DEFORMATIONS AND CONTRACTIONS OF ALGEBRAIC STRUCTURES

ALICE FIALOWSKI

Abstract. I describe the basic notions of versal deformation theory of algebraic structures and compare it with the analytic theory. As a special case, I consider the notion of versal deformation used by Arnold. With the help of versal deformation we get a stratification of the moduli space into projective orbifolds. I compare this with Arnold's stratification in the case of similarity of matrices. The other notion I discuss is the opposite notion of contraction.

1. Introduction

Deformations of analytic and algebraic objects is an old problem both in mathematics and physics. The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. The fundamental idea, which should be credited to Riemann, was to introduce an analytic structure therein.

In mathematics one of the main tools to study a given object is by deforming it to families with "similar" structure. In physics we can develop from known theories new ones, like quantum groups, deformations of Hopf algebras, q-deformed physics, deformation quantization, deformed geometry and gravitation, quantum field theory etc.

In this paper I restrict myself to the case of Lie algebras – one of the most important categories in physics. The paper is based on the talk given at the conference in Moscow in 2013 for Victor Buchstaber's 70th birthday.

2. Basic notions

Let $\mathcal{L}$ be a Lie algebra with Lie bracket μ_0 over a field $\mathbb{K}$.

a) ***Intuitive definition of deformation.*** A deformation of $\mathcal{L}$ is a one-parameter family $\mathcal{L}_t$ of Lie algebras with the bracket (possibly infinite series)

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where φ_i are $\mathcal{L}$-valued 2-cochains, i.e. elements of $\mathrm{Hom}_{\mathbb{K}}(\Lambda^2\mathcal{L}, \mathcal{L}) = C^2(\mathcal{L};\mathcal{L})$, and $\mathcal{L}_t$ is a Lie algebra for each $t \in \mathbb{K}$. Two deformations, $\mathcal{L}_t$ and $\mathcal{L}'_t$ are equivalent if there exists a linear automorphism $\widehat{\psi}_t = \mathrm{id} + \psi_1 t + \psi_2 t^2 + \dots$ of $\mathcal{L}$ where ψ_i are linear maps over $\mathbb{K}$, i.e. elements of $C'(\mathcal{L};\mathcal{L})$ such that

$$\mu'_t(x,y) = \widehat{\psi}_t^{-1}\big(\mu_t(\widehat{\psi}_t(x), \widehat{\psi}_t(y))\big) \quad \text{for} \quad x, y \in \mathcal{L}.$$

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The Jacobi identity for the algebras $\mathcal{L}_t$ implies that the 2-cochain φ_1 is indeed a cocycle. If μ'_t is an equivalent deformation with cochains φ'_i, then

$$\varphi'_1 - \varphi_1 = d_1\psi_1$$

hence every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L};\mathcal{L})$.

We call a deformation *infinitesimal* if it is a deformation up to order 1. It follows that nonequivalent infinitesimal deformations are in one-to-one correspondence with the 2-dimensional chomology classes. (For details see [9].)

The problem with the classical theory is that it is not satisfactory to describe all nonequivalent deformations of a given object. For that purpose one has to introduce *deformations with base*.

b) *General definition*. Consider a deformation $\mathcal{L}_t$ and as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra over $\mathbb{K}$ with identity as base of a deformation. Let us fix an augmentation $\varepsilon : A \to \mathbb{K}$, $\varepsilon(1) = 1$ and set $\operatorname{Ker}\varepsilon = m$, which is a maximal ideal. (For details see [2, 3].)

Definition 2.1. A deformation λ of $\mathcal{L}$ with base (A, m) is a Lie A-algebra structure on $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,\ ]_\lambda$ such that

$$\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structure, compatible with $\varepsilon \otimes \mathrm{id}$.

c) *Formal deformations*. Let A be a complete local algebra. A formal deformation of $\mathcal{L}$ with base A is a Lie A-algebra structure on the completed tensor product $A\widehat{\otimes}\mathcal{L} = \overleftarrow{\lim\limits_{n\to\infty}}\big((A/m^n) \otimes \mathcal{L}\big)$ s.t.

$$\varepsilon\widehat{\otimes}\mathrm{id} : A\widehat{\otimes}\mathcal{L} \longrightarrow \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

d) *Miniversal formal deformations*. It is known that in the category of algebraic varieties the quotient by a group action does not always exist. Specifically, there is no universal deformation in general of a Lie algebra $\mathcal{L}$ with a commutative algebra base B with the property that for any other deformation of $\mathcal{L}$ with base A there exists a unique homomorphism $f : B \to A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of $\mathcal{L}$ with base B *versal*.

Definition 2.2 ([3])**.** A formal deformation η of a Lie algebra $\mathcal{L}$ with a complete local algebra base B is called *miniversal*, if

i) for any formal deformation λ of $\mathcal{L}$ with any complete local base A there exists a homomorphism $f : B \to A$ s.t. the deformation λ is equivalent to the push-out of η by f;
ii) if A satisfies $m^2 = 0$, then f is unique.

Using Schlessinger's general set-up [14], I was able to prove that for formal deformations, under some minor restriction, there exists a miniversal deformation.

