$. For extensions by a 1-dimensional space M, the extension picture stabilizes at $n = 2$, and we are looking at $n = 3$. Thus the five solutions for λ here, which give the codifferentials $d_3, \cdots, d_7$ correspond to the five 3-dimensional codifferentials $d_2, \cdots, d_6$.

5. Extensions of the 2-dimensional semisimple algebra $\mathbb{C}^2$ by a 2-dimensional nilpotent algebra

Let $W = \langle f_3, f_4\rangle$ and $M = \langle f_1, f_2\rangle$.

There are two choices of μ in this case, depending on whether the algebra structure on M is the trivial or nontrivial nilpotent structure. Although we cannot calculate $G_{\delta,\mu}$ without knowing μ, we can say that the matrix G_2 in the expression above for an element of $G_{M,W}$ must be one of the two permutation matrices.

5.1. **Extensions by the nontrivial nilpotent algebra.** In this case $\mu = \psi_1^{22}$. In order for $[\mu, \lambda] = 0$, using equations (2) and (1), we must have

$$(L_k)_1^2 = (R_k)_1^2 = 0, (L_k)_1^1 = (L_k)_2^2 = (R_k)_1^1 = (R_k)_2^2,$$

for all k. It follows that L_k and R_k are upper triangular matrices with the same values on the diagonal, and since they are also nondefective matrices, they must be diagonal, and therefore are either both equal to the identity or both the zero matrix. It follows that by applying a permutation, we obtain either the solution $\lambda = 0$ or $L_3 = R_4 = I$, and $L_k = R_k = 0$ for $k > 3$. In fact, we see that this case stabilizes when $n = 1$, and we are looking at the case $n = 2$. Thus the two solutions d_8 and d_9 correspond to the three dimensional algebras d_7 and d_8. In fact, d_7 arises by the following consideration. Given an algebra on an n-dimensional space, there is an easy way to extend it to a unital algebra on an $n + 1$-dimensional space, by taking any vector not in the original space and making it play the role of the identity. This is how d_7 arises. The algebra d_8 also arises in a natural way as the direct sum of the algebra structures δ and μ. For this μ, these are the only such structures which arise, and this is somewhat typical.

5.2. **Extensions by the trivial nilpotent algebra.** In this case, L_k and R_k are 2×2 matrices. The nontrivial permutation has the effect of interchanging L_3 and L_4 as well as R_3 and R_4. The matrix G_1 acts on all four of the matrices by simultaneously conjugating them.

By permuting if necessary, one can assume that L_3 is either a nonzero matrix, or both L_3 and L_4 vanish. Moreover, by conjugation in case L_3 is not the identity or the zero matrix, we have $L_3 = \operatorname{diag}(1, 0)$, which we will denote by T. Now if $L_3 = I$, then since $L_3L_4 = 0$, we must have $L_4 = 0$, but if $L_3 = T$, then the condition $L_3L_4 = 0$ forces to either be $B = \operatorname{diag}(0, 1)$ or 0. A similar analysis applies to the R matrices. Let us consider a case by case analysis.

If $L_3 = I$ then $L_4 = 0$. Since I is invariant under conjugation, we can still apply a conjugation to put R_3 in the form I, T or 0. If $R_3 = I$, then $R_3 = 0$. If $R_3 = T$, then either $R_4 = B$ or $R_4 = 0$. If $R_3 = 0$, then R_4 may equal I, T or 0. This gives six solutions.

Next, assume $L_3 = T$ and $L_4 = B$. Since we have used up the conjugation in putting L_3 and L_4 in diagonal form, we can only use the fact that since R_3 and R_4 commute with L_3 and L_4, they can be simultaneously diagonalized, so we may assume they are diagonal. Thus R_3 is either I, T B or 0. If $R_3 = I$ then $R_4 = 0$. If $R_3 = T$ then $R_4 = B$ or $R_4 = 0$. If $R_3 = D$ then $R_4 = T$ or $R_4 = 0$. If $R_3 = 0$, then R_4 is either I, T, B or 0. This gives 9 possibilities, but there is one more thing which we have to be careful of. A certain conjugation interchanges T and D, so that if we first apply the nontrivial permutation and then the conjugation which interchanges T and B, we find that the $L_3 = T$, $L_4 = B$, $R_3 = I$ and $R_4 = 0$ is the same as if $R_3 = 0$ and $R_4 = I$. Similarly, $R_3 = T$ and $R_4 = 0$ transforms to $R_3 = 0$ and $R_4 = B$. Finally, $R_3 = B$ and $R_4 = 0$ transforms to $R_3 = 0$ and $R_4 = B$. Thus instead of 9 cases, we only obtain 6.

If $L_3 = T$ and $L_4 = 0$, then we obtain the same 9 cases for R_3 and R_4 as when $L_3 = T$ and $L_4 = B$, except this time, there are no hidden symmetries, so we get exactly 9 cases.

Finally, when $L_3 = L_4 = 0$, if $R_3 = I$ then $R_4 = 0$, while if $R_3 = T$, then $R_4 = B$ or $R_4 = 0$, and if $R_3 = 0$, then $R_4 = 0$, giving 4 more cases.

This gives a total of 25 nonequivalent extensions, and they are also all nonequivalent as algebras, corresponding to $d_{10},\dots d_{34}$. In this case, $n=2$ is not the stable case. It is not hard to see that $n=4$ gives the stable case, corresponding to the 6-dimensional moduli space.

6. Extensions of the 1-dimensional simple algebra $\mathbb{C}$ by a 3-dimensional nilpotent algebra

Here $M=\langle f_1,f_2,f_3\rangle$ and $W=\langle f_4\rangle$, and L_4 and R_4 are 3×3 matrices, which for simplicity, we will just denote by L and R. Elements in $C^{0,1}$ are of the form Let $\beta=\varphi_1^4b_1+\varphi_2^4b_2+\varphi_3^4b_3$.

6.1. **Extensions by the nilpotent algebra $\mu=\psi_2^{13}+\psi_2^{31}+\psi_1^{33}$.** In order for $[\mu,\lambda]=0$, using equations (2) and (1), we must have

$$L=\begin{bmatrix}L_1^1 & 0 & L_3^1\\ L_3^1 & L_1^1 & L_3^2\\ 0 & 0 & L_1^1\end{bmatrix}, R=\begin{bmatrix}L_1^1 & 0 & L_3^1\\ L_3^1 & L_1^1 & R_3^2\\ 0 & 0 & L_1^1\end{bmatrix}.$$

Taking into account the MC equation, we obtain that L and R must be diagonal matrices, so we only get two solutions, depending on whether L and R both vanish or are both equal to the identity matrix. Note that this is the stable case. It corresponds to the codifferentials d_{35} and d_{36}. Notice that we obtain one unital algebra d_{35} and one algebra which is a direct sum, d_{36}.

6.2. **Extensions by the nilpotent algebra $\mu=\psi_2^{13}p+\psi_2^{31}q+\psi_2^{33}$.** Here the situation depends on the projective coordinate $(p:q)$, which is parameterized by $\mathbb{P}^1/\Sigma_2$. There are special cases when $p=q=0$ or $p=1$ and $q=0$. These three cases arise from the compatibility condition $[\mu,\lambda]=0$, which generically has one solution, but has additional solutions when either p or q vanishes, and when both p and q vanish.

6.2.1. *The generic case.* In this case, in order for $[\mu,\lambda]=0$, we must have

$$L=\begin{bmatrix}L_1^1 & 0 & 0\\ L_1^2 & L_1^1 & L_3^2\\ 0 & 0 & L_1^1\end{bmatrix}, R=\begin{bmatrix}L_1^1 & 0 & 0\\ R_1^2 & L_1^1 & R_3^2\\ 0 & 0 & L_1^1\end{bmatrix}.$$

Taking into account the MC equation, we obtain that L and R must be diagonal matrices, so we only get two solutions, depending on whether L and R both vanish or are both equal to the identity matrix. Note that this is the stable case. It corresponds to the algebras $d_{37}(p:q)$ and $d_{38}(p:q)$. Note that both of these are families parameterized by $\mathbb{P}^1/\Sigma_2$.

