# FUNDAMENTAL THEOREM OF FINITE DIMENSIONAL $\mathbb{Z}_2$-GRADED ASSOCIATIVE ALGEBRAS

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Abstract. In this paper, we give an extension of the Fundamental Theorem of finite dimensional algebras to the case of $\mathbb{Z}_2$-graded algebras. Essentially the results are the same as in the classical case, except that the notion of a $\mathbb{Z}_2$-graded division algebra needs to be modified. We classify all finite dimensional $\mathbb{Z}_2$-graded division algebras over $\mathbb{C}$ and $\mathbb{R}$.

## 1. Introduction

Super Lie algebras, or $\mathbb{Z}_2$-graded Lie algebras, have been studied for a long time, and have many applications in mathematics and physics. The notion of a $\mathbb{Z}_2$-graded associative algebra is not so well known, but these algebras are examples of $\mathbb{Z}_2$-graded $A_\infty$ algebras, and thus they arise naturally in the study of $A_\infty$ algebras. In the case of Lie algebras, the $\mathbb{Z}_2$-graded Jacobi identity picks up some signs that depend on the parity of the elements being bracketed, but the associativity relation for $\mathbb{Z}_2$-graded associative algebras does not pick up any signs. So it may seem at first glance that there are no new features which arise in the study of $\mathbb{Z}_2$-graded associative algebras.

The moduli space of equivalence classes of $\mathbb{Z}_2$-graded associative algebras on a vector space of dimension $m|n$ differs from the moduli space of associative structures on the same space ignoring the grading in two important ways. First, a $\mathbb{Z}_2$-graded algebra structure is required to be an even map, which means that not all associative algebra structures are allowed in the $\mathbb{Z}_2$-graded case. Secondly, the moduli space is given by equivalence classes of algebra structures under an action by the group of linear automorphisms of the vector space. For the $\mathbb{Z}_2$-graded case, we only allow even automorphisms, which means that the equivalence classes are potentially smaller in the $\mathbb{Z}_2$-graded case. Since there are fewer allowable $\mathbb{Z}_2$-graded algebra structures, but also fewer equivalences between them, it is not obvious whether the moduli space of $\mathbb{Z}_2$-graded associative algebras on a $\mathbb{Z}_2$-graded vector space is larger or smaller than the moduli space of all associative algebra structures on the vector space.

What is true is that there is a map between the moduli space of $\mathbb{Z}_2$-graded algebra structures on a $\mathbb{Z}_2$-graded vector space and the moduli space of all algebra structures on the underlying space. This map in general is neither injective nor surjective.

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Some standard references for known results about associative algebras are [8, 7, 3, 4, 5, 2].

## 2. Preliminaries

We recall briefly the basic facts about $\mathbb{Z}_2$-graded spaces. A $\mathbb{Z}_2$-graded vector space $V$ over a field $K$ is given by a decomposition $V = V_0 \oplus V_1$, where $V_0$ is the subspace of even, and $V_1$ is the subspace of odd elements. Elements of $V_0$ and $V_1$ are called homogeneous, and for a homogeneous element $v$, its parity $|v|$ is given by $|v| = i$ if $v \in V_i$. A subspace $W$ of $V$ is a graded subspace if $W = W_0 \oplus W_1$, where $W_i = W \cap V_i$, or equivalently, if $W$ has a basis consisting of homogeneous elements. The parity reversion $\Pi V$ is the $\mathbb{Z}_2$-graded space whose underlying vector space is the same, but with the parities reversed; *i.e.*, $(\Pi V)_0 = V_1$ and $(\Pi V)_1 = V_0$.

A linear map $f : V \to W$ of $\mathbb{Z}_2$-graded spaces is even if $|f(v)| = |v|$ for homogeneous $v$, and odd if $|f(v)| = |v| + 1$ instead. The tensor product of $\mathbb{Z}_2$-graded spaces has a natural $\mathbb{Z}_2$-grading, with $|v \otimes w| = |v| + |w|$. A product $m$ on $V$ is an even map $m : V \otimes V \to V$. In other words, if we denote the product by juxtaposition, then for homogenous elements $a, b \in V$, $ab$ is homogeneous and $|ab| = |a||b|$.

The space $\mathbf{GL}(V)$ of automorphisms of $V$ is naturally $\mathbb{Z}_2$-graded, as every linear map is uniquely decomposable as a sum of an even and an odd map. The space of equivalence classes of associative product structures on $V$ under the action of the even automorphisms of $V$ is called the moduli space of associative algebras on $V$. Morphisms of $\mathbb{Z}_2$-graded algebras are required to be even maps. If a $\mathbb{Z}_2$-graded algebra is unital, then the multiplicative identity 1 must be even.

A morphism between two associative $K$-algebras is required to be an even map. As a consequence, the kernel of a morphism is always a graded ideal, in other words an ideal which is also a graded subspace of $A$. If $a \in A$, then the graded ideal $(a)$ generated by $a$ is just the direct sum of the ordinary ideals generated by its homogeneous parts, and we define the graded left ideal $L(a)$, and right ideal $R(a)$ generated by $a$ similarly. If $I$ is a graded ideal in $A$, then $A/I$ has a natural $\mathbb{Z}_2$-grading.

If $a$ is homogeneous, then $Aa$, $aA$ and $AaA$ are graded left, right and two-sided ideals, and if $A$ is unital as well, they are the graded left, right and two-sided ideals generated by $a$. If $A$ is not unital, then $Ka + Aa$, $Ka + aA$, and $Ka + AaA$ are the left, right and two sided ideals generated by $a$. When $a$ is not homogeneous, then one simply decomposes $a = a_0 + a_1$ into its homogeneous parts, and the left, right and two sided ideals generated by $a$ are the sums of the corresponding ideals generated by its homogeneous parts. Note that if $L$ is a graded left ideal, then $L + LA$ is a graded two-sided ideal. As in the ordinary case, the sum of two left, right or two-sided graded ideals is again a graded ideal of the same type.

### 2.1. **Nilpotent Ideals.**

Let $A$ be some fixed $K$-algebra. An element $a \in A$ is nilpotent if $a^m = 0$ for some $m \in N$. A (left, right, two-sided) graded ideal $I$ is nilpotent if $I^m = 0$, for some $m \in N$.

**Proposition 2.1.** *The sum of two (left, right, two-sided) nilpotent graded ideals is a nilpotent (left, right, two-sided) graded ideal.*

*Proof.* Suppose that $I$ and $J$ are nilpotent left graded ideals, $a \in I$, $b \in J$, $m \leq n$ and that $I^m = J^n = 0$. Then we claim that $(a + b)^{2n} = 0$. To see this, note that if we expand the product, then any term will have at least $n$ factors of $a$ or $b$. If there are $n$ factors of $a$, then the term must lie in $I^nA = 0$, and similarly if there are $n$ factors of $b$. Thus $(I + J)^{2n} = 0$. The proof for right or two-sided graded ideals is similar. □

An element $a$ is said to be properly nilpotent if $(a)$ is a nilpotent graded ideal.

**Proposition 2.2.** *Let $a$ be a homogeneous nilpotent element of $A$. Then the following are equivalent.*

(1) *$a$ is properly nilpotent.*
(2) *$Aa$ is a nilpotent left graded ideal.*
(3) *$aA$ is a nilpotent right ideal.*
(4) *$AaA$ is a nilpotent ideal.*

*Proof.* Since $Aa$, $aA$ and $AaA$ are contained in $(a)$, they are nilpotent if $(a)$ is nilpotent. On the other hand, suppose that $Aa$ is nilpotent. Suppose that $(Aa)^n = 0$. Then $(AaA)^n \subseteq (Aa)^nA = 0$. Thus $AaA$ is a nilpotent graded ideal, and thus $Aa + AaA$ is a nilpotent graded left ideal. Note that $(a)^2 \subseteq Aa + AaA$. It follows that $(a)^2$ is nilpotent. But then it is evident that $(a)$ is also nilpotent. The case where $aA$ is nilpotent is similar. Since $(a)^3 \subseteq AaA$, the case when $AaA$ is nilpotent is even simpler. □

**Proposition 2.3.** *Let $a \in A$ and suppose that $a = a_0 + a_1$ is the decomposition of $a$ into its homogeneous parts. Then the following are equivalent.*

(1) *$a$ is properly nilpotent.*
(2) *$Aa_0$ and $Aa_1$ are nilpotent left graded ideals.*
(3) *$a_0A$ and $a_1A$ are nilpotent right ideals.*
(4) *$Aa_0A$ and $Aa_0$ are nilpotent ideals.*

Note that any element of a nilpotent left, right or two-sided ideal is properly nilpotent. Note also that if $L$ is a nilpotent graded left ideal of $A$, then $L + LA$ is a nilpotent graded ideal containing $L$. Thus every nilpotent graded left ideal is contained in a nilpotent graded ideal.

**Definition 2.4.** The radical $\mathrm{Rad}(A)$ of the algebra $A$ is the union of all nilpotent ideals of $A$.

It is easy to see that the radical consists of all properly nilpotent elements of $A$.

**Proposition 2.5.** *If $A$ is a finite dimensional algebra, then* $\mathrm{Rad}(A)$ *is a nilpotent ideal.*

**Example 1.** Consider the ring $A = K[x_1, \dots]/(x_1^2, x_2^3, \dots)$. Every element in the ideal $I = (x_1, \dots)$ is nilpotent, but $I$ is not a nilpotent ideal.

Recall that an element $a$ in an algebra is *idempotent* if $a^2 = a$. In a $\mathbb{Z}_2$-graded algebra, it is clear that a nonzero odd element cannot be idempotent. However, an idempotent need not be homogeneous, as the important example below illustrates.

