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ON THE COHOMOLOGY $H^*(L_k, L_s)$

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Introduction

Let $W_1^{pol} = W_1$ be the infinite dimensional Lie algebra of vector fields $f(x)\frac{d}{dx}$ on the line with polynomial coefficients. The Lie algebra W_1 has an additive algebraic basis consisting of the fields $e_k = x^{k+1}\frac{d}{dx}$, $k \geqq 1$, in which the bracket is described by

$$[e_k, e_\ell] = (\ell - k)e_{k+\ell}.$$

Consider the subalgebras L_k, $k \geqq 0$ of W_1, consisting of the fields such that they and their first k derivatives vanish at the origin. The Lie algebra L_k is generated by the basis elements $\{e_k, e_{k+1}, \dots\}$. The algebras W_1 and L_k are naturally graded by $\deg e_i = i$. Obviously, the infinite dimensional subalgebras L_k of W_1 are nilpotent for $k \geqq 1$.

The cohomology theory of infinite dimensional Lie algebras is worked out in [6]. The cohomology rings $H^*(W_1)$ and $H^*(L_k)$, $k \geqq 0$, with trivial coefficients are computed in [7] and [8]. The main results are the following:

1 $$H^q(W_1) = \begin{cases} \mathbf{C} & \text{for } q = 0,3 \\ 0 & \text{for all other } q, \end{cases}$$

2 $$H^q(L_0) = \begin{cases} \mathbf{C} & \text{for } q = 0,1 \\ 0 & \text{for } q > 1, \end{cases}$$

3 $$\dim H^q(L_k) = \binom{q+k-1}{k-1} + \binom{q+k-2}{k-2} \text{ for } k \geqq 1.$$

In particular,

$$\dim H^q(L_1) = 2 \text{ for } q \geq 1,$$
$$\dim H^q(L_2) = 2q + 1,$$
$$\dim H^q(L_3) = (q+1)^2.$$

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With the help of this last result, one can compute the cohomology of the Lie algebra W_1 with coefficients in different modules. For each $\lambda \in \mathbf{C}$, let F_λ denote the W_1-module of the tensor fields of the form $f(z)dz^{-\lambda}$, where $f(z)$ is a formal power series in z. Then the formula

$$(g\frac{d}{dx})f dx^{-\lambda} = (gf' - \lambda f g')dx^{-\lambda}$$

gives the action of W_1 on F_λ. The module F_λ has an additive basis $\{f_j \mid j = 0, 1, \dots\}$ where $f_j = x^j dx^{-\lambda}$ and the action on the basis elements is

$$e_i f_j = (j - (i+1)\lambda) f_{i+j}.$$

Denote by $\mathcal{F}_\lambda$ the W_1-module which is defined in the same way, except that the index j runs over all integers. Define the adjoint modules F'_λ, $\mathcal{F}'_\lambda$ as modules of linear functionals $F_\lambda \to \mathbf{C}$, $\mathcal{F}_\lambda \to \mathbf{C}$ which are finite in the sense that they take nonzero values only on a finite number of f_j-s. Obviously $\mathcal{F}'_\lambda = \mathcal{F}_{-1-\lambda}$ and $F'_\lambda = \mathcal{F}_{-1-\lambda}/F_{-1-\lambda}$.

Let us define now the L_0-module $F_{\lambda,\mu}$ as the subspace, generated — like F_λ — by the elements f_j, $j = 0, 1, \dots$, on which L_0 acts by

$$e_i f_j = (j + \mu - (i+1)\lambda) f_{i+j}.$$

In this definition μ can be an arbitrary complex number. Let $F'_{\lambda,\mu}$ denote the module, conjugate to $F_{\lambda,\mu}$. Finally define the modules $\mathcal{F}_{\lambda,\mu}$ over W_1 as $F_{\lambda,\mu}$ above, without requiring the positivity of j. Obviously, $\mathcal{F}'_{\lambda,\mu} = \mathcal{F}_{-1-\lambda,-\mu}$ and $F'_{\lambda,\mu} = \mathcal{F}_{-1-\lambda,-\mu}/F_{-1-\lambda,-\mu}$. The cohomology of the Lie algebra W_1 and L_1 with coefficients in the above mentioned tensor field modules are known (see [2], [3]). The computation reduces to that of the cohomology of the algebra L_1 with trivial coefficients. In the case of the Lie algebra W_1, the problem is also solved for cohomology with coefficients of the form $F_\lambda \otimes F_\mu$ [2].