Theorem 2.3 ([3]). *Assume that the space $H^2(\mathcal{L};\mathcal{L})$ is finite-dimensional. Then there exists a miniversal formal deformation of $\mathcal{L}$, and the base of this deformation is formally embedded into $H^2(\mathcal{L};\mathcal{L})$, i.e. it can be described in $H^2(\mathcal{L};\mathcal{L})$ by a finite system of equations.*

Construction of miniversal deformation

1. Universal infinitesimal deformation

Let $A = \mathbb{K} \oplus H^2(L;L)'$. Fix a homomorphism

$$\mu : H^2(L;L) \to C^2(L;L) = \operatorname{Hom}\big(H^2(L;L), L\big).$$

Define a Lie A-algebra structure on the space

$$A \otimes L = \mathbb{K} \otimes L \oplus H^2(L;L)' \otimes L = L \oplus \operatorname{Hom}\big(H^2(L;L), L\big)$$

with the bracket

$$[\ell_1 \otimes \phi_1, \ell_2 \otimes \phi_2] = \big([\ell_1,\ell_2], \psi\big), \quad \text{where}$$
$$\psi(\alpha) = \mu(x)(\ell_1,\ell_2) + \big[\phi_1(\alpha), \ell_2\big] + \big[\ell_1, \phi_2(\alpha)\big]$$
$$\ell_1,\ell_2 \in L, \quad \phi_1,\phi_2 \in \operatorname{Hom}\big(H^2(L;L), L\big), \quad \alpha \in H^2(L;L).$$

This is an infinitesimal deformation with base A.

Proposition 2.4 ([4]). The deformation L_A constructed in this way is universal.

2. The construction follows a recursive procedure. In order to extend the infinitesimal deformation to the next order, we need that all the Massey 2-products are trivial. The Massey products are generated by cohomology classes in $H^3(L;L)$, and the base of a miniversal deformation will be the factor of the algebra of formal power series $\mathbb{K}\big[[H^2(L;L)]'\big]$ by some ideal. The ideal is generated by the nontrivial Massey products, corresponding to the generating cocycles of $H^3(L;L)$.

Suppose that $\dim H^2(L;L) < \infty$.

Let $C_0 = \mathbb{K}$, $C_1 = \mathbb{K} \oplus H^2(L;L)'$, and let

$$0 \to H^2(L;L)' \xrightarrow{i_1} C_1 \xrightarrow{p_1'} \mathbb{K} \to 0$$

be the canonical splitting extension. The universal infinitesimal deformation of L with base C_1 will be denoted here by η_1. Suppose that for some $k \geq 1$ we have already constructed a finite-dimensional commutative algebra C_k and a deformation η_k of L with base C_k. Consider the extension

$$0 \to H^2_{\text{Harr}}(C_k;\mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}_{k+1}'} C_k \to 0$$

using the cocycle f_{C_k}. The operation of C_k on $H^2_{\text{Harr}}(C_k;\mathbb{K})'$ is induced by the operation of C_k on $\mathbb{K}$, and the cocycle

$$f_{C_k} : S^2(C_k) \to H^2_{\text{Harr}}(C_k;\mathbb{K})$$

is defined as the dual of a homomorphism

$$\mu : H^2_{\text{Harr}}(C_k;\mathbb{K}) \to Ch^2(C_k;\mathbb{K}) = (S^2A)',$$

which takes a cohomology class to a cocycle from this class.

According to [4], we obtain the obstruction

$$\mathcal{O}_{\eta_k}(f_{C_k}) \in H^2_{\text{Harr}}(C_k,\mathbb{K})' \otimes H^3(L;L)$$

to the extension of η_k. This gives us a map

$$\omega_k\colon H^2_{\mathrm{Harr}}(C_k,\mathbb{K}) \to H^3(L;L).$$

Set

$$C_{k+1} = \bar{C}_{k+1}/\bar{i}_{k+1}\circ\omega'_k(H^3(L;L)').$$

Obviously, the extension (2) factorizes to an extension

$$0 \to (\operatorname{Ker}\omega_k)' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p'_{k+1}} C_k \to 0.$$

Notice that all the algebras C_k are local. Since C_k is finite-dimensional, the cohomology $H^2_{\mathrm{Harr}}(C_k;\mathbb{K})$ is also finite-dimensional, and hence C_{k+1} is finite-dimensional.

PROPOSITION 4.1.[4] *The deformation η_k admits an extension to a deformation with base C_{k+1}, and this extension is unique up to an isomorphism and an automorphism of an extension* (3).

We choose an extended deformation and denote it by η_{k+1}.

The induction yields a sequence of finite-dimensional algebras

$$\mathbb{K} \xleftarrow{p'_1} C_1 \xleftarrow{p'_2} \dots \xleftarrow{p'_k} C_k \xleftarrow{p'_{k+1}} C_{k+1} \xleftarrow{p'_{k+2}} \dots,$$

and a sequence of deformations η_k of L such that $(p'_{k+1})_*\eta_{k+1} = \eta_k$.

Taking the projective limit, we obtain a formal deformation η of L with base $C = \varprojlim_{k\to\infty} C_k$.

Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{K}[[H(L;L)']]$.

PROPOSITION 4.2.[4] $C_k = \mathbb{K}[[(H(L;L)']]/I_k$ *where*

$$\mathfrak{m}^2 = I_1 \supset I_2 \supset \dots,\ I_k \supset \mathfrak{m}^{k+1}.$$

3. Different definitions of miniversal deformation

The analytic definition of miniversal deformation is different and much earlier. In the Kodaira–Spencer theory a deformation is versal if the Kodaira–Spencer map is surjective, miniversal if the reduced Kodaira–Spencer map is isomorphism.

In this section we consider Arnold's presentation in [1] of versal deformations of matrices under a group action. There he considers deformations of the moduli space of similar matrices, that is $n\times n$ matrices under the action of $\mathbb{GL}(n)$ by conjugation. It is well known that every matrix is conjugate to a matrix in Jordan normal form, and this form is unique up to the ordering of the Jordan blocks.