**Example 2.** Let $V = \langle\theta\rangle$ be an odd vector space, with basis $\theta$, and $T(V)$ be the tensor algebra of $V$. Then $T(V)/(\theta^2 - 1) = K \oplus K\theta$. Then $\frac{1}{2}(1 \pm \theta)$ are nonhomogeneous idempotents. This algebra has no proper, nontrivial graded ideals, although the subspaces generated by the nonhomogeneous idempotents above are nontrivial ideals. (Note that if $\mathrm{char}(K) = 2$, then the element $\frac{1}{2}(1 + \theta)$ is not well defined, and in fact, $1 + \theta$ is nilpotent.)

**Theorem 2.6.** *Every nontrivial finite dimensional $\mathbb{Z}_2$-graded associative algebra is either nilpotent or contains a nonzero* even idempotent element.

*Proof.* If $V$ is a $0|1$-dimensional vector space, then it has no nontrivial algebra structure, so the only $0|1$-dimensional associative algebra is the trivial one. On the other hand, suppose that $V$ is $1|0$-dimensional algebra, and $x$ is a basis element. Then the identity $x^2 = kx$ for some $k \in K$ completely determines the algebra structure. If $k = 0$, it is a trivial algebra, so it is nilpotent. On the other hand, if $k \neq 0$, let $e = k^{-1}x$. Then $e^2 = e$, so the algebra is isomorphic to $K$. Since $e$ is even, it satisfies our requirements.

Now we induct on the *total dimension* $m + n$ for an $m|n$-dimensional space. We have already shown that the theorem holds when $m + n = 1$. Suppose it holds whenever $m + n < k$ and that $m + n = k$. Suppose that $Aa = A$ for some homogeneous element $a \in A$. Let $\{u_1, \cdots, u_{m+n}\}$ be a homogeneous basis of $A$. Then $\{u_1 a, \cdots, u_{m+n} a\}$ is a homogeneous set which spans $A$, and thus must be a homogeneous basis of $A$. It follows that $xa = 0$ for some $x \in A$ iff $x = 0$. Moreover, $ba = a$ for some $b \in A$, and if $e$ and $f$ are the even and odd parts of $b$, it follows that $fa = 0$ and $ea = a$. Thus we have $(e - e^2)a = 0$, so $e = e^2$. Since $A$ is nontrivial, $e \neq 0$.

Therefore, we may assume that $Aa \neq A$ for any homogeneous element $a$. Moreover, since $Aa$ is a $\mathbb{Z}_2$-graded algebra of smaller dimension than $A$, we can assume that $Aa$ is nilpotent for all homogeneous $a$, since otherwise, some $Aa$ contains a nonzero even idempotent by the inductive hypothesis. But then $a$ is properly nilpotent for all homogeneous $a$. Since an element $a$ is properly nilpotent iff its homogeneous parts are properly nilpotent, it follows that $A = \operatorname{Rad}(A)$ and thus $A$ is nilpotent. □

2.2. **Peirce Decomposition by idempotents.** In [6], one of the main tools used in the classification of low dimensional associative algebras was a decomposition of the algebras by idempotents. For our purposes, we will always use even idempotents, but the essential theory is unchanged.

If $e$ is an even idempotent, let

$$\begin{aligned} \mathfrak{L}_e &= \{x \in A | xe = 0\} \\ \mathfrak{R}_e &= \{x \in A | ex = 0\} \\ \mathfrak{C}_e &= \mathfrak{L}_e \cap \mathfrak{R}_e = \{x \in A | xe = ex = 0\}. \end{aligned}$$

Then $\mathfrak{L}_e$ is a graded left ideal, $\mathfrak{R}_e$ is a graded right ideal, and we can decompose $A$ into graded subalgebras as follows.

$$A = eAe \oplus e\mathfrak{L}_e \oplus \mathfrak{R}_e e \oplus \mathfrak{C}_e. \tag{1}$$

Note that $eAe$ is a unital graded algebra.

**Proposition 2.7.**

$$\begin{aligned} \operatorname{Rad}(eAe) &= \operatorname{Rad}(A) \cap eAe = e \operatorname{Rad}(A) e \\ \operatorname{Rad}(\mathfrak{C}_e) &= \operatorname{Rad}(A) \cap \mathfrak{C}_e. \end{aligned}$$

*Proof.* Clearly $\operatorname{Rad}(A) \cap eAe \subseteq \operatorname{Rad}(eAe)$. To show the reverse inclusion, suppose that $x \in \operatorname{Rad}(eAe)$ and $a \in A$. Let $m$ be such that $\operatorname{Rad}(A)^m = 0$. Then $x = exe$, $eaex \in \operatorname{Rad}(eAe)$ so $(ax)^{m+1} = (aexe)^{m+1} = ax(eaex)^m = 0$. It follows that

$(Ae)^{m+1} = 0$, so $Ae$ is a nilpotent left ideal. Thus $x$ is properly nilpotent, so $x \in \mathrm{Rad}(A)$. □

If $uv = vu = 0$, then $u$ and $v$ are said to be *orthogonal.* The following proposition is easy to see.

**Proposition 2.8.** *If $e_1, \cdots, e_r$ are nonzero pairwise orthogonal even elements, then they are a basis for an even subalgebra of $A$ isomorphic to $K^r$. Thus, if $\dim(A) = m|n$, we must have $r \le m$.*

**Definition 2.9.** Let $e$ be an even idempotent in $A$. Then $e$ is said to be *principal* if the only even idempotent orthogonal to $e$ is zero. On the other hand, $e$ is said to be *primitive* if whenever $e = u+v$, where $u$ and $v$ are even orthogonal idempotents, then either $u$ or $v$ vanishes.

**Theorem 2.10.** *Let $A$ be a finite dimensional graded algebra. Then $A$ has a principal idempotent. Moreover, every even idempotent has a decomposition as a sum of primitive idempotents.*

*Proof.* Suppose that $e$ is an even idempotent. If it is not primitive, it can be decomposed as a sum $e = u + v$ of two nonzero even orthogonal idempotents. Suppose that $u$ and $v$ decompose as sums $u = x_1 + \cdots + x_k$ and $v = y_1 + \cdots + y_l$ of orthogonal even idempotents. Then we claim that $x_i$ and $y_j$ are orthogonal, for any $i$ and $j$. For $vy_j = y_j$ and $x_i u = x_i$, so $x_i y_j = x_i u v y_j = 0$, and similarly $y_j x_i = 0$. Because $A$ is finite dimensional, the process of decomposition must terminate, and so we obtain a decomposition of $e$ into primitive elements.

On the other hand, we know that there is some nonzero even idempotent. Therefore, there must be primitive idempotents. Suppose that $\{e_1, \cdots, e_k\}$ is a maximal family of nonzero orthogonal primitive idempotents, and let $e = e_1 + \cdots + e_k$. Then $e$ must be principal. □

**Proposition 2.11.** *Let $A$ be a finite dimensional algebra. An even element $e$ in $A$ is principal iff $\mathfrak{C}_e$ is nilpotent. Moreover, if $e$ is principal, then*

$$\mathfrak{T}_e = e\mathfrak{L}_e \oplus \mathfrak{R}_e e \oplus \mathfrak{C}_e \subseteq \mathrm{Rad}(A).$$

*Proof.* Since $\mathfrak{C}_e$ consists precisely of those elements which are orthogonal to $e$, and it contains a nonzero even idempotent iff it is not nilpotent, the first statement is clear. To see the second statement, note that $\mathfrak{L}_e\mathfrak{R}_e$ is an ideal and $\mathfrak{R}_e\mathfrak{L}_e \subseteq \mathfrak{C}_e$.

Let $k$ be such that $\mathfrak{C}_e^k = 0$. Then $(\mathfrak{L}_e\mathfrak{R}_e)^{k+1} \subseteq \mathfrak{L}_e(\mathfrak{R}_e\mathfrak{L}_e)^k\mathfrak{R}_e = 0$. Thus $\mathfrak{L}_e\mathfrak{R}_e \subseteq \mathrm{Rad}(A)$. Since $\mathfrak{L}_e e = 0$,

$$\mathfrak{L}_e A = \mathfrak{L}_e(eAe \oplus e\mathfrak{L}_e \oplus \mathfrak{R}_e e \oplus \mathfrak{C}_e) = \mathfrak{L}_e\mathfrak{R}_e e \oplus \mathfrak{L}_e\mathfrak{C}_e \subseteq \mathfrak{L}e\mathfrak{R}_e \subseteq \mathrm{Rad}(A).$$

Thus, for any $a \in \mathfrak{L}_e$, $aA \subseteq \mathrm{Rad}(A)$, so $a$ is properly nilpotent. Thus $\mathfrak{L}_e \subseteq \mathrm{Rad}(A)$, and $e\mathfrak{L}_e \subseteq \mathrm{Rad}(A)$. Similarly $\mathfrak{R}_e e \subseteq \mathrm{Rad}(A)$. It follows that $\mathfrak{T}_e \subseteq \mathrm{Rad}(A)$. □

**Proposition 2.12.** *A even idempotent $e$ in $A$ is primitive iff it is the only nonzero even idempotent in $eAe$.*

*Proof.* Suppose that $e$ is a primitive idempotent. Let $u$ be a nonzero even idempotent in $eAe$. Then $eu = ue = u$, so $(e-u)^2 = e - eu - ue + u = e - u$ is an even idempotent. Moreover $u(e-u) = (e-u)u = 0$, so $u$ and $e-u$ are orthogonal idempotents. Since $e = (e-u) + u$, and $e$ is primitive, it follows that $u = 0$.