Considering the adjoint representation as coefficient space, we get a very important application of the cohomology. The elements of the space $H^2(L, L)$ correspond to the infinitesimal deformations of the Lie algebra L (see e.g. [6]). In the case of the Lie algebra L_0, we get that $H^2(L_0; L_0) = 0$, consequently, L_0 is rigid (see [5] and [6]). As an L_1-module, L_1 is $F_{1,1}$, and we have the result $\dim H^q_{(-m)}(L_1; L_1) = \dim H^q_{(-m)}(L_2; \mathbf{C})$ (see [4]). In particular,

$$\dim H^2_{(-m)}(L_1; L_1) = \begin{cases} 1 & \text{for } m = 2, 3, 4 \\ 0 & \text{for } m \neq 2, 3, 4. \end{cases}$$

Here the cohomology space is defined in the graded sense:

$$H^q(L_1; L_1) = \bigoplus_m H^q_{(-m)}(L_1; L_1)$$

where for the cocycle ϕ, representing a class of $H^q_{(-m)}(L_1;L_1)$, the weight of $\phi(e_{i_1},\dots,e_{i_q})$ is $-m+i_1+\dots+i_q$.

The analogous problem for the Lie algebras L_k, $k>1$ seems to be very difficult and has not been solved. Nor has any other cohomology space for these Lie algebras been computed, with other than trivial coefficients. We naturally want to know first of all the cohomology with coefficients in the adjoint representation. In this paper, we prove the finiteness of these cohomology spaces, and also give lower bounds for their dimensions. In fact, we give estimates in the more general cases, where the coefficient module in an L_k-module L_s: we study the cohomology spaces $H^*(L_k;L_s)$ with $k\geqq 1$ and $s\geqq 1$. We make the computations for the lower bound for $k=2$.

Special attention is paid to the cohomology $H^2(L_2;L_2)$ which is, according to the general theory, the space of infinitesimal deformations of the Lie algebra L_2. The computation of $H^2(L_2;L_2)$ is the first step in determining the base of the versal deformation of the Lie algebra L_2; recall that the similar problem for L_1 is completely solved in [4].

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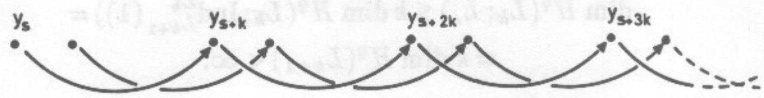
§1. Upper bounds. Finiteness

Suppose first that $s>k$. Then the L_k-module cohomology $H^q(L_k;L_s)$ may be estimated via the cohomology $H^q(L_{k+1})$. Consider the module $\mathrm{Ind}^{L_k}_{L_{k+1}}(1)$. It has the following structure: it has the basis $x_1,x_2,x_3,x_4,\dots$ and $e_kx_i=x_{i+1}$, $e_\ell x_i=0$ for $\ell>k$. The sum

$$\bigoplus_k \mathrm{Ind}^{L_k}_{L_{k+1}}(1)$$

is isomorphic to the module $N_{k,s}$ which has the following structure: it has the basis $y_s,y_{s+1},y_{s+2},\dots$.



The action of the vector field e_k is shown by the arrows, while the fields e_ℓ with $\ell>k$ act trivially. One may assume that $e_ky_t=(t-k)y_{t+k}$. (It is important here that $s>k$, so that the difference $t-k$ cannot vanish.)