A deformation $A(\lambda)$ of an $n\times n$ matrix A is a family of $n\times n$ matrices $A(\lambda)$ where the entries of the matrices are given by power series in a family of parameters λ_i, which converge in a neighborhood of the origin, such that $A(0) = A_0$. We can express this in a more algebraic language as follows. Let B_k be the algebra given by the germs of differentiable functions at the origin on $\mathbb{C}^k$ for some k. Then B_k is a formal algebra, with maximal ideal $\mathfrak{m}$ given by the functions which vanish at the origin. An element $A(\lambda)\in\mathfrak{gl}(n)\otimes B_k$ is a deformation of A_0 provided that $A(\lambda) = A_0+\Psi(\lambda)$, where $\Psi(\lambda)\in\mathfrak{gl}(n)\otimes\mathfrak{m}$. If φ is a germ of a holomorphic function $\mathbb{C}^l\to\mathbb{C}^k$ at the origin, such that $\varphi(0)=0$, and $A(\lambda)$ is a deformation of A_0 with base B_k, then φ determines an induced deformation $\varphi^*(A)(\mu)$ of A_0 with base B_l, given by $\varphi^*(A)(\mu) = A(\varphi(\mu))$.

Two deformations $A(\lambda)$ and $B(\lambda)$ of A_0 with base λ are equivalent if there is a deformation $C(\lambda)$ of the identity matrix, such that

$$B(\lambda) = C(\lambda)^{-1}A(\lambda)C(\lambda.$$

A deformation with base B_k is said to be versal if given any deformation $Q(\mu)$ with base B_l, there is a $\varphi : C^l \to C^k$ such that $Q(\mu)$ is equivalent to $\varphi^*(A)(\mu)$. If φ is determined uniquely, then $A(\lambda)$ is called a universal deformation of A_0.

Just as in the algebraic case, we cannot expect that a universal deformation of A_0 generally exists. A versal deformation $A(\lambda)$ is called miniversal when the dimension of the parameter space B_k is minimal. A versal deformation always exists, and therefore there is always a miniversal deformation as well.

The notion of versality is similar but not identical to the algebraic case as the following example will show. Consider the matrix $A_0 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Let $A(\lambda)$ be the deformation given by $A(\lambda) = \left[\begin{smallmatrix} \lambda_1 & 1+\lambda_2 \\ 0 & \lambda_3 \end{smallmatrix}\right]$. This deformation was given in [1] as an example of a deformation which is not versal. To see this, suppose that $Q(\mu) = \left[\begin{smallmatrix} \mu_1 & 1+\mu_2 \\ \mu_3 & \mu_4 \end{smallmatrix}\right]$. It is not hard to see that $Q(\mu)$ must be a versal deformation. Suppose we define $\varphi : C^4 \to C^3$ by $\varphi(\mu_1, \mu_2, \mu_3, \mu_4) = (\lambda_1, \lambda_2, \lambda_3)$, where

$$\begin{aligned} \lambda_1 &= 1/2\,(\mu_1 + \mu_4 + \sqrt{{\mu_4}^2 - 2\,\mu_4\mu_1 + {\mu_1}^2 + 4\,\mu_3\mu_2 + 4\,\mu_3}) \\ \lambda_2 &= \mu_2 \\ \lambda_3 &= 1/2\,(\mu_1 + \mu_4 - \sqrt{{\mu_4}^2 - 2\,\mu_4\mu_1 + {\mu_1}^2 + 4\,\mu_3\mu_2 + 4\,\mu_3}). \end{aligned}$$

Then $Q(\mu) = G^{-1}A(\varphi(\mu))G$, where the $g_{2,1}$ element of G is

$$-1/2\,\frac{-\mu_1 + \mu_4 + \sqrt{{\mu_4}^2 - 2\,\mu_4\mu_1 + {\mu_1}^2 + 4\,\mu_3\mu_2 + 4\,\mu_3}}{1 + \mu_2}.$$

Thus this mapping is not analytic at zero. As a consequence, it fails one of the conditions for versality.

One of the nice features of this analytic version of deformation theory is that versal deformations can be characterized in terms of transversality. Essentially, the idea is that the action of the group at a point determines a subspace of the tangent space at the point. A deformation $A(\lambda)$ is transversal if in a neighborhood of $A(0)$, the subspace of the tangent space at each point of $A(\lambda)$ parallel to the family is complementary to the subspace determined by the group action.

One of the nice results in [1] is the determination of the number of parameters of a miniversal deformation of a matrix A_0 in terms of the Jordan decomposition of the matrix.

Theorem 3.1 (Arnold [1]). *If A_0 is a matrix, then the number n of parameters of a miniversal deformation of A_0 is given by*

$$n = \sum_{\lambda} n_1 + 3n_2 + 5n_3 + \cdots\,,$$

where the sum is taken over all eigenvalues λ of the matrix, and $n_1 \geq n_2 \geq \cdots$ are the sizes of the Jordan blocks corresponding to λ.

The basic idea of the proof is as follows. If we consider a one parameter subgroup $\exp(tB)$ of $\mathbb{GL}(n)$, then a tangent vector to the action of this subgroup is $\frac{d}{dt}\exp(tB)^*(A) = [A, B]$. Consider the map f defined on the $n \times n$ matrices given by $f(B) = [A, B]$. Then the dimension of the tangent space to the group action at A is $\dim(\mathrm{Im}(f))$, and so its codimension $\dim(\ker(f))$ is just the dimension of the centralizer of A. This means the dimension of the miniversal deformation is given by the dimension of the centralizer of A. Arnold goes on to compute this dimension by looking at a Jordan normal form for A, and explicitly constructing its centralizer, from which he concludes the dimension formula above.

One should note that the centralizer subspace is not, in general, transverse to the tangent space of the group action, so that one cannot use the centralizer to compute the miniversal deformation directly. Nevertheless, Arnold gives an explicit form for a miniversal deformation based on the Jordan normal form of the matrix.