On the other hand, suppose that $e$ is the only even idempotent in $eAe$ and that $e = u+v$ for some orthogonal even idempotents. Then $eue = (u+v)u(u+v) = u^3 = u$ and similarly $eve = v$. Thus both $u$ and $v$ are elements of $eAe$. It follows that one of them must be $e$ and the other must be 0. This shows that $e$ is primitive. □

**Definition 2.13.** A nontrivial associative algebra $A$ is *semisimple* iff $\operatorname{Rad}(A) = 0$.

**Proposition 2.14.** *A finite dimensional semisimple algebra is unital.*

*Proof.* Since $A$ is finite dimensional, it has a principal idempotent $e$. Since $\mathfrak{T}_e \subseteq \operatorname{Rad}(A) = 0$, it follows that $A = eAe$. Thus $e$ is the multiplicative identity in $A$. □

**Proposition 2.15.** *Suppose that $A$ is a finite dimensional nonnilpotent algebra. Then $A/\operatorname{Rad}(A)$ is semisimple.*

*Proof.* Let $\pi : A \to A/\operatorname{Rad}(A)$ be the natural map, $\overline{\mathfrak{N}} = \operatorname{Rad}(A/\operatorname{Rad}(A))$ and $\mathfrak{N} = \pi^{-1}(\overline{\mathfrak{N}})$. Since $\overline{\mathfrak{N}}^k = 0$ for some $k$, it follows that $\pi(\mathfrak{N}^k) = 0$, so $\mathfrak{N}^k \subseteq \operatorname{Rad}(A)$. If $\operatorname{Rad}(A)^l = 0$, then $\mathfrak{N}^{kl} = 0$. But then $\mathfrak{N}$ is nilpotent, so $\mathfrak{N} \subseteq \operatorname{Rad}(A)$. This implies that $\overline{\mathfrak{N}} = \pi(\mathfrak{N}) = 0$. Since $A \neq 0$, it is semisimple. □

**Proposition 2.16.** *Suppose that $e$ is a nonzero even idempotent of a semisimple algebra $A$. Then $eAe$ is also semisimple.*

*Proof.* Since $e \in eAe$, $eAe$ is nontrivial. Moreover, $\operatorname{Rad}(eAe) = e\operatorname{Rad}(A)e = 0$. □

**Definition 2.17.** A nonzero algebra $A$ is called *simple* if it has no nontrivial proper ideals.

Clearly, any simple algebra is also semisimple.

**Proposition 2.18.** *Suppose that $e$ is a nonzero even idempotent of a simple algebra $A$. Then $eAe$ is also simple.*

*Proof.* If $B$ is an ideal in $eAe$, then $B = eBe$. Thus we have

$$B = eBe \subseteq eABAe \subseteq (eAe)B(eAe) \subseteq B.$$

Thus $B = eABAe$. Now $ABA$ is an ideal in $A$, so either $ABA = 0$, in which case $B$=0, or $ABA = A$, in which case $B = eAe$. This shows that $A$ is simple. □

In the non graded case, a division algebra is a unital algebra in which every nonzero element is invertible. This definition is not the right notion for $\mathbb{Z}_2$-graded algebras. We first give the "correct" definition, and then motivate it by relating it to properties of primitive elements.

**Definition 2.19.** A $\mathbb{Z}_2$-graded associative division algebra is a unital associative $\mathbb{Z}_2$-graded algebra in which every nonzero *homogeneous* element is invertible.

**Proposition 2.20.** *Let $e$ be an even idempotent of a finite dimensional simple algebra $A$. Then $D = eAe$ is a division algebra iff $e$ is primitive.*

*Proof.* Suppose that $D$ is a division algebra, and $u$ is a nontrivial even idempotent. Then $e$ is the multiplicative identity of $D$. Since $u$ is a nontrivial homogeneous element, there is some $v$ such that $uv = e$. But then $u = ue = u^2v = uv = e$.

On the other hand, if $e$ is primitive, then $e$ is the only nontrivial idempotent in $D$. Let $a$ be a nonzero homogeneous element of $D$. Then $Da$ is a nontrivial left ideal in $D$, which is a simple algebra, so that $\operatorname{Rad}(D) = 0$. Thus $Da$ is not nilpotent, and so contains an even idempotent, which must be $e$. Therefore $ba = e$ for some $b \in D$. But this means that $a$ has a left inverse. Similarly, $a$ has a right inverse, so it must be invertible. $\square$

Since the direct product $B \times C$ and direct sum $B \oplus C$ of two graded algebras coincide, we will use the former notation, because we will often need to decompose an algebra $A$ as a direct sum of graded subspaces $A = B \oplus C$, without meaning that $A$ is the direct sum of algebras $B$ and $C$.

**Proposition 2.21.** *Suppose that $B$, $C$ are graded subalgebras of $A$ satisfying*

1. $B \cap C = 0$.
2. $BC = CB = 0$.
3. $A = B + C$.

*Then the map $B \times C \to A$ given by $(b, c) \mapsto b + c$ is an graded isomorphism of algebras. In particular, $B$ and $C$ are ideals in $A$.*

*Proof.* The map $B \times C \to A$ is a morphism by the second condition, is injective by the first condition, and is surjective by the third. $\square$

By the above proposition, it is natural to write $A = B \times C$. More generally, we have

**Proposition 2.22.** *Let $A_i$ for $i = 1, \cdots, n$ be graded subalgebras of $A$ such that*

1. $A_i \cap \sum_{j \neq i} A_j = 0$ *for all* $i$.
2. $A_i A_j = 0$, *if* $i \neq j$.
3. $A = \sum_i A_i$.

*Then the map $\prod A_i \to A$ given by $(a_1, \cdots, a_n) \mapsto a_1 + \cdots + a_n$ is an isomorphism of algebras.*

**Corollary 2.23.** *Let $A$ be a unital algebra $e_1, \cdots, e_n$ be a set of pairwise orthogonal even central idempotents of $A$ such that $e = e_1 + \cdots + e_n$ is the identity in $A$. Then if $A_i = Ae_i$, we have $A = \prod_i A_i$.*

*Proof.* Clearly every element $a \in A$ has a unique expression $a = a_1 + \cdots + a_n$ where $a_i = ae_i$. Since the $e_i$-s are pairwise orthogonal, $A_i A_j = 0$ if $i \neq j$. $\square$

**Proposition 2.24.** *Suppose that $B$ is an ideal in an algebra $A$, and $e$ is a multiplicative identity in $B$. Then $A = B \times \mathfrak{C}_e$. Moreover, in any decomposition of $A$ as a direct product $A = B \times C$, where $C$ is a subalgebra of $A$, we must have $C = \mathfrak{C}_e$.*

*Proof.* First, $B \cap \mathfrak{C}_e = 0$, since if $b \in B \cap \mathfrak{C}_e$, then $b = be = 0$. Secondly, if $b \in B$ and $c \in \mathfrak{C}_e$, then $bc = bec = 0$ and similarly, $cb = 0$. Finally, note that since $B$ is an ideal and $e \in B$, $eAe$, $e\mathfrak{L}_e$, and $\mathfrak{R}_e e$ are all contained in $B$. Using the Pierce decomposition, we have $A = eAe \oplus e\mathfrak{L}_e \oplus \mathfrak{R}_e \oplus \mathfrak{C}_e \subseteq B + C \subseteq A$. Thus $A = B \times \mathfrak{C}_e$.

Now suppose that $A = B \times C$ for some subalgebra $C$. Then for any $b \in B$ and $c \in C$, we have $bc = cb = 0$. Thus $C \subseteq \mathfrak{C}_e$ and equality must hold. $\square$

**Definition 2.25.** An algebra $A$ is *decomposable* iff $A = B \times C$ for some subalgebras $B$ and $C$. Otherwise it is *indecomposable*.

**Proposition 2.26.** *A semisimple algebra is simple iff it is indecomposable.*

*Proof.* Clearly, if $A$ is simple, it is indecomposable. On the other hand, suppose that $A$ is not simple, so it has a proper, nontrivial ideal $B$. Let $N = \mathrm{Rad}(B)$. Then $ANA \subseteq ABA \subseteq B$, and $ANA$ is an ideal in $A$. Note that $BNB$ is a nilpotent ideal in $A$, so $BNB = 0$. Moreover $(ANA)^3 \subseteq BNB = 0$, so $N \subseteq ANA = 0$. Thus $B$ is semisimple, so it has an identity $e$. Thus $A = B \times \mathfrak{C}_e$, and $A$ is decomposable. $\square$

**Definition 2.27.** A subalgebra $M$ of an $\mathbb{Z}_2$-graded associative algebra is called a *matrix subalgebra* if $M = \langle e_i^j | i, j = 1 \dots n\rangle$, where $e_i^j e_k^l = \delta_k^j e_i^l$, and $e_i^l$ are homogeneous. In other words, $M = \mathfrak{gl}(V)$ for some graded space $V$.

**Proposition 2.28.** *For a matrix algebra $M$, we have the following properties.*

(1) $e_i^i$ *is even for all* $i$.
(2) $|e_j^i| = |e_i^j|$ *for all* $i$ *and* $j$.

*Proof.* Since $e_i^i$ is homogeneous and idempotent, it must be even. Since $e_i^j e_j^i = e_i^i$, the second remark holds. $\square$

**Definition 2.29.** If $B$ is a graded subalgebra of $A$, then the *centralizer* of $B$ is the subalgebra $C(B)$ consisting of all elements whose homogeneous parts graded commute with homogeneous elements of $B$.

**Proposition 2.30.** *Let $A$ be a graded $K$-algebra and $B$ and $C$ be graded subalgebras. Then the map $B \otimes C \to A$, given by $b \otimes c \mapsto bc$ is an isomorphism precisely when*

(1) $bc = (-1)^{bc} cb$ *for all homogeneous* $b \in B$ *and* $c \in C$.
(2) $\sum_{k=1}^r b_i c_i = 0$ *for* $b_i \in B$ *and* $c_i \in C$, *then* $\{c_1, \cdots, c_r\}$ *is a $K$-linearly dependent subset of* $C$.
(3) $BC = A$.