Now consider the complex $C^*(L_k;L_s)$. It has a natural (decreasing) filtration: $F^{m,q}=F^mC^q(L_k;L_s)$ consists of those $c\in C^q(L_k;L_s)$, for which $c(e_{i_1},\dots,e_{i_q})=0$ for $i_1+\dots+i_q<m$. Consider the groups

$$G^{m,q}=\frac{F^{m,q}\cap d^{-1}F^{m+r,q+1}}{(F^{m,q}\cap dF^{m-r,q-1})+(F^{m+1,q}\cap d^{-1}F^{m+r,q+1})}$$

They lie between the groups $E_r^{m,q-m}$ and $E_{r+1}^{m,q-m}$ of the spectral sequence, corresponding to our filtration, and hence the sum $\oplus_m G^{m,q}$ gives the upper estimate for $H^q(L_k; L_s)$. On the other hand, for $r = k+1$ this sum is precisely the cohomology of the complex with all the summands corresponding to the action of e_t with $t > r$ removed from the formula for the differential. But this is nothing else but $H^q(L_k; N_{k,s})$.

REMARK. The descending filtration $\{F^m\}$ of the complex $C^*(L_k; L_s)$ is infinite, and this could create some convergence problems for the spectral sequence, but in our case everything is good thanks to the finite dimensionality of $H^*(L_k)$. What we have to check is essentially the fact that $\bigcap_m \operatorname{Im}\{H^q(F^m) \to H^q(L_k; L_s)\} = 0$. But actually $H^q(F^m) = 0$ for m being large enough. Indeed, let $H^q_{(m')}(L_k) = 0$ for $m' \geqq m$ (the subscript is related to the grading in L_k). Let then $c = c_m \in F^{m,q}$ be a cocycle, also of some fixed degree d. Then $c(e_{i_1}, \dots, e_{i_q}) = 0$ for $i_1 + \dots + i_q < m$ and let $c(e_{i_1}, \dots, e_{i_q}) = c_{i_1 \dots i_q} e_{m-d}$ for $i_1 + \dots + i_q = m$. Then $\bar{c}$, $\bar{c}(e_{i_1}, \dots, e_{i_q}) = c_{i_1 \dots i_q}$ is a cocycle in $C^q_{(m)}(L_k)$, and hence $\bar{c} = \delta \bar{b}$, $\bar{b} \in C^{q-1}_{(m)}(L_k)$. Define $b_m \in F^{m,q-1}$ by the formula

$$b_m(e_{j_1}, \dots, e_{j_{q-1}}) = \begin{cases} \bar{b}(e_{j_1}, \dots, e_{j_{q-1}})\, e_{m-d} & \text{for } j_1 + \dots + j_{q-1} = m, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently, $c_m - \delta b_m \in F^{m+1}$; we set $c_{m+1} = c_m - \delta b_m$ and acting precisely as before find b_{m+1} with $c_{m+1} - \delta b_{m+1} \in F^{m+2}$; set $c_{m+2} = c_{m+1} - \delta b_{m+1}$, and so on. The series $\sum_{p=m}^{\infty} b_m$ obviously converges (b_m have disjoint supports), and $c = \delta b$ where b is the sum of the series.

Hence the following statement is valid.

THEOREM 1. *If $s > k$,*

$$\dim H^q(L_k; L_s) \leqq k \dim H^q(L_k; \operatorname{Ind}_{L_{k+1}}^{L_k}(1)) = k \dim H^q(L_{k+1}) < \infty.$$

REMARK. The last inequality follows from [8].

The general case (s is arbitrary natural number, may be less than or equal to k) can be traced back to the previous one. If $s \leqq k$, then consider the exact sequence

$$0 \to L_{k+1} + L_s \to L_s / L_{k+1} \to 0.$$

The cohomology spaces $H^q(L_k; L_s/L_{k+1})$, $q = 1, 2, \dots$ are finite dimensional. In particular, LEMMA.

$$\dim H^q(L_k; L_s/L_{k+1}) \leqq \dim H^q(L_k) \dim H^q(L_k; L_s/L_k) = \\ = (k+1-s) \dim H^q(L_k).$$

PROOF. Denote the module L_{k-i+1}/L_{k+1} by M_i. We have the exact sequence

$$0 \to M_{i-1} \to M_i \to \mathbf{C} \to 0$$

from which it follows that $M_i/M_{i-1} = C$. Then

$$L_s/L_{k+1} = M_{k+1-s} \supset M_{k+s} \supset \ldots \supset M_1 \supset M_0 = 0,$$

and we have the next cohomology sequence:

$$\ldots \to H^q(L_k; M_{i-1}) \to H^q(L_k; M_i) \to H^q(L_k; \mathbf{C}) \to \ldots.$$

From this it follows that

$$\dim H^q(L_k; M_i) \leqq \dim H^q(L_k; M_{i-1}) \oplus \dim H^q(L_k)$$

which gives $\dim H^q(L_k; M_i) \leqq i \dim H^q(L_k)$. The lemma is proved.