6.2.2. *The case $p=1$, $q=0$.* In this case, in order for $[\mu,\lambda]=0$, we must have

$$L=\begin{bmatrix}L_1^1 & 0 & L_1^1-L_3^3\\ L_1^2 & L_1^1 & L_3^2\\ 0 & 0 & L_3^3\end{bmatrix}, R=\begin{bmatrix}L_3^3 & 0 & L_3^3-R_3^3\\ R_1^2 & R_3^3 & R_3^2\\ 0 & 0 & R_3^3\end{bmatrix}.$$

Since $[\mu,\beta]=\psi_2^{43}(b_1+b_3)+\psi_2^{14}b_3+\psi_2^{34}b_3$, we can further assume that $L_3^2=0$. Moreover, the eigenvalues of L are L_1^1 and L_3^3, while those of R are L_3^3 and R_3^3, and these numbers must be either 0 or 1, yielding 8 possibilities. In fact, each one of these 8 choices corresponds to a solution solution of the MC equation. Let

$$T_1=\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix},T_2=\begin{bmatrix}0&0&-1\\0&0&0\\0&0&1\end{bmatrix},B_1=\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix},B_2=\begin{bmatrix}0&0&-1\\0&1&0\\0&0&1\end{bmatrix}.$$

Then the 8 solutions are $L=I,R=I$; $L=0,R=0$; $L=I,R=B_1$; $L=T_1,R=B_2$; $L=T_2,R=B_2$; $L=0,R=B_2$; $L=T_2,R=B_1$ and $L=T_1,R=0$, corresponding to the codifferentials $d_{37(1:0)}$, $d_{38(1:0)}$, $d_{39},\cdots,d_{44}$.

This is not the stable case, because given an L,R pair above, there is another such pair, which satisfies the requirements that the products of the L matrices vanish, the products of the R matrices vanish, and the L and R matrices commute. In fact, it is not hard to see that $n=2$ gives the stable case, which will occur for 5-dimensional algebras.

6.2.3. *The case* $p=0$, $q=0$. In this case, in order for $[\mu,\lambda]=0$, we must have

$$L=\begin{bmatrix}L^1_1&0&L^1_3\\L^2_1&L^3_3&L^2_3\\0&0&L^3_3\end{bmatrix},R=\begin{bmatrix}R^1_1&0&R^1_3\\R^2_1&L^3_3&R^2_3\\0&0&L^3_3\end{bmatrix}.$$

Since $[\mu,\beta]=\psi^{43}_2b_3+\psi^{34}_2b_3$, we can further assume that $L^2_3=0$. Moreover, the eigenvalues of L_4 are L^1_1 and L^3_3, while those of R_4 are L^3_3 and R^3_3, and these numbers must be either 0 or 1, yielding 8 possibilities. In fact, each one of these 8 choices corresponds to a unique solution of the MC equation, which has some parameters. However, at this point we still have not taken into account the action of $G_{\delta,\mu}$. An element in G_μ is a matrix of the form $G=\begin{bmatrix}g^1_1&0&g^1_3\\g^2_1&(g^3_3)^2&g^2_3\\0&0&g^3_3\end{bmatrix}$, and it acts on λ by conjugating L and R simultaneously. This action is sufficient to eliminate the parameters in the solutions for L and R. Let $T=\operatorname{diag}(1,0,0)$ and $B=\operatorname{diag}(0,1,1)$.

The 8 solutions are $L=I,R=I$, $L=0,R=0$, $L=I,R=B$, $L=T,R=T$, $L=B,R=I$, $L=0,R=T$, $L=B,R=B$ and $L=T,R=0$, corresponding to the codifferentials $d_{37(0:0)}$, $d_{38(0:0)}$, $d_{45},\cdots,d_{50}$.

This is not the stable case, because given an L,R pair above, there is another such pair, which satisfies the requirements that the products of the L-s vanish, the products of the R-s vanish, and the L and R matrices commute. In fact, it is not hard to see that $n=2$ gives the stable case, which will occur for 5-dimensional algebras.

6.3. **Extensions by the nilpotent algebra** $\mu=\psi^{13}_2-\psi^{31}_2$**.** In order for $[\mu,\lambda]=0$, we must have

$$L=\begin{bmatrix}L^3_3&0&0\\L^2_1&L^3_3&L^2_3\\0&0&L^3_3\end{bmatrix},\begin{bmatrix}L^3_3&0&0\\R^2_1&L^3_3&R^2_3\\0&0&L^3_3\end{bmatrix}$$

Taking into account the MC equation, we obtain that L and R must be diagonal matrices, so we only get two solutions, depending on whether L and R both vanish or are both equal to the identity matrix. Note that this is the stable case. It corresponds to the codifferentials d_{51}, which is the unital extension, and d_{52}, which is the direct sum extension.

6.4. **Extensions by the trivial nilpotent algebra.** Since $\mu = 0$, we don't get any restrictions on λ from the compatibility condition, but since $G_\mu = \mathbf{GL}(3,\mathbb{C})$, we can assume that L is in Jordan normal form, and since L is nondefective, this implies that L is diagonal. From this it follows that L can only be one of I, $T_1 = \operatorname{diag}(1,1,0)$ $T_2 = \operatorname{diag}(1,0,0)$ or 0.

When $L = I$ or $L = 0$, it is invariant under conjugation, so we may conjugate R to be one of the same 4 matrices I, T_1, T_2 or 0. When $L = T_1$, R can also be conjugated to make it diagonal, and we obtain that R is one of the six matrices I, 0, T_1, T_2, $B_1 = \operatorname{diag}(1,0,1)$ or $B_2 = \operatorname{diag}(0,0,1)$. When $L = T_2$, R can again be conjugated to make it diagonal, and it is one of the six matrices I, 0, T_1, T_2, $B_3 = \operatorname{diag}(0,1,1)$,or $B_4 = \operatorname{diag}(0,1,0)$. This gives the 20 codifferentials $d_{53}, \cdots, d_{72}$.

This is not the stable case, and it is not hard to see that the stable case occurs when $\dim(W) = 6$.

M	W	δ	μ	N	Range
1	3	$\psi_4^{44} + \psi_3^{33} + \psi_2^{22}$	ψ_1^{11}	1	d_2
1	3	$\psi_4^{44} + \psi_3^{33} + \psi_2^{22}$	0	5	$d_3, \cdots, d_7$
2	2	$\psi_4^{44} + \psi_3^{33}$	ψ_1^{22}	2	d_8, d_9
2	2	$\psi_4^{44} + \psi_3^{33}$	0	25	$d_{10}, \cdots, d_{34}$
3	1	ψ_4^{44}	$\psi_2^{31} + \psi_2^{13} + \psi_1^{33}$	2	d_{35}, d_{36}
3	1	ψ_4^{44}	$\psi_2^{31}q + \psi_2^{13}p + \psi_2^{33}$	2	$d_{37}(p:q), d_{38}(p:q)$
3	1	ψ_4^{44}	$\psi_2^{13} + \psi_2^{33}$	6	$d_{39}, \cdots, d_{44}$
3	1	ψ_4^{44}	ψ_2^{33}	6	$d_{45}, \cdots, d_{50}$
3	1	ψ_4^{44}	$\psi_2^{13} - \psi_2^{31}$	2	$d_{51}, \cdots, d_{52}$
3	1	ψ_4^{44}	0	20	$d_{53}, \cdots, d_{72}$

TABLE 3. Table of Extensions of δ on W by μ on M

Note that the simple algebra d_1 does not appear in the table above, because it does not arise as an extension.

7. Hochschild Cohomology and Deformations

Suppose that V is a vector space, defined over a field $\mathbb{K}$ whose characteristic is not 2 or 3, equipped with an associative multiplication structure $m : V \otimes V \to V$. The associativity relation can be given in the form

$$m \circ (m \otimes 1) = m \circ (1 \otimes m).$$

The notion of isomorphism or *equivalence* of associative algebra structures is given as follows. If g is a linear automorphism of V, then define

$$g^*(m) = g^{-1} \circ m \circ (g \otimes g).$$

Two algebra structures m and m' are equivalent if there is an automorphism g such that $m' = g^*(m)$. The set of equivalence classes of algebra structures on V is called the *moduli space* of associative algebras on V.

Hochschild cohomology was introduced in [16], and was used by Gerstenhaber in [10] to classify infinitesimal deformations of associative algebras.