In fact, the first condition is equivalent to the map being a morphism of rings, the second to the morphism being injective, and the third to surjectivity. It is natural to write $A = B \otimes C$. Moreover, when only the third condition does not hold, then it is natural to say that $B \otimes C$ is a subalgebra of $A$.

**Proposition 2.31.** *Suppose that $B = \langle e_i \rangle$ and $C = \langle f_j \rangle$ are subalgebras of $A$ such that $\{e_i f_j\}$ is a basis of $A$ and that $bc = (-1)^{bc} cb$ for all homogeneous elements $b \in B$ and $c \in C$. Then the map $B \otimes C \to A$ given by $b \otimes c \mapsto bc$ is an isomorphism.*

**Proposition 2.32.** *Suppose that the multiplicative identity in a matrix subalgebra $M$ of $A$ is the multiplicative identity in $A$. Then $A = M \otimes C(M)$.*

*Proof.* Let $a \in A$ be homogeneous and define $a_j^i = \sum_k (-1)^{e_k^i (a + e_i^j)} e_k^i a e_j^k$. First, we show that $a_j^i \in C(M)$. Note that $|a_j^i| = |a| + |e_j^i|$. We claim that $a_j^i \in C(M)$. This can be shown as follows:

$$e_s^r a_j^i = \sum_k (-1)^{e_k^i (a - e_i^j)} e_s^r e_k^i a e_j^k = (-1)^{e_r^i (a + e_i^j)} e_s^i a e_j^r$$
$$a_j^i e_s^r = \sum_k (-1)^{e_k^i (a - e_i^j)} e_k^i a e_j^k e_s^r = (-1)^{e_s^i (a + e_i^j)} e_s^i a e_j^r$$

Now we want to show that $e_s^r a_j^i = (-1)^{e_s^r a_j^i} a_j^i e_s^r$, so we check

$$(-1)^{e_s^r a_j^i} (-1)^{e_s^i (a + e_i^j)} = (-1)^{(e_r^i + e_s^i)(a + e_i^j) + e_s^i (a + e_i^j)} = (-1)^{e_r^i (a + e_i^j)}.$$

Moreover, we have $\sum_{i,j} a^i_j e^j_i = \sum_{i,j}(-1)^{e^i_i(a+e^j_i)} e^i_i a e^j_j = a$. Now suppose that $\sum_{r,s} a^r_s e^s_r = 0$. Then

$$0 = \sum_{k,r,s} (-1)^{e^i_k a^i_j} e^i_k a^r_s e^s_r e^k_j = \sum_k a^i_j e^k_k = a^i_j,$$

for all $i$ and $j$. This shows that $A = M \otimes C(M)$. □

**Theorem 2.33** (Wedderburn)**.** *Suppose that $A$ is a finite dimensional $\mathbb{Z}_2$-graded simple associative algebra. Then $A = M \otimes D$ where $D$ is a division algebra and $M$ is a matrix algebra. Moreover, any algebra of this form is simple.*

*Proof.* Decompose the identity $e = e_1 + \cdots + e_k$, where $e_i$ are primitive idempotents. Let $A_{ij} = e_i A e_j$. Then $A_{ij}$ is a graded subspace of $A$. Since $e_i$ is primitive, $A_{ii}$ is a division algebra. Since $Ae_jA$ is a nontrivial ideal, it follows that $Ae_jA = A$. It follows that $A_{ij}A_{kl} = \delta^j_k A_{il}$. Since $A_{1j}A_{j1} = A_{11}$, for $j = 1, \cdots, k$, we can choose some homogeneous $e^j_1 \in A_{j1}$ and some homogenous $x_j \in A_{1j}$ such that $a_j = x_j e^j_1 \neq 0$. As $A_{ii}$ is a division algebra with identity $e_i$, and $a_j$ is homogeneous, there is some $b_j \in A_{11}$ such that $b_j a_j = e_i$. Let $e^1_j = b_j x_j$. Then $e^1_j e^j_1 = b_j x_j e^j_1 = b_j a_j = e_1$. Define $e^1_1 = e_1$ and $e^i_j = e^i_1 e^1_j$ if $i$ and $j$ are larger than 1. Clearly $e^i_j$ is homogeneous for all $i$ and $j$, and $e^i_j \in A_{ij}$, so $e_k e^i_j = \delta^k_i e^i_j$. Thus

$$e^i_j e^k_l = e^i_1 e^1_j e^k_1 e^1_l = e^i_1 e^1_j e_j e^k_1 e^1_l = \delta^j_k e^i_1 e^1_j e^j_1 e^1_l = \delta^j_k e^i_l e_1 e^1_l = \delta^j_k e^i_l.$$

As $e^1_i e^i_j e^j_1 = e^1_1$, it follows that $e^i_j \neq 0$. As $e^i_i$ is a nonzero even idempotent in $A_{ii}$, it follows that $e^i_i = e_i$. Thus $M = \langle e^i_j \rangle$ is a matrix subalgebra of $A$ whose identity agrees with that of $A$. It follows that $A = M \otimes C(M)$.

Let $D = C(M)$. We claim that $A_{ii} = De_i$ for all $i$. To see this, suppose that $a \in D$. Then $ae_i = ae_ie_i = e_iae_i \in A_{ii}$, so $De_i \subseteq A_{ii}$. On the other hand, let $a \in A$. Then $a = \sum_{r,s} a^r_s e^r_s$, where $a^r_s \in D$, so $e_iae_i = \sum_{r,s} e_i a^r_s e^r_s e_i = \sum_{r,s} a^r_s e^i_i e^r_s e^i_i = a^i_i e_i$. Thus any element in $A_{ii} = e_iAe_i$ is in $De_i$. This shows that $A_{ii} = De_i$. Thus $De_i$ is a division algebra. But the map $D \to De_i$, given by $d \to de_i$ is an isomorphism of algebras. Thus $D$ is a division algebra.

Now suppose that $A = M \otimes D$, where $M$ is a matrix subalgebra of $A$ and $D$ is a division algebra, and that $I$ is a nonzero graded ideal in $A$. Then there is a nonzero homogeneous element $a$ in $I$. Now, $a = \sum_{i,j} a^i_j e^i_j$ where $a^i_j \in D$ are homogeneous. Some element $a^r_s$ must not vanish. But $a^r_s = e^r_r a e^s_s \in I$, and since $a^r_s$ is invertible, it follows that $1 \in I$. Thus $I = A$. □

**Definition 2.34.** If $A$ is an associative algebra which is even as a graded space, then the $\mathbb{Z}_2$-graded *double* of $A$ is the algebra $A \oplus \Pi A$, where $\Pi A$ is the parity reversion of $A$, with the multiplication $(a, a')(b, b') = (ab + a'b', ab' + a'b)$.

The proof that the double of $A$ is associative is straightforward.

**Theorem 2.35.** *If $A$ is a completely even division algebra over $K$, then its double is also a division algebra.*

*Proof.* This is obvious since every nonzero homogeneous element is invertible. □

**Theorem 2.36.** *If $A$ is a completely even simple algebra over $K$, then its double is also simple.*

*Proof.* Let $I$ be a graded ideal in the double of $A$. Let $I_0$ be the even part of $I$. Note that $AI_0 \subseteq I_0$ and $I_0A \subseteq I_0$. Therefore $I_0$ is an ideal in $A$. It follows that $I_0 = 0$ or $I_0 = A$. If $I_0 = A$, then let $b \neq 0 \in \Pi A$. Then $bA = \Pi A$, so $\Pi A \subseteq I$ and $I = A \oplus \Pi A$. If $I_0 = 0$, then let $I_1$ be the odd part of $I$. If $b \in I_1$, then $bI_1 \subseteq I_0 = 0$, and it follows that $I_1 = 0$. Thus $I = 0$. □

**Proposition 2.37.** *Let $A$ be a $\mathbb{Z}_2$-graded division algebra over $K$. Then $A_0$ is an ordinary division algebra over $K$, and either $A_1 = 0$ or $\dim(A_0) = \dim(A_1)$.*

*Proof.* Since every nonzero element in $A_0$ is invertible, $A_0$ is an ordinary division algebra. If $b \neq 0 \in A_1$, then $b$ is invertible, so the map $A_0 \to A_1$ given by $a \mapsto ba$ has inverse given by multiplication by $b^{-1}$. □

**Theorem 2.38.** *Let $A$ be a division algebra over $K$, and suppose that $D = A_0$. Let $\theta$ be a generator of $A_1$ as a $D$-module. Define $f : D \to D$ by $\theta x = f(x)\theta$. Then $f$ is an automorphism of $D$. Suppose that $\theta^2 = a$. Then $f(a) = a$ and $a \neq 0$. Moreover $f^2(x) = axa^{-1}$ for any $x \in D$, so $f^2$ is an inner automorphism.*

*If $f$ is any automorphism of $D$, $a$ is a nonzero element of $D$ fixed by $f$, and $f^2(x) = axa^{-1}$ for all $x \in D$, then the rules $\theta x = f(x)\theta$ and $\theta^2 = a$ determine a unique structure of a division algebra on $A$.*

*Proof.* If $A$ is a division algebra, then note that

$$f(xy)\theta = \theta(xy) = (\theta x)y = (f(x)\theta)y = f(x)(\theta y) = f(x)(f(y)\theta = (f(x)f(y))\theta,$$

so that it follows that $f(x)f(y) = f(xy)$. Moreover $f(1)\theta = \theta \cdot 1 = \theta$, so $f(1) = 1$. Thus $f$ is an automorphism of $D$. Also

$$a\theta = \theta^2\theta = \theta\theta^2 = \theta a = f(a)\theta,$$

so $f(a) = a$. Finally

$$xf(y)az\theta = (x\theta \cdot y\theta) \cdot z\theta = x\theta \cdot (y\theta \cdot z\theta) = x\theta \cdot yf(z)a = xf(y)f^2(z)a\theta,$$

so $f^2(z) = aza^{-1}$.