A natural question is what is the relation to this idea of transversality and miniversal deformations to the algebraic notion. First, note that the cocycle condition $[d,\varphi]=0$ is really a condition that at least on the infinitesimal level, the direction indicated by φ points along the variety of algebras. This condition does not arise in Arnold's formulation because he was not studying a subvariety of the vector space, so that all tangent directions are allowed. Next, the condition $\varphi=[d,\lambda]$ for a coboundary is precisely the same condition as in the Arnold formulation, because it means that φ is a tangent vector in the direction of the group action. Thus $H^2(d)$ represents the transverse directions to the group action.

The above description is more morally correct than precisely correct. For example, there is a finite dimensional Lie algebra d for which $H^2(d)$ does not vanish, but there are no nontrivial deformations of d. There is an infinitesimal deformation which does not extend to a formal deformation. Thus, in reality, there are no curves in the variety of Lie algebra structures through the algebra d which are transverse to the group action. This means the infinitesimal picture only gives an approximation to the actual deformation picture.

3.1. **A stratification of the moduli space.** Arnold gives a stratification of the space of $n\times n$ matrices in terms of "types" of Jordan decompositions. The authors have been studying moduli spaces of algebras and have found that for small dimensional spaces, the moduli spaces of Lie and associative algebras have a natural stratification by orbifolds of a very simple type. For Lie algebras, a portion of the moduli space is given by the action of a group on the space of $n\times n$ matrices, which is not quite the same action as conjugation by matrices, but is nearly so. We have an action of $\mathbb{GL}(n)\times\mathbb{C}^*$ on matrices given by $(G,x)^*(A)=xG^{-1}AG$. Thus our action is conjugation up to multiplication by a nonzero scalar. When we looked at the picture developed by Arnold, we realized that a similar stratification of the space of matrices up to conjugation can be given, which is consistent with Arnold's computation of the dimension of the miniversal deformations. We will describe the decompositions for dimensions 2, 3 and 4 below.

3.2. **Decomposition of $\mathfrak{gl}(2)$.** The space $\mathfrak{gl}(2)$, under the action of $\mathbb{GL}(2)\times\mathbb{C}^*$ has a stratification by the matrix types

$$A(p:q)=\begin{bmatrix}p&1\\0&q\end{bmatrix},\qquad B_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\qquad B_0=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

The first stratum can be parameterized by $\mathbb{CP}^1/\Sigma_2$, where the action of the symmetric group Σ_2 on $\mathbb{CP}^1$ is given by permutation of the projective coordinates $(p:q)$. In this formulation, the generic point $B(0:0)$ is considered to be an element of $\mathbb{CP}^1$, except that the number of parameters of the miniversal deformation of the generic point can be higher. The other two strata consists of singleton points. For the strata $A(p:q)$, the miniversal deformation has one parameter, while the miniversal deformation of $A(0:0)$ requires 2 parameters. The stratum with the singleton B_1 has 3 parameters in the miniversal deformation. Finally, the zero matrix B_0 also requires three parameters for its miniversal deformation.

The space $\mathfrak{gl}(2)$, under the action of $\mathbb{GL}(2)$ by conjugation has a stratification

$$A(p,q) = \begin{bmatrix} p & 1 \\ 0 & q \end{bmatrix}, \qquad B(p) = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}.$$

The stratum $A(p,q)$ is parameterized by $\mathbb{C}^2/\Sigma_2$, where Σ_2 acts by permutation of the coordinates (p,q). The stratum $B(p)$ is parameterized by $\mathbb{C}$. The miniversal deformation of the stratum $A(p,q)$ requires exactly 2 parameters, while the miniversal deformation of the stratum $B(p)$ requires 4 parameters.

Let us show how the count of the parameters in the Arnold case is obtained. For an element $A(p,q)$ when $p \neq q$, the parameter n is given by $n = 1 + 1$, because each eigenvalue contributes 1 to the total. But when $p = q$, we get a Jordan block of size 2, so that $n = 2$ again. For the matrix $B(p)$, we simply have $n = 1 + 3$ since there are 2 Jordan blocks of size 1.

The first thing to note is that in our formulation, the matrices $A(p,q)$ can have 2 different Jordan decompositions, so two different strata of Arnold's decomposition are combined to make one complete orbifold stratum. Secondly, we see that $A(p,q)$ is related to $A(p:q)$ and that the number of parameters in the projective formulation drops by 1, except for the generic value. A similar relation to $B(p)$ and B_1 occurs, since B_1 requires 1 less parameter than $B(p)$ for a miniversal deformation.

3.3. **Decomposition of $\mathfrak{gl}(3)$.** The space $\mathfrak{gl}(3)$, under the action of $\mathbb{GL}(3) \times \mathbb{C}^*$ has a stratification by the matrix types

$$A(p:q:r) = \begin{bmatrix} p & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & r \end{bmatrix} B(p:q) = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 1 \\ 0 & 0 & q \end{bmatrix} C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The first stratum $A(p:q:r)$ is parameterized by $\mathbb{CP}^2/\Sigma_3$, where Σ_3 acts by permuting the projective coordinates $(p:q:r)$. The miniversal deformation of an element in this stratum has 2 parameters except that $A(0:0:0)$ requires 3 parameters. The second stratum $B(p:q)$ is parameterized by $\mathbb{CP}^1$, and there is no action of Σ_2 on this stratum, which is clear from the fact that p and q play different roles in the matrix. The miniversal deformation has 4 parameters, except that $B(0:0)$ requires 5 parameters. The strata C_1 and C_0 each require 8 parameters for the miniversal deformation.

The space $\mathfrak{gl}(3)$, under the action of $\mathbb{GL}(3)$ by conjugation has a stratification

$$A(p,q,r) = \begin{bmatrix} p & 1 & 0 \\ 0 & q & 1 \\ 0 & 0 & r \end{bmatrix}, \quad B(p,q) = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 1 \\ 0 & 0 & q \end{bmatrix}, \quad C(p) = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}.$$

The first stratum, $A(p,q,r)$ is parameterized by $\mathbb{C}^3/\Sigma_3$, and has a miniversal deformation with 3 parameters. The second stratum, $B(p,q)$ is parameterized by $\mathbb{C}^2$, and requires 5 parameters for a miniversal deformation. Finally, the third stratum $C(p)$ is parameterized by $\mathbb{C}$, and requires 9 parameters.