THEOREM 2. *If $s \leqq k$,*

$$\dim H^q(L_k; L_s) \leqq k \dim H^q(L_{k+1}) + (k+1-s) \dim H^q(L_k).$$

PROOF. Consider the next cohomology sequence:

$$\ldots \to H^q(L_k; L_{k+1}) \to H^q(L_k; L_s) \to H^q(L_k; L_s/L_{k+1}) \to \ldots.$$

From this it follows that

$$\dim H^q(L_k; L_s) \leqq \dim H^q(L_k; L_{k+1}) + \dim H^q(L_k; L_s/L_{k+1}).$$

Using now the lemma, we get the result.

EXAMPLE. For small values of q,

$$\dim H^q(L_k) = \begin{cases} 1 \text{ for} & q=0 \\ k+1 \text{ for} & q=1 \\ \frac{k(k+3)}{2} \text{ for} & q=2. \end{cases}$$

Consequently, we have

$$\dim H^1(L_k; L_k) \leqq k(k+2) + (k+1) = k^2+3k+1$$

and

$$\dim H^2(L_k; L_k) \leqq \frac{k(k+1)(k+4)}{2} + \frac{k(k+3)}{2} = \frac{k}{2}(k^2+6k+7).$$

The first inequality shows that our upper bounds are too crude, because for $H^1(L_k; L_k)$ we have

PROPOSITION.

$$\dim H^1(L_k; L_k) = k.$$

PROOF. Consider the L_k-module L_0, and the following exact sequence:

$$0 \to L_k \to L_0 \to L_0/L_k \to 0.$$

Obviously, the dimension of the trivial module L_0/L_k is k. Then we have the next cohomology sequence:

$$H^0(L_k; L_0) \to H^0(L_k; L_0/L_k) \xrightarrow{\tau} H^1(L_k; L_k) \xrightarrow{\mu} H^1(L_k; L_0) \to .$$

Here $H^0(L_k, L_0) = 0$ as there are no invariants. Further, $\dim H^0(L_k; L_0/L_k) = k$ and τ is monomorphism. Easy to see that μ is zero, consequently $\dim H^1(L_k; L_k) = k$. Remark that the exterior derivations of L_k are the bracket operations with $e_0, e_1, \dots, e_{k+1}$.

For $k = 2$ the second inequality gives

$$\dim H^2(L_2; L_2) \leqq 23.$$

§2. Lower bounds

We can compute the cohomology $H^*(L_k; L_s)$, with the help of the spectral sequence associated to the filtration

$$L_s \supset L_{s+1} \supset L_{s+2} \supset \dots$$

in the coefficient module. As L_{t+1}/L_t is a one-dimensional trivial L_k-module, the first term of this spectral sequence consists of the group $H^q(L_k)$. The spectral sequence itself is a modification of the one introduced by Feigin and Fuks in [2] for computing the cohomology space of the Lie algebra L_1 with coefficients in the modules $F_{\lambda,\mu}$ (see Introduction). Let us give a more convenient construction of this spectral sequence. We have the grading in the cohomology spaces:

$$H^*(L_k; L_s) = \bigoplus_r H^*_{(r)}(L_k; L_s)$$

where $H^*_{(r)}(L_k; L_s)$ is the cohomology of the cochain complex of the form

$$(*) \qquad e_{i_1} \wedge \dots \wedge e_{i_q} \to \alpha_{i_1,\dots,i_q} e_{i_1+\dots+i_q+r}, \quad r \in \mathbf{Z}.$$

