

## ON THE DEFORMATIONS OF $L_1$

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Let  $W_1$  be the Lie algebra of the vector fields on the line with polynomial coefficients, and  $L_1$  its subalgebra, consisting of the fields, which turn into zero with their first derivatives at the origin. In this work we study the deformations of  $L_1$ . Calculation of various invariants of infinite dimensional Lie algebras is rather complicated, even in the case of affine Lie algebras (see [4]).  $L_1$  seems in some sense the next one by difficulty. The main results of the present paper, without proofs, have been published in [3].

The study of deformations of nilpotent subalgebras in the Lie algebra of vector fields was stimulated specifically by [2], where it was proved that the cohomology with trivial coefficients of  $L_1$  and similar nilpotent Lie (super) algebras of geometrical origin are invariant under certain deformations. Moreover,  $L_1$  seems to be the simplest one among the Lie algebras which have higher order obstructions to continuation of deformations, as follows from the properties of its cohomology with coefficients in the coadjoint representation, which are studied in this work.

All these show that the study of deformations of  $L_1$  has to be one of the first steps on the way to study deformations of infinite dimensional Lie algebras.

1. At first we give some facts about the cohomology of  $L_1$  with coefficients in the coadjoint representation.

In  $W_1$  choose the basis  $e_{-1}, e_0, e_1, e_2, \dots$ , where  $e_i = x^{i+1} \partial / \partial x$ . The bracket in this basis is of the form  $[e_i, e_j] = (j-i)e_{i+j}$ . Then  $L_1$  has the basis  $e_1, e_2, \dots$ . We also need the subalgebras  $L_i \subset W_1$ ; the basis in  $L_i$  ( $i \geq 0$ ) consists of the fields  $e_i, e_{i+1}, \dots$ .  $W_1$  and  $L_i$  are naturally graded, the grading (weight) of  $e_i$  equals  $i$ . This grading is inherited by the cohomology spaces with coefficients in the graded modules. Specifically,  $H^k(L_1; L_1) = \bigoplus_m H_{(m)}^k(L_1; L_1)$ .

PROPOSITION 1.  $H^k(L_1; L_1)$ ,  $k > 0$  has dimension  $2k-1$  and is generated by elements of weight  $-\frac{3k^2-k}{2} + i$ ,  $i = 1, 2, \dots, 2k-1$ . (In other words,  $H_{(m)}^k(L_1; L_1) \cong H_{(m)}^{k-1}(L_2; 1)$ ).

Specifically,  $H^1(L_1; L_1)$  is one-dimensional with weight 0;  $H^2(L_1; L_1)$  is three-dimensional and is generated by elements of weights  $-2, -3, -4$ ;  $H^3(L_1; L_1)$  is five-dimensional and is generated by elements of weights  $-7, -8, -9, -10, -11$ .

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PROOF. This proposition follows from the results of [1]. Namely, the homology groups of  $L_1$  with coefficients in the modules  $\mathcal{F}_{\lambda,\mu}, F_{\lambda,\mu}, F'_{\lambda,\mu}$  are calculated in [1]. Here  $\mathcal{F}_{\lambda,\mu}, \lambda, \mu \in \mathbb{C}$  is the module with the basis  $f_i, i \in \mathbb{Z}, e_i f_j = (j + \mu - \lambda(i+1))f_{i+j}; F_{\lambda,\mu}$  is the submodule of  $\mathcal{F}_{\lambda,\mu}$  with the basis  $f_0, f_1, f_2, \dots; F'_{\lambda,\mu}$  is the module, conjugate to  $F_{\lambda,\mu}$ . The adjoint representation in this notation is  $F_{1,1}$ . Remark that  $H^i(L_1, F_{1,1})$  is dual to  $H_i(L_1, F'_{1,1})'$ . By using Theorem 4.3 in [1] we get  $H_i(L_1, F'_{1,1}) = H_{i-1}(L_1, F_{-2,-1}) \oplus H_i(L_1, \mathcal{F}_{-2,-1})$ . From Theorem 4.1 [1] it follows that  $H_i(L_1, \mathcal{F}_{-2,-1}) = 0$  for each  $i$ . Using Theorem 4.2 [1] and the subsequent we receive that  $\dim H_i(L_1, F_{-2,-1}) = \dim H_i(L_2)$ . Q.e.d.

The spaces  $H^*(L_k; 1)$  are calculated in [5]. By comparing the results of those calculations with the above one we obtain the required Proposition. Cocycle  $\varphi$  representing  $H^1(L_1; L_1)$  has the form  $\varphi(e_i) = ie_i$ . The elements of  $H^1(L_1; L_1)$ , as we know, are exterior derivations. Each of these derivations define a Lie algebra, containing  $L_1$  as its ideal of codimension 1. In the present case we obtain  $L_0$ .