On the other hand suppose that $f$ is an automorphism of $D$ such that $f(a) = a$ and $f^2(x) = axa^{-1}$. Every element of $D$ is of the form $x$ or $x\theta$, where $x \in D$. We define the product in $D$ by

$$\begin{aligned} x \cdot y\theta &= xy\theta \\ x\theta \cdot y &= xf(y)\theta \\ x\theta \cdot y\theta &= xf(y)a. \end{aligned}$$

□

Since the only automorphism of $K$ over $K$ is the identity, a division algebra over $K$ with nontrivial odd part whose even part is $K$ is determined by an element $a \neq 0 \in K$. Then the division algebra structure on $K \oplus K$ given by $\pi x \pi y = axy$ will be called the double of $K$ determined by $a$, and be denoted by $K_a$. We have $K_a = K_{a'}$ iff $a' = ab^2$ for some $b \in K$. As a consequence, if $K$ is algebraically closed, there is a unique structure of a double of $K$.

**Theorem 2.39.** *The only $\mathbb{Z}_2$-graded division algebras over $\mathbb{C}$ are $\mathbb{C}$, considered as a completely even space, and its double $\mathbb{C} \oplus \Pi\mathbb{C}$.*

*Proof.* Since the only even division algebra over $\mathbb{C}$ is $\mathbb{C}$, a division algebra which is not completely even is a double of $\mathbb{C}$, which has a unique structure, since $\mathbb{C}$ is algebraically closed. $\square$

The following result is well known.

**Proposition 2.40.** *If $A$ is a division algebra over the reals, then the even part $A_0$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

**Proposition 2.41.** *Let $A$ be a real division algebra with $A_0 = \mathbb{R}$. Then there are three possibilities:*

(1) *$A = \mathbb{R}$.*
(2) *$A = \mathbb{R} \oplus \Pi\mathbb{R}$, the double of $\mathbb{R}$. In other words, $A = \langle 1, \theta\rangle$, with $\theta^2 = 1$.*
(3) *$A = \mathbb{R}_{-1} = \mathbb{C}$, with $i$ of odd parity. In other words, $A = \langle 1, i\rangle$, with $i^2 = -1$.*

*Proof.* This follows because there are precisely two equivalence classes in $\mathbb{R}$ modulo squares, 1 and $-1$. Thus there are only two doubles of $\mathbb{R}$, up to isomorphism. $\square$

**Proposition 2.42.** *Let $A$ be a real division algebra with $A_0 = \mathbb{C}$. Then there are four possibilities:*

(1) *$A = \mathbb{C}$.*
(2) *$A = \mathbb{C} \oplus \Pi C$, the double of $\mathbb{C}$. In other words $A = \langle 1, \theta\rangle$ with $\theta^2 = 1$ and $\theta\alpha = \alpha\theta$.*
(3) *$A = \mathbb{H}$, with $j$ taken to be odd. In other words, $A = \langle 1, j\rangle$ with $j^2 = -1$, and $j\alpha = \bar{\alpha}j$.*
(4) *$A = \langle 1, \theta\rangle$, with $\theta^2 = 1$ and $\theta\alpha = \bar{\alpha}\theta$.*

*Proof.* The first case arises when $A_1 = 0$. There are precisely two automorphisms of $\mathbb{C}$ over $\mathbb{R}$, the identity, and complex conjugation. Both of them have square equal to the identity. Moreover, if $a \neq 0 \in \mathbb{C}$, $axa^{-1} = x$, so that we can choose any $a \in \mathbb{C}$, with $\theta^2 = a$. However, if we replace $\theta$ with $\hat{\theta} = b\theta$, then $\hat{\theta}^2 = bf(b)a$. If $f$ is the identity, then we obtain $b^2a$, which means that if we choose $b = \sqrt{a^{-1}}$, we obtain $\hat{\theta}^2 = 1$. This gives the second case. If $f$ is conjugation, then since $f$ must fix $a$, $a \in \mathbb{R}$. If $a < 0$, then let $b = \sqrt{-a^{-1}}$, and replace $\theta$ by $j = b\theta$, we get $j^2 = -1$, which gives the third case. The fourth case arises when $a$ is positive. $\square$

**Proposition 2.43.** *Let $A$ be a real division algebra with $A_0 = \mathbb{H}$. Then there are three possibilities:*

(1) *$A = \mathbb{H}$.*
(2) *$A = \mathbb{H} \oplus \Pi H$, the double of $\mathbb{H}$. In other words $A = \langle 1, \theta\rangle$ with $\theta^2 = 1$ and $\theta q = q\theta$.*
(3) *$A = \langle 1, \theta\rangle$, with $\theta^2 = -1$ and $\theta q = q\theta$.*

*Proof.* The proof relies on two well known properties of $\mathbb{H}$. First, the center of $\mathbb{H}$ is $\mathbb{R}$. Secondly, every automorphism of $\mathbb{H}$ is an inner automorphism. Thus, we have $f(q) = bqb^{-1}$ for some $b \in \mathbb{H}$. Replace $\theta$ with $\hat{\theta} = b^{-1}\theta$, and then $\hat{\theta}q = b^{-1}f(q)\theta = b^{-1}bqb^{-1}\theta = q\hat{\theta}$, so we can assume that $f$ is the identity. But then $aqa^{-1} = f^2(q) = q$, which implies that $a \in \mathbb{R}$. Up to isomorphism, we can choose either $a = 1$ or $a = -1$, which gives the two cases where $A_1 \neq 0$. $\square$

## 3. Connected Algebras

The (graded) *center* $Z(A)$ of a $\mathbb{Z}_2$-graded algebra $A$ is the graded subalgebra generated by all homogeneous $c \in A$ such that $ca = (-1)^{ac}ac$ for all homogeneous $a \in A$. A unital $K$-algebra is said to be *connected* over $K$ if its center is $K$.

**Proposition 3.1.** *Suppose that* $\operatorname{char}(K) \neq 2$. *Then the center of a* $\mathbb{Z}_2$*-graded division algebra over* $K$ *is a field.*

*Proof.* If $D$ is a division algebra and $a \in C(D)$ is odd, then $a^2 = -a^2$. Thus $a^2 = 0$. It follows that $a = 0$. Thus $C(D)$ is completely even, and it is a division algebra, so it must be a field. □

**Example 3.** Let $K = \mathbb{Z}_2$, and $A = K \oplus \Pi K$ be the double of $K$. Then $A$ is a division algebra, and it is graded commutative. However $(1+\theta)(1-\theta) = 0$, so $A$ is not a field.

Because of the example above, we will assume from now on that the field $K$ does not have characteristic 2. Later on, we will encounter another reason to exclude fields of characteristic 2, because in the proof of one of the key lemmas, we need to multiply by $\frac{1}{2}$.

If $D$ is a division algebra over $K$, and $F$ is the center of $D$, then $D$ is connected as an $F$-algebra.

**Proposition 3.2.** *Let $A$ be an algebra containing a* connected *division algebra $D$ and $C$ be a subalgebra of $C(D)$ with identity $e$ coinciding with the identity of $D$. Then the map $D \otimes C \to A$ given by $a \otimes c \mapsto ac$ is an isomorphism onto the subalgebra $DC$ of $A$.*

*Proof.* First note that if $a \in Dc_1 + \cdots + Dc_k$, then $Da \subseteq Dc_1 + \cdots + Dc_k$. Therefore, we can find a homogeneous subset $u_1, \cdots, u_k$ of $C$, such that $DC = \bigoplus Du_i$. We claim that $C = \langle u_1, \cdots, u_k \rangle$. Clearly, the $u_i$-s are linearly independent over $K$. Let $c$ be a homogeneous element of $C$. Then $c = a_1u_1 + \cdots + a_ku_k$ for some homogeneous $a_i \in D$. Since $c \in C$, if $a \in D$ is homogeneous, we have

$$0 = ca - (-1)^{ac}ac = (a_1a - (-1)^{aa_1}aa_1)u_1 + \cdots + (a_ka - (-1)^{aa_k}aa_k)u_k.$$

It follows that $a_ia = (-1)^{aa_i}aa_i$, so $a_i \in Z(D) = K$. But this means that $c$ is in the span of the $u_i$. Thus $\dim(C) = k$ and since $\dim Du_k = \dim D$, it follows that $\dim DC = \dim(D)\dim(C) = \dim(D \otimes C)$. This shows that the map $D \otimes C \to DC$ is an isomorphism, since it is clearly surjective. □

**Theorem 3.3.** *Suppose $D$ is a connected division algebra. Let $D^\circ$ be the opposite algebra of $D$. Then the map $\varphi : D \otimes D^\circ \to \mathfrak{gl}(D)$ given by $\varphi(a \otimes b)(x) = (-1)^{bx}axb$ is an isomorphism.*

*Proof.* First note that $\varphi$ is a well defined morphism, and that $D' = \varphi(D \otimes 1)$ is a connected subalgebra of $\mathfrak{gl}(D)$ isomorphic to $D$, since if $a \in D$, then $\varphi(a \otimes 1)(1) = a$, so $\varphi$ maps $D \otimes 1$ injectively into $\mathfrak{gl}(D)$. Let $C = \varphi(1 \otimes D^\circ)$, and note that $C \subseteq (C(D))$ and $I = \varphi(1 \otimes 1)$ is the identity of $\mathfrak{gl}(D)$ which lies both in $D'$ and $C$. As a consequence, the previous lemma shows that the map $\varphi$ is injective, since it is an isomorphism onto its image. Finally, note that $\dim(D \otimes D^\circ) = \dim(D)^2 = \dim(\mathfrak{gl}(D))$, so $\varphi$ is an isomorphism. □