Denote the cochain groups of this complex by $C^q_{(r)}(L_k; L_s)$. These cochain groups are similar to the cochain groups $C^q(L_k)$, with the following difference: in $(*)$ one should have $i_1 + \ldots + i_q + r \geqq s$. So we have the following situation. The cochain groups $C^*(L_k)$ are also graded:

$$C^q(L_k) = \bigoplus_r C^q_{(r)}(L_k),$$

and we have the isomorphism

$$(**) \qquad C^q_{(r)}(L_k; L_s) \cong \bigoplus_{t \geqq s-r} C^q_{(t)}(L_k).$$

This isomorphism is given by the mapping

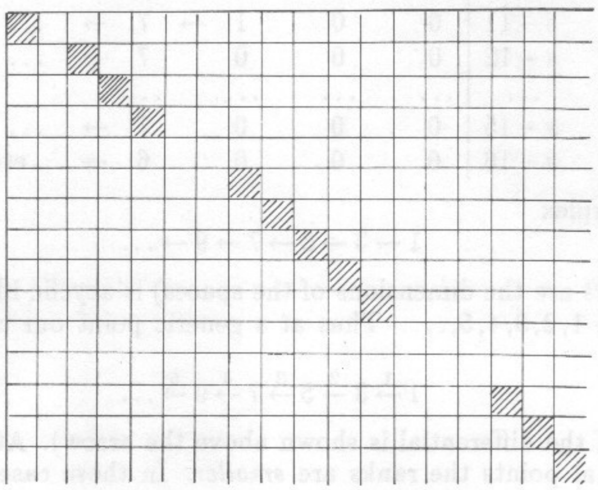
$$\left[e_{i_1} \wedge \ldots \wedge e_{i_q} \to \alpha_{i_1 \ldots i_q} e_{i_1 + \ldots + i_q + r}\right] \in C^q_{(r)}(L_k; L_s) \leftrightarrow$$
$$\leftrightarrow \left[e_{i_1} \wedge \ldots \wedge e_{i_q} \to \alpha_{i_1 \ldots i_q}\right] \in C^q_{(t)}(L_k), \qquad t = i_1 + \ldots + i_q.$$

We assign to the summand $C^q_{(t)}(L_k)$ in the expansion $(**)$ the filtering index t. So we have the spectral sequence with:

$$E_0^{p,q} = \begin{cases} C^{p+q}_{(-p)}(L_k) & \text{for } p \geqq s - r, \\ 0 & \text{for } p < s - r; \end{cases}$$

$$E_1^{p,q} = \begin{cases} H^{p+q}_{(-p)}(L_k) & \text{for } p \geqq s - r, \\ 0 & \text{for } p < s - r. \end{cases}$$

The spectral sequence converges in usual sense to $H^*_{(q)}(L_k; L_s)$. For $r \geqq s$ the E^1 term of our spectral sequence does not depend on r; for example, if $k = 2$ it has the form



(Here ■ means a one-dimensional space; the differentials act by the usual way to the right and then down.)

Let $E^n_\alpha = \bigoplus\limits_{p+q=n} E^{p,q}_\alpha$ and $\varrho^n = \varrho^n(r)$ be equal to the sum of ranks of differentials $d^n_\alpha \colon E^n_\alpha \to E^{n+1}_\alpha$ (for $\alpha \geqq 1$). Clearly

$$\dim H^n_{(r)}(L_k; L_s) = \dim E^n_1 - \varrho^{n-1} - \varrho^n .$$

For almost all $r \geqq s$, the numbers $\varrho^n(r)$ are the same (as the differential depends on r polynomially). These generic values of $\varrho^n(r)$ may be easily calculated since we know that $\dim H^q(L_k; L_s)$ is finite for any q (see Section 1) and thus $H^q_{(r)}(L_k; L_s) = 0$ for almost all r.