We define the Hochschild coboundary operator D on $\operatorname{Hom}(\mathcal{T}(V),V)$ by

$$D(\varphi)(a_0,\cdots,a_n)=a_0\varphi(a_1,\cdots,a_n)+(-1)^{n+1}\varphi(a_0,\cdots,a_{n-1})a_n + \sum_{i=0}^{n-1}(-1)^{i+1}\varphi(a_0,\cdots,a_{i-1},a_ia_{i+1},a_{i+2},\cdots,a_n).$$

We wish to transform this classical viewpoint into the more modern viewpoint of associative algebras as being given by codifferentials on a certain coalgebra. To do this, we first introduce the *parity reversion* ΠV of a $\mathbb{Z}_2$-graded vector space V. If $V=V_e\oplus V_o$ is the decomposition of V into its even and odd parts, then $W=\Pi V$ is the $\mathbb{Z}_2$-graded vector space given by $W_e=V_o$ and $W_o=V_e$. In other words, W is just the space V with the parity of elements reversed.

Given an ordinary associative algebra, we can view the underlying space V as being $\mathbb{Z}_2$-graded, with $V=V_e$. Then its parity reversion W is again the same space, but now all elements are considered to be odd. One can avoid this gyration for ordinary spaces, by introducing a grading by exterior degree on the tensor coalgebra of V, but the idea of parity reversion works equally well when the algebra is $\mathbb{Z}_2$-graded, whereas the method of grading by exterior degree does not.

Denote the tensor (co)-algebra of W by $\mathcal{T}(W)=\bigoplus_{k=0}^{\infty}W^k$, where W^k is the k-th tensor power of W and $W^0=\mathbb{K}$. For brevity, the element in W^k given by the tensor product of the elements w_i in W will be denoted by $w_1\cdots w_k$. The coalgebra structure on $\mathcal{T}(W)$ is given by

$$\Delta(w_1\cdots w_n)=\sum_{i=0}^{n}w_1\cdots w_i\otimes w_{i+1}\cdots w_n.$$

Define $d:W^2\to W$ by $d=\pi\circ m\circ(\pi^{-1}\otimes\pi^{-1})$, where $\pi:V\to W$ is the identity map, which is odd, because it reverses the parity of elements. Note that d is an odd map. The space $C(W)=\operatorname{Hom}(\mathcal{T}(W),W)$ is naturally identifiable with the space of coderivations of $\mathcal{T}(W)$. In fact, if $\varphi\in C^k(W)=\operatorname{Hom}(W^k,W)$, then φ is extended to a coderivation of $\mathcal{T}(W)$ by

$$\varphi(w_1\cdots w_n)=\sum_{i=0}^{n-k}(-1)^{(w_1+\cdots+w_i)\varphi}w_1\cdots w_i\varphi(w_{i+1}\cdots w_{i+k})w_{i+k+1}\cdots w_n.$$

The space of coderivations of $\mathcal{T}(W)$ is equipped with a $\mathbb{Z}_2$-graded Lie algebra structure given by

$$[\varphi,\psi]=\varphi\circ\psi-(-1)^{\varphi\psi}\psi\circ\varphi.$$

The reason that it is more convenient to work with the structure d on W rather than m on V is that the condition of associativity for m translates into the codifferential property $[d,d]=0$. Moreover, the Hochschild coboundary operation translates into the coboundary operator D on $C(W)$, given by

$$D(\varphi)=[d,\varphi].$$

This point of view on Hochschild cohomology first appeared in [20]. The fact that the space of Hochschild cochains is equipped with a graded Lie algebra structure was noticed much earlier [10, 11, 12, 13, 14].

For notational purposes, we introduce a basis of $C^n(W)$ as follows. Suppose that $W=\langle w_1,\cdots,w_m\rangle$. Then if $I=(i_1,\cdots,i_n)$ is a *multi-index*, where $1\le i_k\le m$, denote $w_I = w_{i_1} \cdots w_{i_n}$. Define $\varphi_i^I \in C^n(W)$ by

$$\varphi_i^I(w_J) = \delta_J^I w_i,$$

where δ_J^I is the Kronecker delta symbol. In order to emphasize the parity of the element, we will denote φ_i^I by ψ_i^I when it is an odd coderivation.

For a multi-index $I = (i_1, \cdots, i_k)$, denote its *length* by $\ell(I) = k$. Then since W is a completely odd space, the parity of φ_i^I is given by $|\varphi_i^I| = k + 1 \pmod 2$. If K and L are multi-indices, then denote $KL = (k_1, \cdots, k_{\ell(K)}, l_l, \cdots, l_{\ell(L)})$. Then

$$\begin{aligned}(\varphi_i^I \circ \varphi_j^J)(w_K) &= \sum_{K_1K_2K_3=K} (-1)^{w_{K_1}\varphi_j^J} \varphi_i^I(w_{K_1}, \varphi_j^J(w_{K_2}), w_{K_3}) \\ &= \sum_{K_1K_2K_3=K} (-1)^{w_{K_1}\varphi_j^J} \delta^I_{K_1 j K_3} \delta^J_{K_2} w_i,\end{aligned}$$

from which it follows that

$$\varphi_i^I \circ \varphi_j^J = \sum_{k=1}^{\ell(I)} (-1)^{(w_{i_1}+\cdots+w_{i_{k-1}})\varphi_j^J} \delta_j^k \varphi_i^{(I,J,k)}, \tag{3}$$

where (I, J, k) is given by inserting J into I in place of the k-th element of I; *i.e.*, $(I, J, k) = (i_1, \cdots, i_{k-1}, j_1, \cdots, j_{\ell(J)}, i_{k+1}, \cdots, i_{\ell(I)})$.

Let us explain the notion of an infinitesimal deformation in terms of the language of coderivations. We say that

$$d_t = d + t\psi$$

is an infinitesimal deformation of the codifferential d precisely when $[d_t, d_t] = 0 \mod t^2$. This condition immediately reduces to the cocycle condition $D(\psi) = 0$. Note that we require d_t to be odd, so that ψ must be an odd coderivation. One can introduce a more general idea of parameters, allowing both even and odd parameters, in which case even coderivations play an equal role, but we will not adopt that point of view in this paper.

For associative algebras, we require that d and ψ lie in $C^2(W)$. Since in this paper, our algebras are ordinary algebras, so that the parity of an element in $C^n(W)$ is $n + 1$, elements of $C^2(W)$ are automatically odd.

We need the notion of a versal deformation, in order to understand how the moduli space is glued together. To explain versal deformations we introduce the notion of a deformation with a local base. For details see [2, 3]. A local base A is a $\mathbb{Z}_2$-graded commutative, unital $\mathbb{K}$-algebra with an augmentation $\epsilon : A \to \mathbb{K}$, whose kernel $\mathfrak{m}$ is the unique maximal ideal in A, so that A is a local ring. It follows that A has a unique decomposition $A = \mathbb{K} \oplus \mathfrak{m}$ and ϵ is just the projection onto the first factor. Let $W_A = W \otimes A$ equipped with the usual structure of a right A-module. Let $T_A(W_A)$ be the tensor algebra of W_A over A, that is $T_A(W_A) = \bigoplus_{k=0}^{\infty} T_A^k(W_A)$ where $T_A^0(W_A) = A$ and $T_A^{k+1}(W_A) = T^k(W_A)_A \otimes_A W_A$. It is not difficult to show that $T_A^k(W_A) = T^k(W) \otimes A$ in a natural manner, and thus $T_A(W_A) = T(W) \otimes A$.

Any A-linear map $f : T_A(W) \to T_A(W)$ is induced by its restriction to $T(W) \otimes \mathbb{K} = T(W)$ so we can view an A-linear coderivation δ_A on $T_A(W_A)$ as a map $\delta_A : T(W) \to T(W) \otimes A$. A morphism $f : A \to B$ induces a map

$$f_* : \mathrm{Coder}_A(T_A(W_A)) \to \mathrm{Coder}_B(T_B(W_B))$$

given by $f_*(\delta_A) = (1 \otimes f)\delta_A$, moreover if δ_A is a codifferential then so is $f_*(A)$. A codifferential d_A on $T_A(W_A)$ is said to be a deformation of the codifferential d on $T(W)$ if $\epsilon_*(d_A) = d$.

If d_A is a deformation of d with base A then we can express

$$d_A = d + \varphi$$

where $\varphi : T(W) \to T(W) \otimes \mathfrak{m}$. The condition for d_A to be a codifferential is the Maurer-Cartan equation,

$$D(\varphi) + \frac{1}{2}[\varphi, \varphi] = 0$$

If $\mathfrak{m}^2 = 0$ we say that A is an infinitesimal algebra and a deformation with base A is called infinitesimal.