**Proposition 3.4.** *If $A$ is a connected simple algebra, then $A \otimes A^\circ$ is a matrix algebra.*

*Proof.* Recall that $\mathfrak{gl}(V \otimes W) = \mathfrak{gl}(V) \otimes \mathfrak{gl}(W)$. In other words, the tensor product of two matrix algebras is a matrix algebra. Moreover, $(A \otimes B)^\circ = A^\circ \otimes B^\circ$. and $\mathfrak{gl}(V)^\circ = \mathfrak{gl}(V)$. Also, if $A \otimes B$ is connected, then both $A$ and $B$ must be connected. Moreover, if $M$ is a matrix algebra over $K$, then $M$ is connected. From Wedderburn's theorem, we can express $A = M \otimes D$ for some matrix algebra $M$ over $K$ and a division algebra $D$, which must be connected. Then

$$A \otimes A^\circ = (M \otimes D) \otimes (M \otimes D)^\circ = M \otimes D \otimes M \otimes D^\circ = M \otimes M \otimes (D \otimes D^\circ).$$

Thus $A \otimes A^\circ$ is a matrix algebra. □

**Theorem 3.5.** *An algebra $A$ is a connected, simple $K$-algebra precisely when $A_E$ is a connected, simple $E$-algebra for every extension field $E$ of $K$.*

*Proof.* Clearly if $A_E$ is a connected simple $E$-algebra for every extension field $E$ of $K$, then $A = A_K$ is connected and simple $K$-algebra. On the other hand, if $A$ is a connected simple $K$-algebra, then $M = A \otimes A^\circ$ is a matrix algebra over $K$. Then $M_E$ is a matrix algebra over $E$. In fact, if $M = \mathfrak{gl}_K(V)$, then $M_E = \mathfrak{gl}_E(V_E)$. Moreover, $M_E = A_E \otimes_E (A^\circ)_E$. If a tensor product of two algebras is simple, then both algebras must be simple, so it follows that $A_E$ is a simple $E$-algebra. □

**Theorem 3.6.** *Suppose that $D$ is a division algebra over $K$, and that $D_0 \neq K$. Then there is an extension $E$ of $K$ such that $D_E$ is not a division algebra.*

*Proof.* Let $a \in D_0$ not lie in $K$. Then the minimal polynomial $m_a(x)$ of $a$ over $K$ is irreducible, and not linear. The minimal polynomial of $a \otimes 1$ in $D_E$, for any extension field $E$ of $K$, is just $m_a(x)$, because $\deg(m_a(x)) = \dim_k(K(a)) = \dim_E(E(a \otimes 1))$. If we let $E$ be a field in which $m_a(x) = f(x)g(x)$ is a nontrivial factorization, then $f(a)g(a) = 0$ but $f(a)$ and $g(a)$ are nonzero. Thus $D_E$ is not a division algebra. □

**Theorem 3.7.** *An algebra $A$ is a connected and simple $K$-algebra iff there is an extension $E$ of $K$, such that $A_E = M \otimes D$, where $M$ is a matrix algebra over $D$ and either $D = E$ or $D = E_a$ is a double of $E$.*

*Proof.* If $D_0 = K$, then we are done, so assume that $D_0 \neq K$. Let $E$ be an extension of $K$ such that $D_E$ is not a division algebra. Since $D$ is connected, $D_E$ must be simple. Thus $D_E = M \otimes_E D'$ for some nontrivial matrix algebra $M$ over $E$ and a connected division algebra $D'$ over $E$. Since $\dim_K(D) = \dim_E(D_E) = \dim_E(M)\dim_E(D')$, it follows that $\dim_E(D') < \dim_K(D)$. Thus by a straightforward induction argument, we can continue the process until we arrive at an algebra of the form $M \otimes_E D'$, for some field $E$ and a division algebra $D$ such that $D_0 = E$. □

**Example 4.** Let $A = \mathbb{C}$, considered as an even $\mathbb{R}$-algebra. Then $A$ is not connected, since $Z(A) = \mathbb{C}$. Consider the algebra $A_{\mathbb{C}}$. The elements $\frac{1}{2}(1 \otimes 1 \pm i \otimes i)$ are orthogonal idempotents, so that $A_{\mathbb{C}} = \mathbb{C} \times \mathbb{C}$, and therefore is semisimple, but not simple, as an algebra over $\mathbb{C}$.

**Example 5.** Let $A = \mathbb{C}$, this time with $i$ considered as an odd element. Then since $i^2 \neq 0$, $i$ is not in the graded center of $A$, so $Z(A) = \mathbb{R}$, and $A$ is connected as an $\mathbb{R}$-algebra. Now $A_{\mathbb{C}} = \mathbb{C} \oplus \pi C = \langle 1, \theta \rangle$, where $1 = 1 \otimes 1$ and $\theta = i \otimes i$. Note that $\theta^2 = 1$, so that indeed, $A$ is the double of $\mathbb{C}$, and thus is a division algebra.

**Example 6.** Let $A=\mathbb{H}$, considered as a completely even algebra. Then $A$ is a connected $\mathbb{R}$ algebra. The elements $e_1=\frac{1}{2}(1\otimes 1+i\otimes i)$, $e_2=\frac{1}{2}(1\otimes 1-i\otimes i)$ are orthogonal idempotents in $A_{\mathbb{C}}$. Identifying $\mathbb{H}$ with $\mathbb{H}\otimes 1$ in $A_{\mathbb{C}}$, we note that $e_1j=je_2$ and $e_2j=je_1$, and that $A_{\mathbb{C}}=\langle e_1,e_2,e_1j,e_2j\rangle$. If we let $e_{ii}=e_i$ and $e_{12}=e_1j$, $e_{21}=-e_2j$, then $e_{ij}$ determine a matrix subalgebra of $A_{\mathbb{C}}$, which must equal all of $A_{\mathbb{C}}$. Thus $\mathbb{H}\otimes\mathbb{C}=\mathfrak{gl}(\mathbb{C}^2)$.

**Example 7.** Let $A=\mathbb{H}$, where this time, we consider $j$ to be odd. Then $A_0=\mathbb{C}$. Note that the center of $A_0$ is $\mathbb{C}$, but the center of $\mathcal{A}$ is just $\mathbb{R}$, so $A$ is connected as an $\mathbb{R}$-algebra. The same decomposition of $A_{\mathbb{C}}$ as in the previous example works, but this time $A_{\mathbb{C}}=\mathfrak{gl}(\mathbb{C}\oplus\Pi\mathbb{C})$.

## 4. Separable Algebras

**Proposition 4.1.** *Let $A$ be a $K$-algebra, $E$ be an extension field of $K$, and $\varphi: A\to B$ be a $K$-algebra morphism, where $B$ is a $E$-algebra. Then $\varphi$ extends to an $E$-algebra morphism $\varphi: A\otimes_K E\to B$, given by $\varphi(a\otimes x)=ax$.*

**Theorem 4.2.** *Let $F=K(a)$ be a separable, finite extension of $K$ and $E$ be a splitting field of the minimal polynomial $p(x)$ of $a$. If $n=\deg(a)$, then $F\otimes E\cong E^n$ as $E$-algebras.*

*Proof.* Suppose that $p(x)=(x-a_1)\cdots(x-a_n)$ in $E$, where $a_1,\cdots,a_n$ are the roots of $p(x)$ in $E$, which are distinct, since $a$ is separable. Now $E^n=Ee_1\oplus\cdots Ee_n$, where $1=e_1+\cdots+e_n$ is a decomposition of the identity in $E^n$ into primitive, pairwise orthogonal idempotents.

Let $y=a_1e_1+\cdots+a_ne_n$. Then $f(y)=f(a_1)e_1+\cdots+f(a_n)e_n$, for any polynomial $f$ with coefficients in $E$, so in particular, $p(y)=0$. As a consequence, there is a morphism $\varphi: F\to E^n$ given by $\varphi(f(a))=f(y)$ for $f(a)\in F$, which extends to a morphism $\varphi: F\otimes E\to E^n$. Define $p_i(x)\in E[x]$ by $p(x)=(x-a_i)p_i(x)$. It follows that $p_i(a_i)\neq 0$, but $p_i(a_j)=0$ if $i\neq j$. Thus, we obtain that $p_i(y)=p_i(a_i)e_i$. It follows that $e_i=p_i(y)p_i(a_i)^{-1}\in\varphi(F\otimes E)$. Since the $e_i$-s form a basis of $E^n$ as an $E$-algebra, we see that $\varphi$ is surjective. But then, since $\dim_E(F\otimes E)=n=\dim_E(E^n)$, it follows that $\varphi$ must be an isomorphism. □

If we denote the element $p_i(a)p_i(a_i)^{-1}$ in $F\otimes E$ as $e_i$, then it is easy to see that $ae_i=a_ie_i$, because both elements have the same image in $E^n$ under the isomorphism $\varphi$. Moreover the $e_i$-s give a decomposition of the identity in $F\otimes E$ into orthogonal idempotents.

**Theorem 4.3.** *Suppose that $F=K(a)$, where the minimal polynomial is $m_a(x)=x^{p^k}-b$, where $\operatorname{char}(K)=p$. Then $F\otimes F$ has a nilpotent element.*

*Proof.* We have $(a\otimes 1-1\otimes a)^{p^k}=0$ but $a\otimes 1\neq 1\otimes a$ in $F\otimes F$. □

If $E$ is an extension field of $K$, and $A$ is a $K$-algebra, then $A$ is naturally identified with the subalgebra $A\otimes 1$ of $A_E$.