Consider the case $k = 2$. In this case the dimensions of E^α_1 are the following:

r	E^0_1		E^1_1		E^2_1		E^3_1		$\dots$
$s+2$	1	→	3	→	5	→	7	→	
$s+1$	1	→	3	→	5	→	7	→	$\dots$
s	1	→	3	→	5	→	7	→	$\dots$
$s-1$	0		3	→	5	→	7	→	$\dots$
$s-2$	0		2	→	5	→	7	→	$\dots$
$s-3$	0		1	→	5	→	7	→	$\dots$
$s-4$	0		0		5	→	7	→	$\dots$
$s-5$	0		0		5	→	7	→	$\dots$
$s-6$	0		0		5	→	7	→	$\dots$
$s-7$	0		0		5	→	7	→	$\dots$
$s-8$	0		0		4	→	7	→	$\dots$
$s-9$	0		0		3	→	7	→	$\dots$
$s-10$	0		0		2	→	7	→	$\dots$
$s-11$	0		0		1	→	7	→	$\dots$
$s-12$	0		0		0		7	→	$\dots$
$\dots$	$\dots$		$\dots$		$\dots$		$\dots$		
$s-15$	0		0		0		7	→	$\dots$
$s-16$	0		0		0		6	→	$\dots$ etc.

The complex

$$1 \to 3 \to 5 \to 7 \to 9 \to \dots$$

(the numbers are the dimensions of the spaces) is acyclic iff the differentials are of ranks $1, 2, 3, 4, 5, \dots$. Thus at a generic point our complex is of the form

$$1 \xrightarrow{1} 3 \xrightarrow{2} 5 \xrightarrow{3} 7 \xrightarrow{4} 9 \xrightarrow{5} \dots$$

(the rank of the differential is shown above the arrow). At a finite number of exceptional points the ranks are *smaller*. In those cases the complex is truncated. If a matrix $A = \blacksquare$ has rank $\leqq r$ then the rank of the truncated matrix $\underbrace{\blacksquare}_{k}$ cannot be larger than $\max(r, k)$.

The cohomology space will be the smallest, if the ranks are maximal as we described (the smaller the rank, the larger the cohomology space). The maximal possible ranks are the following (the rank does not drop anywhere):

		cohomology H^0	H^1	H^2	H^3 ...
$r \geqq s$	$1 \xrightarrow{1} 3 \xrightarrow{2} 5 \xrightarrow{3} 7 \xrightarrow{4} \cdots$	0	0	0	0
$s-1$	$3 \xrightarrow{2} 5 \xrightarrow{3} 7 \xrightarrow{4} \cdots$	0	1	0	0
$s-2$		0	1	0	0
$s-3$	$2 \xrightarrow{2} 5 \xrightarrow{3} 7 \xrightarrow{4} \cdots$	0	0	0	0
$s-4$	$1 \xrightarrow{1} 5 \xrightarrow{3} 7 \xrightarrow{4} \cdots$	0	0	1	0
$s-5$	$0 \quad 5 \xrightarrow{3} 7 \xrightarrow{4} \cdots$	0	0	2	0
$s-6$		0	0	2	0
$s-7$		0	0	2	0
$s-8$	$4 \xrightarrow{3} 7 \xrightarrow{4} \cdots$	0	0	1	0
$s-9$	$3 \xrightarrow{3} 7 \xrightarrow{4} \cdots$	0	0	0	0
$s-10$	$2 \xrightarrow{2} 7 \xrightarrow{4} \cdots$	0	0	0	1
$s-11$	$1 \xrightarrow{1} 7 \xrightarrow{4} \cdots$	0	0	0	2
$s-12$	$7 \xrightarrow{4} \cdots$	0	0	0	3
$s-13$	$7 \xrightarrow{4} \cdots$				3
$s-14$	$7 \xrightarrow{4} \cdots$				3
$s-15$	$7 \xrightarrow{4} \cdots$				3
$s-16$	$6 \xrightarrow{4} \cdots$				2
	$5 \rightarrow \cdots$				1
	$4 \rightarrow \cdots$				0
	$\cdots$				$\cdots$
		0	2	8	18

Here the ranks are maximal, but everywhere we must have

$$\text{rank } \leqq 3 \text{ and}$$
$$\text{rank } \leqq \text{ number of rows.}$$