A typical example of an infinitesimal base is $\mathbb{K}[t]/(t^2)$; moreover, the classical notion of an infinitesimal deformation: $d_t = d + t\varphi$ is precisely an infinitesimal deformation with base $\mathbb{K}[t]/(t^2)$.

A local algebra A is complete if

$$A = \varprojlim_k A/\mathfrak{m}^k$$

A complete, local augmented $\mathbb{K}$-algebra is called formal and a deformation with a formal base is called a formal deformation, see [3]. An infinitesimal base is automatically formal, so every infinitesimal deformation is a formal deformation.

An example of a formal base is $A = \mathbb{K}[[t]]$ and a deformation of d with base A can be expressed in the form

$$d_t = d + t\psi_1 + t^2\psi_2 + \dots$$

This is the classical notion of a formal deformation. It is easy to see that the condition for d_t to be a formal deformation reduces to

$$D(\psi_{n+1}) = -\frac{1}{2}\sum_{k=1}^{n}[\psi_k, \psi_{n+1-k}], \quad n = 0, \dots$$

An automorphism of W_A over A is an A-linear isomorphism $g_A : W_A \to W_A$ making the diagram below commute:

$$\begin{array}{ccc} W_A & \xrightarrow{g_A} & W_A \\ \downarrow{\scriptstyle \epsilon_*} & & \downarrow{\scriptstyle \epsilon_*} \\ W & \xrightarrow{I} & W \end{array}$$

The map g_A is induced by its restriction to $T(W) \otimes \mathbb{K}$ so we can view g_A as a map

$$g_A : T(W) \to T(W) \otimes A$$

so we ca express g_A in the form

$$g_A = I + \lambda$$

where $\lambda : T(W) \to T(W) \otimes \mathfrak{m}$. If A is infinitesimal then $g_A^{-1} = I - \lambda$.

Two deformations d_A and d'_A are said to be equivalent over A if there is an automorphism g_A of W_A over A such that $g_A^*(d_A) = d'_A$. In this case we write $d'_A \sim d_A$.

An infinitesimal deformation d_A with base A is called universal if whenever d_B is an infinitesimal deformation with base B, there is a unique morphism $f : A \to B$ such that $f_*(d_A) \sim d_B$.

Theorem 7.1 ([4]). *If* $\dim H^2_{odd}(d) < \infty$ *then there is a universal infinitesimal deformation* d^{inf} *of* d*, given by*

$$d^{inf} = d + \delta^i t_i$$

where $H^2_{odd}(d) = \langle \bar{\delta^i} \rangle$ *and* $A = \mathbb{K}[t_i]/(t_i t_j)$ *is the base of deformation.*

In the theorem above $\bar{\delta}^i$ is the cohomology class determined by the cocycle δ^i.

A formal deformation d_A with base A is called versal if given any formal deformation of d_B with base B there is a morphism $f : A \to B$ such that $f_*(d_A) \sim d_B$. Notice that the difference between the versal and the universal property of infinitesimal deformations is that f need not be unique. A versal deformation is called *miniversal* if f is unique whenever B is infinitesimal. The basic result about versal deformations is:

Theorem 7.2 ([2, 3, 5]). *If* $\dim H^2_{odd}(d) < \infty$ *then a miniversal deformation of* d *exists.*

The following result can be used in some special cases to compute the versal deformations.

Theorem 7.3. *Suppose* $H^2_{odd}(d) = \langle \bar{\delta^i} \rangle$ *and* $[\delta^i, \delta^j] = 0$ *for all* i, j *then the infinitesimal deformation*

$$d^{inf} = d + \delta^i t_i$$

is miniversal, with base $A = \mathbb{K}[[t_i]]$.

The construction of the moduli space as a geometric object is based on the idea that codifferentials which can be obtained by deformations with small parameters are "close" to each other. From the small deformations, we can construct 1-parameter families or even multi-parameter families, which are defined for small values of the parameters, except possibly when the parameters vanish.

If d_t is a one parameter family of deformations, then two things can occur. First, it may happen that d_t is equivalent to a certain codifferential d' for every small value of t except zero. Then we say that d_t is a jump deformation from d to d'. It will never occur that d' is equivalent to d, so there are no jump deformations from a codifferential to itself. Otherwise, the codifferentials d_t will all be nonequivalent if t is small enough. In this case, we say that d_t is a smooth deformation. (In detail, see [6].)

In [6], it was proved for Lie algebras that given three codifferentials d, d' and d'', if there are jump deformations from d to d' and from d' to d'', then there is a jump deformation from d to d''. The proof of the corresponding statement for associative algebras is essentially the same.

Similarly, if there is a jump deformation from d to d', and a family of smooth deformations d'_t, then there is a family d_t of smooth deformations of d, such that every deformation in the image of d'_t lies in the image of d_t, for sufficiently small values of t. In this case, we say that the smooth deformation of d factors through the jump deformation to d'.

In the examples of complex moduli spaces of Lie and associative algebras which we have studied, it turns out that there is a natural stratification of the moduli space of n-dimensional algebras by orbifolds, where the codifferentials on a given strata are connected by smooth deformations which don't factor through jump deformations. These smooth deformations determine the local neighborhood structure.

The strata are connected by jump deformations, in the sense that any smooth deformation from a codifferential on one strata to another strata factors through a jump deformation. Moreover, all of the strata are given by projective orbifolds.

In fact, in all the complex examples we have studied, the orbifolds either are single points, or $\mathbb{CP}^n$ quotiented out by either Σ_{n+1} or a subgroup, acting on $\mathbb{CP}^n$ by permuting the coordinates.

8. Deformations of the elements in our moduli space

We have ordered the codifferentials so that a codifferential only deforms to a codifferential earlier on the list. Partially, this was accomplished the ordering of the different choices of M and W. That such an ordering is possible is due to the fact that jumps between families have a natural ordering by descent.

The radical of an algebra A is the same as the radical of its opposite algebra, ideals in an algebra are the same as the ideals in its opposite algebra A°, and the quotient of the opposite algebra by an ideal is naturally isomorphic to the quotient of the opposite algebra by the same ideal, it follows that the semisimple quotient of an algebra is the same as its opposite algebra. Also the center of an algebra coincides the center of its opposite algebra. Moreover, if an algebra A deforms to an algebra B, then its opposite algebra A° deforms to B°. A commutative algebra is isomorphic to its opposite algebra, but an algebra may be isomorphic to its opposite algebra without being equal to it. For example, a matrix algebra is always isomorphic to its opposite algebra, and the simple $1|1$-dimensional algebra is isomorphic to its opposite, but neither of these algebras is commutative.

We shall summarize most of the relevant information about the algebras in tables below. Since there are too many codifferentials to list in a single table, we will split them up into several tables. In one set of tables, we will give the codifferential which represents the algebra, as well as information about the cohomology spaces H^0 through H^3. In another set of tables, we will note which algebras are pairs of opposite algebras, give a basis for the center of the algebra, and indicate which algebras it deforms to. It would take up too much space to give the versal deformations for each of these algebras, but all of them were computed using the constructive method we have outlined above.

8.1. **The algebras $d_1 \dots d_9$.** The algebra d_1 represents the matrix algebra $\mathfrak{gl}(2,\mathbb{C})$. As such, it is simple, and so has no ideals, no deformations, and its center consists of the multiples of the identity, so has dimension 1. Thus $\dim H^0 = 1$ and $\dim H^n = 0$ otherwise.

The algebra d_2 is the semisimple algebra which is the direct sum of four copies of $\mathbb{C}$. Being semisimple, it is also cohomologically rigid, but it is commutative, so $\dim H^0 = 4$.

The algebras d_3 d_4 and d_5 are all rigid, with center of dimension 2. The algebras d_4 and d_5 are opposite algebras.

The algebra d_6 is the direct sum of C^2 with the algebra given by adjoining an identity to turn the 1-dimensional trivial algebra into a 2-dimensional unital algebra. It is unital, commutative, is not rigid, and in fact has a jump deformation to d_2.