As in the 2-dimensional example, the number of parameters in the Arnold correspondence is 1 larger than the projective correspondence, except for the generic element, for which the numbers are the same. Note for example, that the stratum $A(p,q,r)$ consists of the matrices with 3 Jordan blocks and 3 distinct eigenvalues, 2 Jordan blocks, one of size 2 and one of size 1, with 2 distinct eigenvalues, and one Jordan block of size 3, with one eigenvalue. In all three cases, the formula equation (3.1) gives the same number of parameters for the miniversal deformation.

3.4. **Decomposition of $\mathfrak{gl}(4)$.** The space $\mathfrak{gl}(4)$, under the action of $\mathbb{GL}(4)\times\mathbb{C}^*$ has a stratification by the matrix types

$$A(p:q:r:s)=\begin{bmatrix}p&1&0&0\\0&q&1&0\\0&0&r&1\\0&0&0&s\end{bmatrix},\quad E_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},E_2=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$B(p:q:r)=\begin{bmatrix}p&0&0&0\\0&p&1&0\\0&0&q&1\\0&0&0&r\end{bmatrix},C(p:q)=\begin{bmatrix}p&1&0&0\\0&q&0&0\\0&0&p&1\\0&0&0&q\end{bmatrix},D(p:q)=\begin{bmatrix}p&0&0&0\\0&p&0&0\\0&0&p&1\\0&0&0&q\end{bmatrix}.$$

The first stratum $A(p:q:r:s)$ is parameterized by $\mathbb{CP}^3/\Sigma_4$, and has a miniversal deformation with 3 parameters, except for $A(0:0:0:0)$ which requires 4. The second stratum $B(p:q:r)$ is parameterized by $\mathbb{CP}^2/\Sigma_2$, where Σ_2 acts by interchanging the coordinates q and r. This stratum requires 5 parameters, except for $B(0:0:0)$ where we need 6. The third stratum $C(p:q)$ is parameterized by $\mathbb{CP}^1/\Sigma_2$, and requires 7 parameters, except for $C(0:0)$ which requires 8. The fourth stratum $D(p:q)$ is parameterized by $\mathbb{CP}^1$, and requires 9 parameters, except for $D(0:0)$, which requires 10. Finally, the strata E_1 and E_0 require 15 parameters.

The space $\mathfrak{gl}(4)$, under the action of $\mathbb{GL}(4)$ has a stratification by the matrix types

$$A(p,q,r,s)=\begin{bmatrix}p&1&0&0\\0&q&1&0\\0&0&r&1\\0&0&0&s\end{bmatrix},\quad E_(p)=\begin{bmatrix}p&0&0&0\\0&p&0&0\\0&0&p&0\\0&0&0&p\end{bmatrix}$$

$$B(p,q,r)=\begin{bmatrix}p&0&0&0\\0&p&1&0\\0&0&q&1\\0&0&0&r\end{bmatrix},C(p,q)=\begin{bmatrix}p&1&0&0\\0&q&0&0\\0&0&p&1\\0&0&0&q\end{bmatrix},D(p,q)=\begin{bmatrix}p&0&0&0\\0&p&0&0\\0&0&p&1\\0&0&0&q\end{bmatrix}.$$

The stratification is as follows: $A(p,q,r,s)$ is parameterized by $\mathbb{C}^4/\Sigma_4$, $B(p,q,r)$ is parameterized by $\mathbb{C}^3/\Sigma_2$, $C(p,q)$ is parameterized by $\mathbb{C}^2/\Sigma_2$, $D(p,q)$ is parameterized by $\mathbb{C}^2$, and finally $E(p)$ is parameterized by $\mathbb{C}$. In each case, the number of parameters is 1 larger than the generic number in the projective picture.

3.5. **How to understand the strata?** In general, one can distinguish two types of one-parameter deformations. A *jump deformation* from an object A to and object B is a deformation $A(\lambda)$ for which $A(0)=A$ and $A(\lambda)$ is equivalent to B for $\lambda\neq 0$. The other type of deformation family $A(\lambda)$ is the following. If $A(\lambda)$ runs along a family of noequivalent objects, it is called a *smooth deformation* family.

As the examples have shown, it is always possible to combine several of the strata given by Jordan types in the Arnold model to give a single stratum which has a nice description as an orbifold. What is not so obvious is that the deformation picture is completely captured by the orbifold stratification, by which we mean that deformations either occur along a stratum, or are given by a jump deformation to a point in another stratum, or "factor through a jump deformation" to a point in another stratum. By "factoring through a jump deformation" we mean that a deformation goes along a neighborhood of a point in the stratum to which there is a jump deformation. A jump deformation from A to B is a deformation $A(\lambda)$ for which $A(0) = A$ and $A(\lambda)$ is equivalent to B for $\lambda \neq 0$.

A simple example of a jump deformation can be given for 2×2 matrices. The deformation $A(t) = \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$ is a jump deformation from $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ to $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, because $A(t)$ is equivalent to B for all $t \neq 0$.

We also noticed the following. When n is even, all the strata have miniversal deformations with an even number of parameters, and when n is odd, they all have an odd number of parameters. This fact is easy to prove. In our examples, the number of parameters of a miniversal deformation were different for each stratum. However, this is an artifact of the size of our matrices, and is not true in general. For $n = 6$, the following two matrices represent different strata, but each of them has a miniversal deformation with 12 parameters.

$$\begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & p & 1 & 0 & 0 \\ 0 & 0 & 0 & q & 1 & 0 \\ 0 & 0 & 0 & 0 & r & 1 \\ 0 & 0 & 0 & 0 & 0 & s \end{bmatrix}, \qquad \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 1 & 0 & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & r \end{bmatrix}.$$

The construction of the strata can be accomplished in a straightforward manner, with one stratum for each partition of the number n. Thus, the number of strata of the moduli space of $n \times n$ matrices under the action of $\mathbb{GL}(n)$ by conjugation is exactly the number of partitions of n.