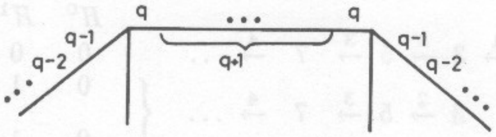
From these computations one obtains:

THEOREM 3.
$$\dim H^q(L_k; L_s) \geqq 2q^2$$
for any q, not depending on s. In particular, we obtain

$$\dim H^2(L_2; L_2) \geqq 8,$$
$$\dim H^3(L_2; L_2) \geqq 18.$$

PROOF. From the above computations we get

$$\dim H^q(L_2, L_s) = \frac{q(q+1)}{2} + q(q+1) + \frac{q(q-1)}{2} = 2q^2.$$



Comparing the above result with the estimation from Section 1:

$$\dim H^q(L_2; L_s) \leqq 2H^q(L_3) = 2(q+1)^2 \text{ for } s \geqq 2.$$

This means that the dimension is between $2q^2$ and $2(q+1)^2$. The lower bound is perhaps closer to the real value of the dimension.

Let us write out separately the results for the 2-dimensional cohomology space of the Lie algebra L_2 with coefficients in the adjoint representation:

$$8 \leqq \dim H^2(L_2; L_2) \leqq 23.$$

The computation in Section 2 for L_k with $k > 2$ is analogous, but, because of the unwieldy formulas for $\dim H^q_{(r)}(L_k)$, the answer would be very complicated. But it seems likely that the following is true.

CONJECTURE. *For any k and s,*

$$\dim H^q(L_k; L_s) = k \dim H^{q-1}(L_{k+1}).$$

If this is true, then the estimate of Theorem 3 is actually exact. But the similar estimates for L_k with $k > 2$ are only close to reality. For example, the procedure of this section gives for $q = 1, 2, 3, 4$

$$\dim H^q(L_3; L_s) \geqq 3, 15, 41, 87,$$

while the conjecture asserts that

$$\dim H^q(L_3; L_s) = 3, 15, 42, 90.$$

§3. Another method for computing $H^2(L_2; L_2)$

With the help of results from [3] we have another method of computing the cohomology $H^2(L_2; L_2)$. We have

$$H^2(L_2; L_2) = H^2(L_1; M)$$

where M is the "coinduced" module $\mathrm{Coind}_{L_2}^{L_1} L_2$. This module is generated by the elements

$$\begin{array}{c} e_2,\ e_3,\ e_4,\dots \\ xe_2,\ xe_3,\ xe_4,\dots \\ x^2e_2,\ x^2e_3,\ x^2e_4,\dots \\ \vdots \end{array}$$

The action of the Lie algebra is the following. The action of the field e_1 is multiplication by x, while in the upper row $e_2, e_3, \dots$ act in the natural way. The actions of the other elements are defined using the previous ones. For instance, for $n \geqq 2$, $k \geqq 2$ we have

$$(k-m)e_{m+k} = e_m(e_k) = \frac{1}{m-2}[xe_{m-1}(e_k) - e_{m-1}(xe_k)]$$

from which it follows that

$$e_m(xe_k) = (k-m)xe_{m+k} - (m-1)(k-m-1)e_{m+k+1}.$$

Analogously,

$$\begin{aligned} e_m(x^ne_k) &= xe_m(x^{n-1}e_k) - (m-1)e_{m+1}(x^{n-1}e_k) = \\ &= (k-m)x^ne_{m+k} + n(m-1)(k-m-1)x^{n-1}e_{m+k+1} + \\ &\quad + \text{lower degree terms at } x \end{aligned}$$

which gives a recurrent method of determining $e_m(x^ne_k)$.

Having the structure of the module M, we apply the complex

$$M \leftarrow M \overset{1}{\bigoplus} M \leftarrow M \overset{2}{\bigoplus} M \leftarrow M \overset{3}{\bigoplus} M \leftarrow \dots$$

(see [3]). To find the space $H^2(L_2; L_2)$, it is enough to know the singular vectors in the Verma module $V_{0,0}$ of degree 5, 7, 12, 15. We should remark that although the above computation can be worked out, and should give a precise answer, it leads to complicated formulas.

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