**Theorem 4.4.** *Let $A$ be a unital algebra with unit $e$. Suppose that $F=K(a)$ is a subfield of $C(A)$, where $a$ is separable over $K$ and that the identity of $F$ coincides with $e$. Let $E=K(a_1,\cdots,a_n)$ be a splitting field of $m_a(x)=(x-a_1)\cdots(x-a_n)$. Let $e_i=p_i(a)p_i(a_i)^{-1}\in A_E$, where $p_i(x)(x-a_i)=m_a(x)$. Let $A_i=Ae_i$. Then $A_i$*

*is a $K(a_i)$ subalgebra of $A_E$ which is isomorphic to $A$ as a $K$-algebra. Moreover $A_Ee_i \cong A_i \otimes_{K(a_i)} E$ and*

$$A_E = A_Ee_1 \times \cdots \times A_Ee_n.$$

*Moreover, $C(A_E) = C(A_Ee_1) \times \cdots \times C(A_Ee_n)$, so that*

$$\dim_E(C(A_Ee_i)) = \dim_K(C(A))/n.$$

*Proof.* To see that $A_i \cong A$, let $A = \langle u_1, \cdots, u_n \rangle$ over $K$. Set $v_j = u_je_i$, so that the $v_j$-s give a $K$-basis of $A_i$. If $u_ju_k = c^m_{jk}u_m$, where $c^m_{jk}$ are the structure constants of the algebra $A$ in terms of the basis $u_j$, then

$$v_jv_k = u_je_iu_ke_i = u_ju_ke_i = c^m_{jk}u_me_i = c^m_{jk}v_m,$$

because $e_i$ is central in $A_E$. Therefore, the algebra $A_i$ is given by the same structure constants in term of the basis $v_j$, so the two algebras are isomorphic. Now $ae_i = a_ie_i$, so $a_ie_i \in A_i$. It follows that $A_i$ is a $K(a_i)$-algebra.

The map $A_i \times E \to A_Ee_i$, given by $(xe_i, y) \mapsto xye_i$, is clearly $K(a_i)$ bilinear, so it induces a map $A_i \otimes_{K(a_i)} E \to A_Ee_i$. Now

$$\dim_E(A_i \otimes_{K(a_i)} E) = \dim_{K(a_i)} A_i = m/n = \dim_E(A_E)/n = \dim_E(A_Ee_i),$$

where the last equality follows since $A_Ee_i \cong A_Ee_j$, and the decomposition $A_E = A_Ee_1 \times \cdots A_Ee_n$ holds because the $e_i$ are a decomposition of the identity of $e$ in terms of central, even, pairwise orthogonal idempotents. The statement about the dimension of $C(A_E)$ follows immediately. □

**Theorem 4.5.** *Suppose that $A$ is a unital algebra and $F = K(b)$ is a subfield of $C(A)$ whose identity coincides with the identity of $A$. If $b$ is not separable over $K$, then there is an extension field $E$ of $K$ such that $A_E$ contains a central nilpotent element.*

*Proof.* Suppose that $m_b(x) = g(x^{p^k})$ is the minimal polynomial of $b$ over $K$, where $g(x)$ is separable over $K$. Let $a = b^{p^k}$, so that $m_a(x) = g(x)$ is the minimal polynomial of $a$ over $K$. Let $E' = K(a_1, \cdots, a_n)$ be a splitting field for $g(x) = (x - a_1) \cdots (x - a_n)$, so $A_{E'} = A_{E'}e_n \times \cdots \times A_{E'}e_n$ is the decomposition of $A_{E'}$ given in Theorem (4.4), with $ae_i = a_ie_i$. Let $E = K(b')$, where the minimal polynomial of $b'$ over $E'$ is $p(x) = x^{p^k} - a_1$. (Since $m_b(b') = 0$, it follows that $m_b(x)$ is the minimal polynomial of $b'$ over $K$, so $p(x)$ must be irreducible over $E'$.) Then $A_E = A_Ee_1 \times \cdots A_Een$, and $b'e_1 \in A_Ee_1$ is not equal to $be_1$. But $(be_1 - b'e_1)^{p^k} = ae_1 - a_1e_1 = 0$, so $be_1 - b'e_1$ is a central nilpotent element of $A_E$. □

**Proposition 4.6.** *Suppose that $A$ is a semisimple $K$-algebra. Then $C(A)$ is also semisimple.*

*Proof.* If $a \in \mathrm{Rad}(C(A))$, then $a$ is nilpotent. Since $a$ commutes with elements of $A$, it follows that $a$ must be properly nilpotent. □

**Proposition 4.7.** *The center of a simple algebra is simple.*

*Proof.* Let $A$ be simple. Then its center $C$ is semisimple, so is a direct sum of simple algebras. Thus $C = C_1 \oplus \cdots \oplus C_n$, for some simple algebras $C_i$. If we let $e_i$ be the identity of $C_i$, then $A = A_1 \oplus \cdots \oplus A_n$, where $A_i = Ae_i$ is an ideal in $A$, since $e_i$ is central. But $A$ is simple, so there can only be one term, and thus $C$ is simple. □

**Proposition 4.8.** *A simple $\mathbb{Z}_2$-graded commutative $K$-algebra must be a field.*

*Proof.* Let $D$ be a $\mathbb{Z}_2$-graded commutative algebra. Then by Wedderburn's theorem, it must be a division algebra, since any matrix algebra of dimension larger than 1 is not commutative. Next, note that if $a$ is an odd element of $D$, then $a^2 = 0$, so $a$ is nilpotent, and since $D$ is commutative, it must be properly nilpotent. Thus $a = 0$ and $D$ is a completely even algebra. Thus $D$ must be a field, since every nonzero element is invertible. □

**Proposition 4.9.** *The center of a semisimple $\mathbb{Z}_2$-graded algebra is a direct sum of fields. In particular, the center of a simple $\mathbb{Z}_2$-graded algebra is a field.*

*Proof.* Since the center of a semisimple algebra is a direct sum of the centers of its simple components, we need only show the second statement. Since the center of a simple algebra is also simple, the previous result shows that it is a field. □

**Definition 4.10.** A $K$-algebra $A$ is called *separable* iff $A_E = A \otimes E$ is semisimple for every extension field of $K$.

Note that a separable algebra must be semisimple, because $A = A_K$.

**Theorem 4.11.** *A simple $K$-algebra is separable precisely when its center is a separable field extension of $K$.*

*Proof.* Let $F$ be the center of $A$, which is a field. If $F$ is not separable then we already know that there is an extension of $E$ of $K$ such that $A_E$ has a central nilpotent element. Since a central nilpotent element is properly nilpotent, it follows that $\operatorname{Rad}(A_E) \neq 0$, so $A_E$ is not semisimple. Thus $A$ is not separable. On the other hand, if $F = K(a)$ is separable over $K$, then if $E$ is the splitting field of the minimal polynomial, then we can decompose $A_E = A_E e_1 \times \cdots A_E e_n$. We know that $A_E e_i$ is a simple algebra, and the its center has dimension 1 over $E$. So $A_E e_i$ is a connected, simple $E$-algebra. Thus, for every extension field $M$ of $E$, $(A_E e_i)_M$ is a simple $M$ algebra. It follows that $A_E$ is semisimple for any extension field $E$ of $K$, and thus $A$ is separable. □

## 5. Fundamental Theorem of Finite Dimensional Algebras

**Lemma 5.1.** *Let $A$ be a $K$-algebra and $\bar{A} = A/\operatorname{Rad}(A)$. If $f$ is an even idempotent in $\bar{A}$, there is an even idempotent $e$ in $A$ such that $\bar{e} = f$. If $\eta$ is an odd element in $f\bar{A}f$ satisfying $\eta^2 = af$ for some $a \neq 0 \in K$, then there is an odd element $\theta \in eAe$ such that $\theta^2 = ae$ and $\bar{\theta} = \eta$.*

*Proof.* Let $\bar{B} = \langle f, \eta \rangle$. Then $\bar{B}$ is a subalgebra of $\bar{A}$ and $f$ is the only even idempotent of $\bar{B}$. Let $B = \pi^{-1}(\bar{B})$, so $B$ is a subalgebra of $A$. The subalgebra $B$ cannot be nilpotent, so there is some even idempotent $e$ in $B$. Thus $\bar{e}$ is an even idempotent of $\bar{B}$, so it follows that $\bar{e} = f$. Note that $\pi(eAe) = f\bar{A}f$, so there is some odd $b \in eAe$ such that $\bar{b} = \eta$. Now $\bar{b}^2 = af$, thus $b^2 = ae + n$, where $n$ is an even element of $e\operatorname{Rad}(A)e$. Since

$$ab + bn = b(ae + n) = b^3 = (ae + n)b = ab + nb,$$

it follows that $bn = nb$. Let $f(n) = \sum_{k=1}^{\infty} b_k n^k$, which is a finite sum, since $n$ is nilpotent. Let $\theta = b(e + f(n))$. We want to solve $\theta^2 = ae$. We have

$$\theta^2 = ae + n + 2af(n) + 2nf(n) + af^2(n) + nf^2(n).$$

It is easy to see that we can solve for coefficients $b_k$, so that $n + 2af(n) + 2nf(n) + af^2(n) + nf^2(n) = 0$. In fact, we need $1 + 2ab_1 = 0$, and for $k > 1$, the coefficient of $n^k$ is $2ab_k$ plus terms involving the coefficients $b_i$ for $i < k$. Finally, note that $\bar{\theta} = \eta$. □

**Lemma 5.2.** *Let $A$ be a $K$-algebra and $\bar{A} = A/\operatorname{Rad}(A)$. Suppose that $f$ is an even idempotent of $\bar{A}$ which decomposes as a sum of pairwise orthogonal even idempotents $f = f_1 + \cdots + f_n$, and that the odd element $\eta \in f\bar{A}f$ satisfies $\eta^2 = ae$ for some $a \neq 0 \in K$, and $f_i\eta = \eta f_i$, for all $i$. Let $e$ be an even idempotent of $A$ such that $\bar{e} = f$. Then $e$ decomposes as a sum of pairwise orthogonal idempotents $e_i$ such that $\bar{e}_i = f_i$, and there is an odd element $\theta \in eAe$ such that $\theta^2 = ae$ and $e_i\theta = \theta e_i$, for all $i$.*