It is important to understand that the stratification we have given is not simply a decomposition of the space in terms of pieces which consist of elements whose miniversal deformations have the same number of parameters, but in fact, the deformation theory is consistent with this stratification and no other.

4. Contractions

A contraction is a procedure somewhat opposite to deformation. Contractions typically transform a Lie algebra into a "more abelian" Lie algebra, and deformations, which are understood in a strict sense, lead to Lie algebra with more intricate Lie brackets.

Contractions are important in physics because they explain in terms of Lie algebra why some theories arise as a limit regime of more "exact" theories. Motivated by the need to relate the symmetries underlying Einstein's mechanics and Newtonian mechanics, Inönü and Wigner introduced the concept of contractions [10]. It consists in multiplying the generators of the symmetry by "contraction parameters" such that when these parameters reach some singularity point, one obtains a non-isomorphic Lie algebra with the same dimension. The method has been generalized a few years later by Saletan [13]. Contractions were used by Lévy-Leblond to emphasize that the condition of largely timelike intervals is just as crucial as the infinite velocity of light, in order to contract the Poincaré algebra to the Galilei algebra [12]. Another physical example is the contraction of the de Sitter algebras to the Poincaré algebra, in the limit of large radius.

The mathematics literature contains various concepts similar to contractions: "degeneration", "orbit closure", "perturbation" etc. Orbit closures arise in many areas of mathematics where algebraic or topological transformation groups are considered, such as invariant theory, representation theory, theory of singularities etc. For algebraic structures on a fixed finite-dimensional vector space, degeneration means that the orbits under the action of the general linear group are the isomorphism classes, and so orbit closure coincides with the closure of these classes.

When computing a contraction, one has a particular Lie algebra in mind, and wants to know all Lie algebras which can "jump" to the one we have in mind.

Definition 4.1. Let g_t be a family of automorphisms of V, defined in a punctured neighborhood of zero. If $\lim_{t\to 0}\big(g_t^*(\mathcal{L}')\big) = \mathcal{L}$ and $\mathcal{L}$ is not equivalent to $\mathcal{L}'$, then the Lie algebra $\mathcal{L}$ is a contraction of $\mathcal{L}'$.

The commutation relations of a contracted Lie algebra $\mathcal{L}'$ of $\mathcal{L}$ are given by the limit

$$[x,y]' = \lim_{t\to t_0} g_t^{-1}\big([g_t(x), g_t(y)]\big),$$

where g_t is a non-singular linear transformation of $\mathcal{L}$ with t_0 being a singularity point of its inverse g_t^{-1}. In mathematical terms, the orbits under the action of the general linear group are the Lie algebra isomorphism classes, and the bracket $[\ ,\ ]'$ is a contraction of $[\cdot,\cdot]$ if it is in the Zariski closure of the orbit of $[\cdot,\cdot]$.

To determine all possible contractions of a Lie algebra $\mathcal{L}'$ by finding all automorphisms g_t such that $\lim_{t\to 0}\big(g_t^*(\mathcal{L}')\big) = \mathcal{L}$ exists can be a daunting task. What can be done?

A non-trivial contraction always induces a non-trivial (inverse) jump deformation. The converse is not always true: there are deformations which do not admit an inverse contraction. For example, one can never have a contraction inside a parameterized family of Lie algebras, but deformations within a family exist. Also, nothing can be contracted to the parameterized family, whereas there can be many non-trivial deformations, as we will see on an example later. We should emphasize that the irreversibility occurs only when we have a family of smooth deformations. In other words, there is one-to-one correspondence between contractions and jump deformations.

¿From constructing a miniversal deformation of a Lie algebra, one can determine all jump deformations. So one can say that the miniversal deformations contain all the information about contractions as well.

The point of view of deformation theory is a bit different from the point of view of contractions. When computing a contraction, one has a particular Lie algebra in mind, and wants to know all Lie algebras which can jump to the one you have in mind. This is quite different from the perspective of deformation theory, where one is interested in seeing what the object of question deforms to. Both perspectives give valuable insights.

Here I present some examples and point out some of the advantages and disadvantages of these two approaches. Some contractions can be computed by use of diagonal matrices.

Theorem 4.2 ([15]). *If there is a contraction from $\mathcal{L}'$ to $\mathcal{L}$, where $\mathcal{L}$ and $\mathcal{L}'$ are Lie algebra structures on a finite-dimensional space V, then there is a basis of V and an automorphism g_t of V, which has diagonal matrix form* $\mathrm{diag}\big(t^{\lambda_1}, ..., t^{\lambda_n}\big)$, *where λ_i are integers, such that $g_t^*(\mathcal{L}')$ is equivalent to $\mathcal{L}$.*

This Theorem makes it possible to determine all Lie algebras $\mathcal{L}$ which arise as contractions of $\mathcal{L}'$, even if the classification of Lie algebra structures on V is not known. However, it is not true that one can compute all contractions given a fixed basis using diagonal matrices.