Codifferential	H^0	H^1	H^2	H^3
$d_1 = \psi^{11}_1 + \psi^{12}_2 + \psi^{23}_1 + \psi^{24}_2 + \psi^{32}_4 + \psi^{31}_3 + \psi^{43}_3 + \psi^{44}_4$	1	0	0	0
$d_2 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_2 + \psi^{11}_1$	4	0	0	0
$d_3 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_2 + \psi^{21}_1 + \psi^{13}_1$	2	0	0	0
$d_4 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_2 + \psi^{12}_1$	2	0	0	0
$d_5 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_2 + \psi^{21}_1$	2	0	0	0
$d_6 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_2 + \psi^{21}_1 + \psi^{12}_1$	4	1	1	1
$d_7 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_2$	4	1	1	1
$d_8 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_1 + \psi^{31}_1 + \psi^{32}_2 + \psi^{13}_1 + \psi^{23}_2$	4	2	2	2
$d_9 = \psi^{33}_3 + \psi^{44}_4 + \psi^{22}_1$	4	2	2	2

TABLE 4. The cohomology of the algebras $d_1 \dots d_9$

The algebra d_7 is the direct sum of the trivial 1-dimensional algebra (which we denote as $\mathbb{C}_0$) with the semisimple 3-dimensional algebra $\mathbb{C}^3$. It is commutative but not unital, and also has a jump deformation to d_2.

The algebra d_8 arises by as a direct sum of $\mathbb{C}$ and the algebra which arises from adjoining an identity to the 2-dimensional nontrivial nilpotent algebra. It is unital and commutative. It has jump deformations to d_2 and d_6.

The algebra d_9 is the direct sum of the nontrivial 2-dimensional nilpotent algebra and $\mathbb{C}^2$. It is not unital, but is commutative. It has deformations to d_2, d_6 and d_7.

8.2. **The algebras $d_{10} \dots d_{34}$.** The algebras $d_{10} \dots d_{20}$ are all nonunital, noncommutative, and rigid. The pairs of opposite algebras are d_{10} and d_{11}, d_{12} and d_{13}, d_{15} and d_{16}, and d_{18} and d_{19}. The algebras d_{14}, d_{17} and d_{20} are all isomorphic to their opposite algebras.

The algebra d_{21} is unital, but not commutative, and it has a jump deformation to d_1.

The algebras d_{22} and d_{23} are nonunital, noncommutative opposite algebras, with d_{22} having a jump deformation to d_5, while d_{23} jumps to its opposite algebra d_4. Similarly d_{25} and d_{26} are also nonunital, noncommutative opposite algebras which jump to the same two elements in the same order.

The algebra d_{24} is a nonunital, noncommutative algebra which is isomorphic to its opposite algebra, and it jumps to d_3. The algebras d_{27} and d_{28} are unital, noncommutative opposite algebras both of which have jump deformations to d_3.

The algebras d_{29} and d_{30} are nonunital, nonunital opposite algebras, with d_{29} jumping to d_3 and d_5, while d_{30} jumps to d_3 and d_4.

The algebra d_{31} is nonunital but is commutative, and it has jump deformations to d_2, d_6 and d_7, all of which are commutative. Note that a commutative algebra may deform to a noncommutative algebra, but the converse is impossible.

The algebra d_{32} is both unital and commutative and jumps to d_2 and d_6. Note that a unital algebra can only deform to another unital algebra, and both d_2 and d_6 are unital.

The algebra d_{33} which arises by first taking the trivial 2-dimensional algebra, adding a multiplicative identity to make it unital, and then taking a direct sum with $\mathbb{C}$, is both unital and commutative. It deforms to d_2, d_3, d_6 and d_8. Note that

Codifferential	H^0	H^1	H^2	H^3
$d_{10} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{32}_2 + \psi^{14}_1$	0	0	0	0
$d_{11} = \psi^{33}_3 + \psi^{44}_4 + \psi^{41}_1 + \psi^{13}_1 + \psi^{23}_2$	0	0	0	0
$d_{12} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{42}_2$	0	0	0	0
$d_{13} = \psi^{33}_3 + \psi^{44}_4 + \psi^{13}_1 + \psi^{24}_2$	0	0	0	0
$d_{14} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{24}_2$	0	0	0	0
$d_{15} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{14}_1 + \psi^{42}_2$	0	0	0	0
$d_{16} = \psi^{33}_3 + \psi^{44}_4 + \psi^{13}_1 + \psi^{41}_1 + \psi^{24}_2$	0	0	0	0
$d_{17} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{32}_2 + \psi^{14}_1 + \psi^{24}_2$	1	3	0	0
$d_{18} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{32}_2$	1	3	0	0
$d_{19} = \psi^{33}_3 + \psi^{44}_4 + \psi^{13}_1 + \psi^{23}_2$	1	3	0	0
$d_{20} = \psi^{33}_3 + \psi^{44}_4 + \psi^{23}_2 + \psi^{31}_1$	1	1	0	1
$d_{21} = \psi^{33}_3 + \psi^{44}_4 + \psi^{13}_1 + \psi^{32}_2 + \psi^{41}_1 + \psi^{24}_2$	1	1	1	1
$d_{22} = \psi^{33}_3 + \psi^{44}_4 + \psi^{32}_2 + \psi^{23}_2 + \psi^{31}_1$	2	1	1	1
$d_{23} = \psi^{33}_3 + \psi^{44}_4 + \psi^{32}_2 + \psi^{23}_2 + \psi^{13}_1$	2	1	1	1
$d_{24} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{14}_1$	2	1	1	1
$d_{25} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{42}_2 + \psi^{24}_2$	2	1	1	1
$d_{26} = \psi^{33}_3 + \psi^{44}_4 + \psi^{13}_1 + \psi^{42}_2 + \psi^{24}_2$	2	1	1	1
$d_{27} = \psi^{33}_3 + \psi^{44}_4 + \psi^{13}_1 + \psi^{41}_1 + \psi^{42}_2 + \psi^{24}_2$	2	1	1	1
$d_{28} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{14}_1 + \psi^{42}_2 + \psi^{24}_2$	2	1	1	1
$d_{29} = \psi^{33}_3 + \psi^{44}_4 + \psi^{32}_2$	2	1	2	2
$d_{30} = \psi^{33}_3 + \psi^{44}_4 + \psi^{23}_2$	2	1	2	2
$d_{31} = \psi^{33}_3 + \psi^{44}_4 + \psi^{32}_2 + \psi^{23}_2$	4	2	2	2
$d_{32} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{13}_1 + \psi^{42}_2 + \psi^{24}_2$	4	2	2	2
$d_{33} = \psi^{33}_3 + \psi^{44}_4 + \psi^{31}_1 + \psi^{13}_1 + \psi^{32}_2 + \psi^{23}_2$	4	4	6	12
$d_{34} = \psi^{33}_3 + \psi^{44}_4$	4	4	8	16

TABLE 5. The cohomology of the algebras $d_{10} \dots d_{34}$

d_3 is not commutative, illustrating the fact that a commutative algebra can deform to a noncommutative algebra.

The final algebra in this group, d_{34}, is the direct sum of the trivial 2-dimensional algebra $\mathbb{C}^2_0$ with $\mathbb{C}^2$, so it is not unital, but is commutative. This algebra has a lot of deformations, with jump deformations to d_2, d_4, d_5, d_6, d_7 and d_9. Note that even though d_{34} is isomorphic to its opposite, it has jump deformations to d_4 and d_5, which are not their own opposites. However, they are opposite algebras, illustrating the fact that if an algebra which is isomorphic to its opposite deforms to another algebra, it also deforms to the opposite of that algebra.

8.3. **The algebras $d_{35} \dots d_{52}$.** The algebra d_{35} which arises by adjoining an identity to the 3-dimensional nilpotent algebra $d_{19} = \psi^{31}_2 + \psi^{13}_2 + \psi^{33}_1$ is both unital and commutative. It has jump deformations to d_2, d_6, d_8 and d_{32}.

The algebra d_{36} which is the direct sum of $\mathbb{C}$ and the 3-dimensional nilpotent algebra d_{19} above, is nonunital but commutative, and it deforms to d_2 d_6, d_7, d_8, d_9, and d_{31}.