*Proof.* We note that the proof of the existence of the $e_i$ reduces to the case $n = 2$. Let $e \in A$ be an even idempotent such that $\bar{e} = f$. Now $f_1 \in f\bar{A}f = \pi(eAe)$, so as in lemma 1, we can find an even idempotent $e_1 \in eAe$ such that $\bar{e}_1 = f_1$. Then $e_2 = e - e_1$ is an idempotent orthogonal to $e_1$ satisfying $e = e_1 + e_2$ and $\bar{e}_2 = f_2$. Thus we can decompose $e = e_1 + \cdots + e_n$ as desired. Now let $\eta_i = \eta f_i$. Then $\eta_i \in f_i\bar{A}f_i$ and $\eta_i^2 = af_i$. Thus we can find $\theta_i \in e_iAe_i$ such that $\theta_i^2 = ae_i$. Let $\theta = \theta_1 + \cdots + \theta_n$. It is easy to check that $\theta$ has the desired properties. □

**Lemma 5.3.** *Let $f, \eta, f_i, e, \theta, e_i$ be as in the previous lemma, and suppose that some homogeneous element $v \in f_i\bar{A}\bar{f}_j$ satisfies $v\eta = (-1)^v\eta$. Then there is some homogeneous $u \in e_iAe_j$ such that $\bar{u} = v$ and $u\theta = (-1)^u\theta u$.*

*Proof.* Since the image of $e_iAe_j$ is $f_i\bar{A}f_j$, there is some homogeneous $u_1 \in e_iAe_j$ such that $\bar{u}_1 = v$. Let $u = \frac{1}{2}(u_1 + (-1)^v a^{-1}\theta u_1\theta)$. Then $\bar{u} = v$. Moreover $\theta u = \frac{1}{2}(\theta u_1 + (-1)^v u_1\theta)$, while $u\theta = \frac{1}{2}(u_1\theta + (-1)^v\theta u_1) = (-1)^u\theta u$. Thus $u$ has the desired properties. □

**Theorem 5.4.** *Suppose that $A$ is a $K$-algebra and $\bar{A} = A/\operatorname{Rad}(A)$ has a subalgebra of the form $\bar{B} = M \otimes D$, where $M$ is a matrix algebra over $K$, and $D = K$ or $D = K^a = \langle 1, \theta\rangle$ is a double of $K$. Then there is a subalgebra $B$ of $A$ which maps isomorphically onto $\bar{B}$.*

*Proof.* We prove the case when $D = K^a$, since it will be clear how to adapt the proof to the case when $D = K$.

Let $M = \langle f_{ij}\rangle$, with $f_{ij}f_{jk} = \delta_{ij}f_{il}$. Let $f$ be the identity in $\bar{B}$. Then if $f_i = f_{ii}$, we have $f = f_1 + \cdots + f_n$ is a sum of pairwise orthogonal even idempotents. We have $\bar{B} = \langle f_{ij}, \eta f_{ij}\rangle$, with $\eta^2 = af$ and $\eta f_{ij} = (-1)^{f_{ij}} f_{ij}\eta$. First, we find even idempotents $e, e_i \in A$ such that $e = e_1 + \cdots + e_n$ is a pairwise orthogonal decomposition of $e$, $\bar{e} = f$ and $\bar{e}_i = f_i$, and an odd element $\theta \in eAe$ such that $\bar{\theta} = \eta$ $e_i\theta = \theta e_i$ and $\theta^2 = ae$.

For each $i$, there is some $e_{1i} \in e_1Ae_i$, such that $\bar{e}_{1i} = f_{1i}$ and $\theta e_{1i} = (-1)^{e_{1i}} e_{1i}\theta$. Let $u_{i1} \in e_iAe_1$ be such that $\bar{u}_{i1} = f_{i1}$ and $\theta u_{i1} = (-1)^{u_{i1}} u_{i1}\theta$. Now $\bar{e}_{1i}\bar{u}_{i1} = f_1$, so it follows that $e_{1i}u_{i1} = e_1 + n_i$ for some even $n_i \in e_1 \operatorname{Rad}(A)e_1$. Moreover $\theta n_i = n_i\theta$. Let $e_{i1} = u_{i1}(e_1 + \sum_{k=1}^{\infty}(-1)^i n_i^k)$. It is easy to check that $e_{1i}e_{i1} = e_1$ and that

$\theta e_{i1} = (-1)^{e_{i1}} e_{i1}\theta$. Finally, define $e_{ij} = e_{i1}e_{1j}$. Note that $e_i e_{ij} = e_{ij}e_j = eij$, since $e_{ij} \in e_i A e_j$.

$$e_{ij}e_{jk} = e_{i1}e_{1j}e_{k1}e_{1l} = \delta_{jk}e_{il}.$$

Moreover $e_{ii} \in e_i A e_i$ is an idempotent, and $\bar{e}_{ii} = f_{ii} = \bar{e}_i$. As a consequence $e_i - e_{ii} \in \operatorname{Rad}(A)$. But $(e_i - e_{ii})^2 = e_i - 2e_{ii} + e_{ii} = e_i - e_{ii}$, so $e_i - e_{ii}$ is idempotent, thus $e_{ii} = e_i$. By our construction $\theta e_{ij} = (-1)^{e_{ij}} e_{ij}\theta$. It follows that the algebra $B$ generated by $e_{ij}$ and $\theta$ is isomorphic to $\bar{B}$. □

**Theorem 5.5.** *Suppose that $= A/\operatorname{Rad}(A)$ is a separable $K$-algebra. Then there is a subalgebra $S$ of $A$ such that which projects isomorphically onto $A/\operatorname{Rad}(A)$, so that $A = \operatorname{Rad}(A) \rtimes S$ expresses $A$ as a semidirect product.*

The proof we give here is essentially the same as in [1].

*Proof.* First, note that if $\dim_K(A) = 1$, then either $A = K$ or $A$ is the trivial algebra, so the theorem applies. We induct on $\dim(A)$. Suppose that the theorem holds for all algebras of smaller dimension. Let $N = \operatorname{Rad}(A)$ and suppose that $N^2 \neq 0$. Then $A/N^2$ is an algebra of smaller dimension, and $N/N^2 = \operatorname{Rad}(A/N^2)$ with $(A/N^2)/(N/N^2) = A/N$. Therefore, we know that $A/N^2 = N/N^2 \oplus \bar{T}$, where $\bar{T}$ is a subalgebra of $A/N^2$. Let $T$ be the inverse image of $\bar{T}$ in $A$ under the natural projection. Now $T$ is a proper subalgebra of $A$ containing $N^2$, and moreover $S/N^2$ is semisimple, so $\operatorname{Rad}(T) \subseteq N^2$. Since $N^2$ is a nilpotent ideal in $T$, it follows that $N^2 = \operatorname{Rad}(T)$. By induction, we have $T = N^2 \rtimes S$, where $S$ is a subalgebra of $T$, and thus of $A$, which projects isomorphically onto $A/N$.

We still have to consider the case when $N^2 = 0$, which we show independently of the inductive argument above. Let $E$ be an extension of $K$ such that $(A/N)_E = A_E/N_E$ is a direct sum of algebras of the form $M \otimes D$, where $M$ is a matrix algebra over $E$ and $D$ is either $E$ or a double of $E$. Let $\langle 1, \xi_1, , \cdots, \xi_n\rangle$ and $\langle \bar{u}_1, \cdots, \bar{u}_m\rangle$ be bases of $E$ and $A/N$ over $K$. By the previous theorem, there is a subalgebra $B$ of $A_E$ which projects isomorphically onto $(A/N)_E$. Since $\dim_E(B) = \dim_K(A/N)$, there is a basis $\langle v_1, \cdots, v_m\rangle$ of $B$ over $E$. Now $A \subseteq A_E$, so we can express

$$u_i = a_i^j v_j + n_i,$$

where $a_i^j \in E$ and $n_i \in N_E$. We can express

$$\bar{u}_i\bar{u}_j = c_{ij}^k \bar{u}_k,$$

where $c_{ij}^k \in K$ are the structure constants of the algebra $A/N$. Let $u'_i = u_i - n_i$, so that $\bar{u}'_i = \bar{u}_i$. Note that $u'_i \in B$, so that in fact $B = \langle u'_i, \cdots, u'_m\rangle$. Now $\overline{u'_i u'_j - c_{ij}^k u'_k} = 0$ so $u'_i u'_j - c_{ij}^k u'_k$ lies in $N_E \cap B = 0$. Thus

$$u'_i u'_j = c_{ij}^k u'_k.$$

We can express $n_i = n_i^0 + n_i^j \xi_j$, where $n_i^0$ and $n_i^j$ lie in $N$. Let $y_i = u_i - n_i^0$. Then $y_i \in A$ and

$$\begin{aligned} y_i y_j &= (u'_i - n_i^k\xi_k)(u'_j - n_j^k\xi_k) = u'_i u'_j - (u'_i n_j^k + n_i^k u'_j)\xi_k \\ &= c_{ij}^l u'_l - (u'_i n_j^k + n_i^k u'_j)\xi_k = c_{ij}^l y_l + (c_{ij}^l n_l^k + u'_i n_j^k + n_i^k u'_j)\xi_k. \end{aligned}$$

By the linear independence of $\xi_k$ and 1 over $K$, it follows that $y_i y_j = c_{ij}^k y_k$. But this means that the subalgebra $S = \langle y_1, \cdots, y_m\rangle$ of $A$ is isomorphic to $A/N$. □

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