5. Examples

a) 3-***dimensional complex Lie algebras.***

$\mathbb{C}^3$:	$[x_i, x_j] = 0, \qquad i, j = 1, 2, 3$
$\mathfrak{n}_3(\mathbb{C})$:	$[x_1, x_2] = x_3$
$\mathfrak{r}_3(\mathbb{C})$:	$[x_1, x_2] = x_2, [x_1, x_3] = x_2 + x_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}), (\lambda \in \mathbb{C}^*, \lvert\lambda\rvert \leq 1)$:	$[x_1, x_2] = x_2, [x_1, x_3] = \lambda x_3$
$\mathfrak{sl}_2(\mathbb{C})$:	$[x_1, x_2] = x_3, [x_2, x_3] = x_1, [x_3, x_1] = x_2$

Table 1: Three-dimensional complex Lie algebras

The results of contractions and deformations of three-dimensional complex Lie algebras are displayed in Fig. 1 (see [5]). The lines and arrows should be interpreted as follows: an arrow points toward the deformation, whereas a simple line connect Lie algebras related by both deformation and contraction, with the deformed Lie algebra lying upward. The left-pointing arrow symbol over $\mathfrak{r}_{3,\lambda\neq\pm1}(\mathbb{C})$ means that it deforms inside the family.

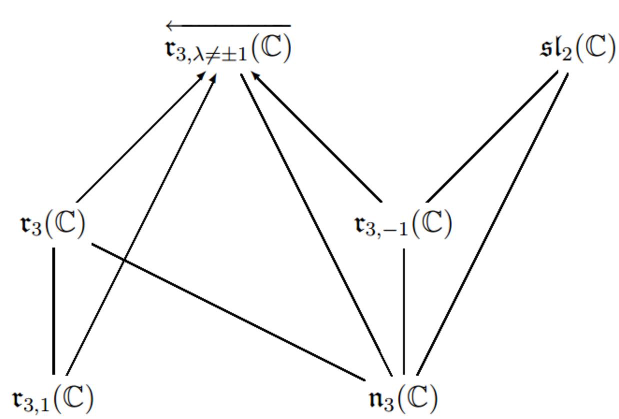

Fig.1 Contractions and deformations of the three-dimensional complex Lie algebras

The family of Lie algebras $\mathfrak{r}_{3,\lambda\neq\pm 1}(\mathbb{C})$ has a non-trivial deformation into itself. Note that the two Lie algebras $\mathfrak{r}_{3,1}(\mathbb{C})$ and $\mathfrak{r}_{3,-1}(\mathbb{C})$ are singled out for two reasons. First, $\mathfrak{r}_{3,1}(\mathbb{C})$ can be deformed into $\mathfrak{r}_3(\mathbb{C})$ whereas $\mathfrak{r}_{3,\lambda\neq 1}(\mathbb{C})$ cannot, and $\mathfrak{r}_{3,-1}(\mathbb{C})$ can deform into $\mathfrak{sl}_2(\mathbb{C})$, whereas $\mathfrak{r}_{3,\lambda\neq -1}(\mathbb{C})$ cannot. Second, $\mathfrak{r}_{3,1}(\mathbb{C})$ is special because it cannot be contracted to $\mathfrak{n}_3(\mathbb{C})$, unlike $\mathfrak{r}_{3,\lambda\neq 1}(\mathbb{C})$.

b) *Infinite dimensional Lie algebras.* In this section, we discuss deformations and contractions of some infinite dimensional Lie algebras. The physics literature about applications of infinite dimensional Lie algebras, namely to conformal field theory is enormous. Their interest stems from critical phenomena in two dimensions. Whereas the Witt and Virasoro algebras describe local invariance of conformal field theories on the (zero genus) Riemann sphere, the Lie algebras of Krichever–Novikov type discussed hereafter corresponds to higher genus. The physical interpretation of the contraction parameters introduced within these Lie algebras remain to be further explored. The difficulty encountered when deforming the known Lie algebras is that formal deformations are no longer sufficient to describe general deformations. The examples discussed below are formally rigid, so that they admit no non-trivial formal deformations. Nevertheless, there exist very interesting non-trivial global deformations. In the global deformation theory, we no longer have the tool of computing cohomology in order to get deformations, so the picture is much more difficult, and there are very few results so far. (See [7, 8].)

This is where a combination of the contractions and deformations proves really fruitful, since it leads to new infinite dimensional objects, as we will show hereafter. In a domain where so few objects are known explicitly, each new object should be of interest both in mathematics and in physics, particularly, in conformal field theory. The deformations we consider here are over affine varieties, which are very special global deformations.

Witt, Virasoro, and Krichever–Novikov algebras

First, let us consider the Witt algebra $\mathfrak{W}$

$$[l_n, l_m] = (m-n)l_{n+m}, \qquad n, m \in \mathbb{Z}.$$

Its only one-dimensional central extension is the Virasoro algebra $\mathfrak{V}$.

Krichever and Novikov invented the algebras of Virasoro type in [11].

It was shown recently that these infinite-dimensional Lie algebras can be interpreted as global deformations of the Witt or Virasoro algebra.

For simplicity, we will consider deformations of the Witt algebra, but this can be generalized is a natural way to the Virasoro algebra.

Despite its infinitesimal and formal rigidity, which prevents any non-trivial *formal* deformation, the Witt algebra $\mathfrak{W}$ can be non-trivially *globally* deformed into Krichever–Novikov type algebras $\mathfrak{KN}$ [7]. Such a phenomenon does not appear with Lie algebras of finite dimension.