The family of algebras $d_{37}(p:q)$ is parameterized by $\mathbb{P}^1/\Sigma_2$, which means that it is a projective family, in the sense that $d_{37}(p:q) \sim d_{37}(up:uq)$ when $u \in \mathbb{C}^*$,

Codifferential	H^0	H^1	H^2	H^3
$d_{35} = \psi_4^{44} + \psi_2^{31} + \psi_2^{13} + \psi_1^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	4	3	3	3
$d_{36} = \psi_4^{44} + \psi_2^{31} + \psi_2^{13} + \psi_1^{33}$	4	3	3	3
$d_{37}(p:q) = \psi_4^{44} + q\psi_2^{31} + p\psi_2^{13} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	2	2	1	0
$d_{37}(1:1) = \psi_4^{44} + \psi_2^{31} + \psi_2^{13} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	4	4	5	6
$d_{37}(1:-1) = \psi_4^{44} - \psi_2^{31} + \psi_2^{13} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	2	2	1	1
$d_{37}(1:0) = \psi_4^{44} + \psi_2^{13} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	2	2	3	5
$d_{37}(0:1) = \psi_4^{44} + \psi_2^{31} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	2	2	3	5
$d_{37}(0:0) = \psi_4^{44} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	4	5	7	13
$d_{38}(p:q) = \psi_4^{44} + q\psi_2^{31} + p\psi_2^{13} + \psi_2^{33}$	2	2	3	3
$d_{38}(1:1) = \psi_4^{44} + \psi_2^{31} + \psi_2^{13} + \psi_2^{33}$	4	4	5	7
$d_{38}(1:-1) = \psi_4^{44} - \psi_2^{31} + \psi_2^{13} + \psi_2^{33}$	2	2	3	4
$d_{38}(1:0) = \psi_4^{44} + \psi_2^{13} + \psi_2^{33}$	2	2	5	8
$d_{38}(0:1) = \psi_4^{44} + \psi_2^{31} + \psi_2^{33}$	2	2	5	8
$d_{38}(0:0) = \psi_4^{44} + \psi_2^{33}$	4	5	9	17
$d_{39} = \psi_4^{44} + \psi_2^{13} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_1^{43} - \psi_1^{34} + \psi_2^{24} + \psi_3^{34}$	1	1	0	0
$d_{40} = \psi_4^{44} + \psi_2^{13} + \psi_2^{33} - \psi_1^{43} + \psi_3^{43} + \psi_1^{14} + \psi_1^{34}$	1	1	1	0
$d_{41} = \psi_4^{44} + \psi_2^{13} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_1^{43}$	0	1	1	1
$d_{42} = \psi_4^{44} + \psi_2^{13} + \psi_2^{33} - \psi_1^{34} + \psi_2^{24} + \psi_3^{34}$	0	1	1	1
$d_{43} = \psi_4^{44} + \psi_2^{13} + \psi_2^{33} - \psi_1^{43} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	0	1	1	1
$d_{44} = \psi_4^{44} + \psi_2^{13} + \psi_2^{33} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_1^{34}$	0	1	1	1
$d_{45} = \psi_4^{44} + \psi_2^{33} + \psi_1^{41} + \psi_2^{24} + \psi_2^{42} + \psi_3^{34} + \psi_3^{43}$	2	2	2	2
$d_{46} = \psi_4^{44} + \psi_2^{33} + \psi_1^{14} + \psi_2^{24} + \psi_2^{42} + \psi_3^{34} + \psi_3^{43}$	2	2	2	2
$d_{47} = \psi_4^{44} + \psi_2^{33} + \psi_1^{14}$	2	2	3	3
$d_{48} = \psi_4^{44} + \psi_2^{33} + \psi_1^{41}$	2	2	3	3
$d_{49} = \psi_4^{44} + \psi_2^{33} + \psi_1^{14} + \psi_1^{41}$	4	3	3	3
$d_{50} = \psi_4^{44} + \psi_2^{33} + \psi_2^{24} + \psi_2^{42} + \psi_3^{34} + \psi_3^{43}$	4	3	3	3
$d_{51} = \psi_4^{44} - \psi_2^{31} + \psi_2^{13} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	2	4	6	8
$d_{52} = \psi_4^{44} - \psi_2^{31} + \psi_2^{13}$	2	4	5	8

TABLE 6. The cohomology of the algebras $d_{35} \dots d_{52}$

and is invariant under the action of Σ_2 by interchanging coordinates, in the sense that $d_{37}(p:q) \sim d_{37}(q:p)$.

We remark that there is an element $d_{37}(0:0)$ corresponding to what is called the generic point in $\mathbb{P}^1$. This point is usually omitted in the definition of $\mathbb{P}^1$, because including this generic point makes $\mathbb{P}^1$ a non-Hausdorff space. In fact, this non-Hausdorff behavior is reflected in the deformations of the point $d_{37}(0:0)$, so the inclusion of the corresponding codifferential in the family here is quite natural. With the families, there is a generic deformation pattern, and then there are some special values of the parameter $(p:q)$ for which the deformation pattern is not generic in the sense that there are additional deformations. Generically, this family consists of unital but not commutative algebras. Because the opposite algebra to $d_{37}(p:q)$ is $d_{37}(q:p)$ which is isomorphic to the original algebra, all of the elements of this family are isomorphic to their own opposite algebras.

Generically, an element in this algebra deforms in a smooth way to other elements in the family, and these are the only deformations. We say that the deformations are along the family. In fact, in every family of codifferentials, there are always smooth deformations along the family. In this case, these are the only deformations which occur generically.

The element $d_{37}(1:0)$ has additional jump deformations to d_3, d_{27} and d_{28}. The element $d_{1:1}$ is commutative, and has additional jump deformations to d_2, d_6 d_8, d_{32} and d_{35}, all of which are unital, commutative algebras.

If an element in a family has a deformation to an algebra, then the generic element in the family will also deform to it. Moreover, the generic element always has jump deformations to all other elements in the family, so $d_{37}(0:0)$ has jump deformations to $d_{37}(p:q)$ for all $(p:q)$ except $(0:0)$. Thus we automatically know that $d_{37}(0:0)$ has jump deformations the elements to which $d_{37}(1:0)$ and $d_{37}(1:1)$ deform. In addition, there is a jump deformation from $d_{37}(0:0)$ to d_{33}. We also note that $d_{37}(0:0)$ is commutative.

The family $d_{38}(p:q)$ is also parameterized projectively by $\mathbb{P}^1/\Sigma_2$. Generically, the elements of the family are not commutative, and the only deformations are smooth deformations along the family.

The algebra $d_{38}(1:0)$ also has jump deformations to d_3, d_4, d_5, d_{22}, d_{23}, d_{29}, and d_{30}. The algebra $d_{38}(1:1)$ is commutative and has additional jump deformations to d_2, d_6, d_7, d_8, d_9, d_{31}, and d_{36}. Finally, the generic element, which is commutative, in addition to all deformations above, and jump deformations to every other element of the family, also has jumps to d_3, d_{33} and d_{34}.

The algebras $d_{39}\dots d_{48}$ are neither unital nor commutative. The algebras d_{39} are isomorphic to their opposite algebras. The algebra d_{39} is rigid, while the algebra d_{40} has a jump deformation to d_1. The algebras d_{41} and d_{42} are opposite algebras, with d_{41} having a jump deformation to d_{10}, while d_{42} deforms to the opposite algebra d_{11}. The algebras d_{43} and d_{44} are opposites, with d_{43} jumping to d_{13} and d_{44} jumping to its opposite d_{12},

The algebras d_{45} and d_{46} are opposites, with d_{45} jumping to d_5, d_{22} and d_{25}, while d_{46} jumps to their opposite algebras d_4, d_{23} and $d26$. The algebras d_{47} and d_{48} are opposite algebras, with d_{47} having jump deformations to d_3, d_4, d_{24}, d_{26}, d_{27}, and d_{30}, while d_{48} jumps to d_3, d_5, d_{24}, d_{25}, d_{28}, and d_{29}.

The algebras d_{49} and d_{50} are nonunital but are commutative, with d_{49} jumping to d_2, d_6, d_7, d_9, d_{31}, and d_{32}, while d_{50} jumps to d_2, d_6, d_7, d_8, and d_{31}.

The algebra d_{51} is unital, with jump deformations to d_1, d_{21}, $d_{37}(1:-1)$, and deforms smoothly near $d_{37}(1:-1)$. This type of smooth deformation is said to factor through the jump deformation to $d_{37}(1:-1)$.

The algebra d_{52} is neither unital nor commutative, but is isomorphic to its own opposite algebra. It has jump deformations to d_{20} and $d_{38}(1:-1)$, as well as deforming smoothly in a neighborhood of $d_{38}(1:-1)$.

8.4. **The algebras $d_{53}\dots d_{72}$.** The algebras $d_{53}\dots d_{68}$ are neither unital nor commutative. The algebras d_{53} and d_{54}, and d_{55} and d_{56}, are pairs of opposite algebras, and they are all rigid. The algebra d_{57} is isomorphic to its opposite algebra, and jumps to d_{14}, d_{20} and d_{39}. the algebras d_{58} and d_{59} are opposites, with the former jumping to d_3, d_5, d_{22}, d_{27}, and d_{29}, while the latter jumps to d_3, d_4, d_{23}, d_{28}, and d_{30}.

The algebra d_{60} is isomorphic to its opposite algebra, and jumps to d_1, d_{15}, d_{16}, d_{20}, d_{21}, and d_{40}. The algebras d_{61} and d_{62} are opposites, with d_{61} jumping to d_{12}, d_{18} and d_{44}, while d_{62} jumps to d_{13} d_{19} and d_{43}. The algebras d_{63} and d_{64}

Codifferential	H^0	H^1	H^2	H^3
$d_{53} = \psi_4^{44} + \psi_1^{14} + \psi_2^{24} + \psi_3^{43}$	0	4	0	8
$d_{54} = \psi_4^{44} + \psi_1^{41} + \psi_2^{42} + \psi_3^{34}$	0	4	0	8
$d_{55} = \psi_4^{44} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43}$	0	8	0	0
$d_{56} = \psi_4^{44} + \psi_1^{14} + \psi_2^{24} + \psi_3^{34}$	0	8	0	0
$d_{57} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{24} + \psi_3^{43}$	1	2	3	5
$d_{58} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{42}$	2	2	3	4
$d_{59} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{24}$	2	2	3	4
$d_{60} = \psi_4^{44} + \psi_1^{41} + \psi_3^{34}$	1	2	4	6
$d_{61} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{42} + \psi_3^{43}$	1	4	4	4
$d_{62} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{24} + \psi_3^{34}$	1	4	4	4
$d_{63} = \psi_4^{44} + \psi_1^{41} + \psi_2^{42}$	1	4	5	5
$d_{64} = \psi_4^{44} + \psi_1^{14} + \psi_2^{24}$	1	4	5	5
$d_{65} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{24} + \psi_2^{42}$	4	5	7	13
$d_{66} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{24} + \psi_2^{42} + \psi_3^{43}$	2	4	8	16
$d_{67} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{24} + \psi_2^{42} + \psi_3^{34}$	2	4	8	16
$d_{68} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41}$	4	5	9	17
$d_{69} = \psi_4^{44} + \psi_1^{41}$	2	4	10	20
$d_{70} = \psi_4^{44} + \psi_1^{14}$	2	4	10	20
$d_{71} = \psi_4^{44} + \psi_1^{14} + \psi_1^{41} + \psi_2^{24} + \psi_2^{42} + \psi_3^{34} + \psi_3^{43}$	4	9	24	72
$d_{72} = \psi_4^{44}$	4	9	27	81

TABLE 7. The cohomology of the algebras $d_{53} \dots d_{72}$

are opposite algebras, with d_{63} having jump deformations to d_{10}, d_{17}, d_{18}, and d_{41}, while d_{64} jumps to d_{11}, d_{17}, d_{19}, and d_{42}.

The algebra d_{65} is its own opposite algebra, and it jumps to d_2, d_3, d_6, d_7, d_8, d_{24}, d_{31}, d_{33}, and d_{50}. The algebras d_{66} and d_{67} are opposite algebras, with d_{66} having jump deformations to d_5, d_{10}, d_{15}, d_{22}, d_{25}, and d_{45}, and d_{67} jumping to d_4, d_{11}, d_{16}, d_{23}, d_{26}, and d_{46}.

The algebra d_{68} is not unital, but is commutative, and it jumps to d_2, d_4, d_5, d_6, d_7, d_9, d_{25}, d_{26}, d_{31}, d_{32}, d_{34} and d_{49}. The opposite algebras d_{69} and d_{70} are neither unital nor commutative, with the former deforming to d_3, d_5, d_{11}, d_{14}, d_{15}, d_{24}, d_{25}, d_{28}, d_{29}, d_{48}, and the latter to d_3, d_4, d_{10}, d_{13}, d_{14}, d_{16}, d_{24}, d_{26}, d_{27}, d_{30}, and d_{47}.

The algebra d_{71} is the algebra arising by adjoining an identity to the trivial 3-dimensional algebra $\mathbb{C}_0^3$, so it is unital and commutative, and it has deformations to every unital algebra, that is, to d_1, d_2, d_3, d_6, d_8, d_{17}, d_{21}, d_{27}, d_{28}, d_{32}, d_{33}, d_{35}, every element of the family $d_{37}(p:q)$, and d_{51}.

Finally, the algebra d_{72} is the direct sum of $\mathbb{C}$ and $\mathbb{C}_0^3$, so it is nonunital and commutative. It has jump deformations to $d_2 \dots d_9$, d_{18}, d_{19}, d_{20}, d_{22}, d_{23}, d_{29}, d_{30}, d_{31}, d_{33}, d_{34}, d_{36}, all members of the family $d_{38}(p:q)$, and d_{52}.

9. Unital Algebras

There are 15 unital algebras, including all of the elements in the family $d_{37}(p:q)$. According to [18], unital complex 4-dimensional associative algebras were classified by P. Gabriel [9], and this classification is in agreement with the unital algebras we

Codifferential	Gabriel Number	Structure
d_1	10	$\mathfrak{gl}(2,\mathbb{C})$
d_2	1	$\mathbb{C}^4$
d_3	13	$\mathbb{C}\oplus\{\left[\begin{smallmatrix} a & b\\ 0 & c\end{smallmatrix}\right]\,\lvert a,b,c\in\mathbb{C}\}$
d_6	2	$\mathbb{C}^2\oplus\mathbb{C}[x]/(x^2)$
d_8	4	$\mathbb{C}\oplus\mathbb{C}[x]/(x^3)$
d_{17}	17	$\left\{\left[\begin{smallmatrix} a & 0 & 0\\ 0 & a & 0\\ c & d & b\end{smallmatrix}\right]\,\lvert a,b,c,d\in\mathbb{C}\right\}$
d_{21}	11	$\left\{\left[\begin{smallmatrix} a & 0 & 0 & 0\\ 0 & a & 0 & d\\ c & 0 & b & 0\\ 0 & 0 & 0 & b\end{smallmatrix}\right]\,\lvert a,b,c,d\in\mathbb{C}\right\}$
d_{27}	15	$\left\{\left[\begin{smallmatrix} a & c & d\\ 0 & a & 0\\ 0 & 0 & b\end{smallmatrix}\right]\,\lvert a,b,c,d\in\mathbb{C}\right\}$
d_{28}	14	$\left\{\left[\begin{smallmatrix} a & 0 & 0\\ c & a & 0\\ d & 0 & b\end{smallmatrix}\right]\,\lvert a,b,c,d\in\mathbb{C}\right\}$
d_{32}	3	$\mathbb{C}[x]/(x^2)\oplus\mathbb{C}[y]/(y^2)$
d_{33}	6	$\mathbb{C}\oplus\mathbb{C}[x,y]/(x^2,xy,y^2)$
d_{35}	5	$\mathbb{C}[x]/(x^4)$
$d_{37}(p:q)$	$18(u)$	$\mathbb{C}[x,y]/(x^2,y^2,yx,-uxy),\quad u\neq -1$
$d_{37}(1:-1)$	19	$\mathbb{C}[x,y]/(y^2,x^2+yx,xy+yx)$
$d_{37}(1:1)$	7	$\mathbb{C}[x,y]/(x^2,y^2)$
$d_{37}(1:0)$	16	$\mathbb{C}[x,y]/(x^2,y^2,yx)$
$d_{37}(0:0)$	8	$\mathbb{C}[x,y]/(x^3,xy,y^2)$
d_{51}	12	$\bigwedge\mathbb{C}^2$
d_{71}	9	$\mathbb{C}[x,y,z]/(x,y,z)^2$

TABLE 8. The structure of 4-dimensional unital algebras

determined by our methods. Note that no nilpotent algebra can be unital, so the classification of the nonnilpotent algebras given here is sufficient to determine all of the unital algebras.