An example of Krichever–Novikov algebras $\mathfrak{KN}$ is a two-dimensional family of Lie algebras parameterized over $\mathbb{C}^2$. The generators are given by the fields:

$$V_{2n+1} = (X-e_1)^n Y \tfrac{d}{dX},$$
$$V_{2n} = 2(X-e_1)^{n-1}(X-e_2)(X+e_1+e_2)\tfrac{d}{dX},$$

which satisfy the following Lie brackets:

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)(V_{n+m} + 3\ e_1\ V_{n+m-2} + \\ \qquad +(e_1 - e_2)(2e_1 + e_2)V_{n+m-4}), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)\ 3e_1\ V_{n+m-2} + \\ \qquad +(n-m-2)(e_1-e_2)(2e_1+e_2)V_{n+m-4}, & n \text{ odd}, m \text{ even}. \end{cases}$$

Note that for $e_1 = 0$, $e_2 = 0$, and $Y = X$, we recover the Witt algebra, and the fields reduce to

$$l_n = X^{n+1}\frac{d}{dX}.$$

A different $\mathfrak{KN}$ algebra can be obtained as a one-parameter global deformation, by taking the following field basis:

$$V_{2n} \equiv X(X-\alpha)^n(X+\alpha)^n\frac{d}{dX}, \qquad V_{2n+1} \equiv (X-\alpha)^{n+1}(X+\alpha)^{n+1}\frac{d}{dX}.$$

One calculates the Lie brackets:

$$(1)\quad [V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n, m \text{ odd}, \\ (m-n)(V_{n+m} + \alpha^2 V_{n+m-2}), & n, m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)\alpha^2 V_{n+m-2}, & n \text{ odd}, m \text{ even}. \end{cases}$$

There are many other ways as well, if we specify the base of the deformation being different affine lines in $\mathbb{C}^2$.

Clearly, they can be contracted back to the Witt algebra. Let us show it on the second type, equation (1), by utilizing a very elegant and simple Weimar-Woods contraction ([15]): if we define

$$(2)\qquad l_n \equiv \varepsilon^n V_n, \qquad \text{for all } n \in \mathbb{Z},$$

then equation (1) becomes

$$\begin{aligned} [l_n, l_m]_\varepsilon &= \varepsilon^{n+m}[V_n, V_m] \\ &= \begin{cases} (m-n)l_{n+m}, & n, m \text{ odd}, \\ (m-n)(l_{n+m} + \varepsilon^2\alpha^2 l_{n+m-2}), & n, m \text{ even}, \\ (m-n)l_{n+m} + (m-n-1)\varepsilon^2\alpha^2 l_{n+m-2}, & n \text{ odd}, m \text{ even}. \end{cases} \end{aligned}$$

Then, it is clear that, in the limit where ε approaches zero, we retrieve the commutation relations of $\mathfrak{W}$. Therefore, the operations of deformation and contraction are mutually reversible in this case.

In addition to retrieving the Witt algebra $\mathfrak{W}$, one may contract $\mathfrak{KN}$ to other, so far unknown, Lie algebras. Let us discuss an example of such a contraction of $\mathfrak{KN}$ which turns out to be a deformation of the respective contraction of $\mathfrak{W}$. Moreover, this exotic contration of $\mathfrak{KN}$ may be contracted back to the corresponding contraction of $\mathfrak{W}$ by utilizing equation (2). In order to do so, let us define $\mathcal{U}_\varepsilon$ *à la* Weimar-Woods:

$$(3)\qquad \mathcal{U}_\varepsilon \equiv \varepsilon^{n_0}\mathrm{id}_{\mathfrak{g}_0} + \varepsilon^{n_1}\mathrm{id}_{\mathfrak{g}_1},$$

where 0 and 1 denote the even and odd sectors of the powers of $\mathfrak{KN}$, respectively. Then, if we take the Lie brackets (1) as a specific example, we obtain the modified

brackets:
(4)

$$[V_n, V_m]_\varepsilon = \begin{cases} \varepsilon^{2n_1-n_0}(m-n)V_{n+m}, & n,m \text{ odd}, \\ \varepsilon^{n_0}(m-n)(V_{n+m}+\alpha^2 V_{n+m-2}), & n,m \text{ even}, \\ \varepsilon^{n_0}[(m-n)V_{n+m}+(m-n-1)\alpha^2 V_{n+m-2}], & n \text{ odd}, m \text{ even}. \end{cases}$$

Clearly, we must have non-negative values of n_0 and $2n_1 - n_0$. We obtain the trivial abelian Lie algebra when these expressions take on positive values. Another trivial contraction is given by $n_0 = n_1 = 0$; then it leaves the commutators of equation (1) unchanged.

Two more contractions may be obtained with the splitting of equation (3). One is the Inönü–Wigner contraction ([10]), given by $n_0 = 0$ and $n_1 = 1$ (or any n_1 positive); then the contracted commutation relations read:

$$(5)\quad [V_n, V_m] = \begin{cases} 0, & n,m \text{ odd}, \\ (m-n)(V_{n+m}+\alpha^2 V_{n+m-2}), & n,m \text{ even}, \\ (m-n)V_{n+m}+(m-n-1)\alpha^2 V_{n+m-2}, & n \text{ odd}, m \text{ even}. \end{cases}$$

This defines a new family of infinite dimensional Lie algebras. In the spirit of sequences of contractions, if we further contract these Lie algebras with $\mathcal{U}_\varepsilon$ such as defined in equation (2), then we obtain

$$[l_n, l_m] = \begin{cases} 0, & n,m \text{ odd}, \\ (m-n)l_{n+m}, & n \text{ or } m \text{ even}. \end{cases}$$

This algebra is clearly non-isomorphic to $\mathfrak{W}$. Indeed, it is a contraction of $\mathfrak{W}$, obtained with equation (3), utilized this time with 0 being the even sector, and 1 the odd sector of $\mathfrak{W}$.

The second contraction of the $\mathfrak{KN}$ algebra of equation (1) is constructed by choosing $n_0 > 0$ and $2n_1 - n_0 = 0$ in equation (4):

$$[V_n, V_m] = \begin{cases} (m-n)V_{n+m}, & n,m \text{ odd}, \\ 0, & n,m \text{ even}, \\ 0, & n \text{ odd}, m \text{ even}. \end{cases}$$

Again, this algebra is not isomorphic to the Witt algebra $\mathfrak{W}$.

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Alice Fialowski, Eötvös Loránd University, Budapest, Hungary
E-mail address: fialowsk@cs.elte.hu