10. COMMUTATIVE ALGEBRAS

There are 20 distinct nonnilpotent commutative algebras, of which 9 are unital. Every commutative algebra is a direct sum of algebras which are ideals in quotients of polynomial algebras. Every finite dimensional unital commutative algebra is a quotient of a polynomial algebra, while every finite dimensional nonunital algebra is an ideal in such an algebra. The algebra $\mathbb{C}$ is representable as $\mathbb{C}[x]/(x)$, while the trivial algebra $\mathbb{C}_0$ is representable as the ideal $x\mathbb{C}[x]/(x^2)$. In Table 7, the ideal (x,y) in $\mathbb{C}[x,y]/(x^2-xy,y^2)$ has dimension 3 as a vector space over $\mathbb{C}$, and the algebra $d_{38}(1:1)$ is expressed as a direct sum of $\mathbb{C}$ and that ideal, which gives a 4-dimensional algebra.

For completeness here, in Table 8, we give the nilpotent commutative algebras as well. The codifferential number given relates to the dsscription of codifferentials which will appear in a sequel. These algebras were classified by Hazlett [15], and also given in [17]. There are 8 nontrivial commutative algebras.

We note that commutative algebras may deform into noncommutative algebras, but noncommutative algebras never deform into a commutative algebras. The fact

Codifferential	Structure
d_2	$\mathbb{C}^4$
d_6	$\mathbb{C}^2 \oplus \mathbb{C}[x]/(x^2)$
d_7	$\mathbb{C}^3 \oplus \mathbb{C}_0$
d_8	$\mathbb{C} \oplus \mathbb{C}[x]/(x^3)$
d_9	$\mathbb{C}^2 \oplus x\mathbb{C}[x]/(x^3)$
d_{31}	$\mathbb{C} \oplus \mathbb{C}_0 \oplus \mathbb{C}[x]/(x^2)$
d_{32}	$\mathbb{C}[x]/(x^2) \oplus \mathbb{C}[y]/(y^2)$
d_{33}	$\mathbb{C} \oplus \mathbb{C}[x,y]/(x^2,xy,y^2)$
d_{34}	$\mathbb{C}^2 \oplus \mathbb{C}_0^2$
d_{35}	$\mathbb{C}[x]/(x^4)$
d_{36}	$\mathbb{C} \oplus x\mathbb{C}[x]/(x^4)$
$d_{37}(1:1)$	$\mathbb{C}[x,y]/(x^2,y^2)$
$d_{37}(0:0)$	$\mathbb{C}[x,y]/(x^3,xy,y^2)$
$d_{38}(1:1)$	$\mathbb{C} \oplus (x,y) \leq \mathbb{C} \oplus \mathbb{C}[x,y]/(x^2-xy,y^2)$
$d_{38}(0:0)$	$\mathbb{C} \oplus \mathbb{C}_0 \oplus x\mathbb{C}[x]/(x^3)$
d_{49}	$\mathbb{C}[x]/(x^2) \oplus y\mathbb{C}[y]/(y^3)$
d_{50}	$\mathbb{C}_0 \oplus \mathbb{C}[x]/(x^3)$
d_{68}	$\mathbb{C}_0^2 \oplus C[x]/(x^2)$
d_{71}	$\mathbb{C}[x,y,z]/(x,y,z)^2$
d_{72}	$\mathbb{C} \oplus \mathbb{C}_0^3$

TABLE 9. The structure of nonnilpotent 4-dimensional commutative algebras

that commutative algebras have noncommutative deformations plays an important role in physics, and deformation quantization describes a certain type of deformation of a commutative algebra into a noncommutative one.

Codifferential	Structure
d_{74}	$x\mathbb{C}[x]/(x^5)$
$d_{75}(1:1)$	$(x,y) \leq \mathbb{C}[x,y]/(x^2-y^2,yx^2,xy^2)$
$d_{75}(0:0) = d_{86}(1:1)$	$\mathbb{C}_0 \oplus (x,y) \leq \mathbb{C}_0 \oplus \mathbb{C}[x,y]/(x^2-y^2,xy)$
d_{76}	$(x,y) \leq \mathbb{C}[x,y]/(y^3-x^2,xy)$
$d_{79}(1:1)$	$(x,y) \leq \mathbb{C}[x,y]/(y^2,x^2y,x^3)$
$d_{79}(0:0) = d_{86}(0:0)$	$\mathbb{C}_0^2 \oplus x\mathbb{C}[x]/(x^3)$
d_{83}	$\mathbb{C}_0 \oplus x\mathbb{C}[x]/(x^4)$
d_{85}	$(x,y,z) \leq \mathbb{C}[x,y,z]/(x^2-y^2,y^2-yz,xy,xz,z^2)$
d_0	$\mathbb{C}_0^4$

TABLE 10. Nilpotent 4-dimensional commutative algebras

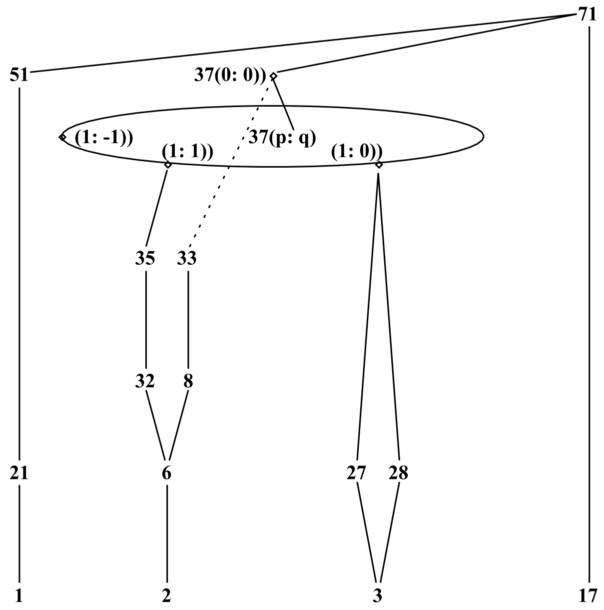

FIGURE 1. Deformations between unital algebras

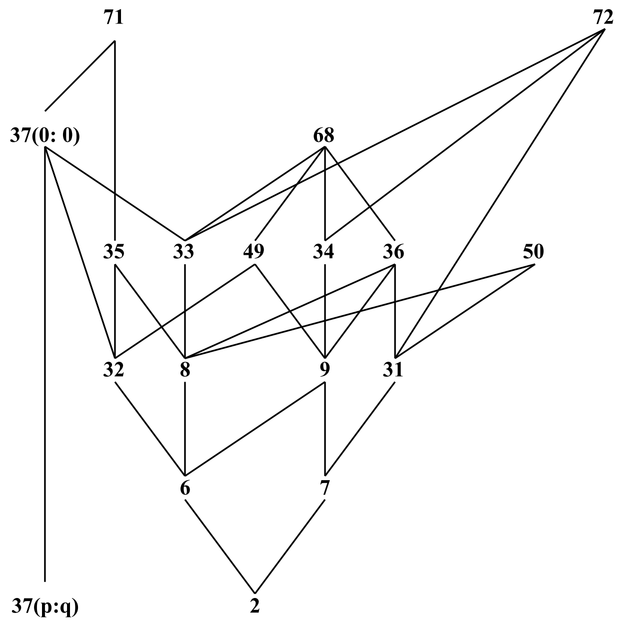

FIGURE 2. Deformations between nonnilpotent commutative algebras

11. Levels of algebras

It would be difficult to construct a picture showing the jump deformations for all 72 families of nonnilpotent complex 4-dimensional algebras, as we did for the unital and commutative algebras, because there are too many of them. Instead, we give a table showing the levels of each algebra. To define the level, we say that a rigid algebra has level 1, an algebra which has only jump deformations to an algebra on level one has level two and so on. To be on level $k+1$, an algebra must have a jump deformation to an algebra on level k, but no jump deformations to algebras on a level higher than k. For families, if one algebra in the family has a jump to an element on level k, then we place the the entire family on at least level $k+1$. Thus, even though generically, elements of the family $d_{37}(p:q)$ deform only to members of the same family, there is an element in the family which has a jump to an element on level 4. For the generic element in a family, we consider it to be on a higher level than the other elements because it has jump deformations to the other elements in its family.

Level	Codifferentials
1	1,2,3,4,5,10,11,12,13,14,15,16,17,18,19,20,39,53,54,55,56
2	6,7,21,22,23,24,25,26,27,28,29,30,40,41,42,43,44,57
3	8,9,31,32,45,46,47,48,58,59,60,61,62,63,64
4	33,34,35,36,49,50,66,67,69,70
5	65,68,37$(p:q)$, 38$(p:q)$
6	37$(0:0)$,38$(0:0)$, 51,52
7	71,72

Table 11. The levels of the algebras

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