Let us denote the three homogeneous elements of weights  $-2, -3$  and  $-4$  in  $H^2(L_1; L_1)$  by  $\alpha, \beta$  and  $\gamma$ , respectively. We give explicitly the cocycles  $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ , which represent the cohomology classes  $\alpha, \beta, \gamma$ :

$$\begin{aligned} \bar{\alpha}(e_2, e_3) &= 4e_3, \\ \bar{\alpha}(e_2, e_j) &= je_j \quad (j \geq 4), \quad \bar{\alpha}(e_3, e_j) = -(j-1)e_{j+1} \quad (j \geq 4), \\ \bar{\alpha}(e_i, e_j) &= 0 \quad \text{for other } i, j; \\ \bar{\beta}(e_2, e_3) &= 8e_2, \quad \bar{\beta}(e_2, e_4) = 4e_3, \quad \bar{\beta}(e_3, e_4) = -10e_4, \\ \bar{\beta}(e_2, e_j) &= (j+1)e_{j-1}, \quad \bar{\beta}(e_3, e_j) = -2je_j, \quad \bar{\beta}(e_4, e_j) = (j-1)e_{j+1} \quad \text{for } j \geq 5, \\ (1) \quad \bar{\beta}(e_i, e_j) &= 0 \quad \text{for other } i, j; \\ \bar{\gamma}(e_2, e_3) &= 14e_1, \quad \bar{\gamma}(e_2, e_5) = 8e_3, \quad \bar{\gamma}(e_3, e_4) = -24e_3, \\ \bar{\gamma}(e_3, e_5) &= -16e_4, \quad \bar{\gamma}(e_4, e_5) = 18e_5, \\ \bar{\gamma}(e_2, e_j) &= (j+2)e_{j-2}, \quad \bar{\gamma}(e_3, e_j) = -3(j+1)e_{j-1} \quad \left. \vphantom{\bar{\gamma}(e_2, e_j)} \right\} \quad \text{for } j \geq 6, \\ \bar{\gamma}(e_4, e_j) &= 3je_j, \quad \bar{\gamma}(e_5, e_j) = -(j-1)e_{j+1} \\ \bar{\gamma}(e_i, e_j) &= 0 \quad \text{for other } i, j. \end{aligned}$$

Indeed, it is easy to prove that every two-dimensional cohomology class of  $L_1$  with coefficients in  $L_1$  is represented by a single cocycle  $\omega$ , which turns into 0 at the field  $e_1$  (i.e.  $\omega(e_1, e_j) = 0$  for each  $j$ ). Then for  $\omega(e_i, e_j)$  with  $1 < i < j$  we can write a system of equations, which has a unique solution (up to a constant multiplier). By giving it explicitly we give another proof of the part of Proposition 1 related to the two-dimensional cohomology.

REMARK. A simpler formula for the cohomology class  $\alpha$  may be obtained in the following way. Let  $\varepsilon \in H^1(L_1; 1)$  be an element of weight  $-2$  and  $v \in H^1(L_1; L_1)$  the cohomology class of the cocycle  $e_i \mapsto ie_i$ ; then  $\alpha = \varepsilon v$ . Classes  $\beta$  and  $\gamma$  have no simple description of this kind.

2. Obstructions of deformations of  $L_1$ .

To find the versal family of  $L_1$  we use the methods of [6]. In this work the Lie operations of Massey are defined and the next process of finding the tangent cone to the base of the versal deformation is proposed. Let  $K_1 \subset H^2(L_1; L_1)$  (where  $H^2(L_1; L_1)$  is the tangent space to the base of the versal deformation) be the cone, consisting of all  $\xi$  such that  $\langle \xi, \xi \rangle = 0$ . Let further  $K_2$  be the cone, consisting of those elements  $\xi \in K_1$  that  $\langle \xi, \xi, \xi \rangle \geq 0$ , and in general,  $K_i = \{ \xi \in K_{i-1} | \langle \xi, \dots, \xi \rangle_{i+1} \geq 0 \}$ . The tangent cone to the base of the versal family is the intersection of all  $K_i$ .

In general, let  $\mathcal{L}$  be a Lie algebra,  $[ , ]_t$  the deformation of the bracket. Let us expand this deformation into a Taylor series:

$$[x, y]_t = [x, y]_0 + t\omega_1(x, y) + t^2\omega_2(x, y) + \dots, \quad x, y \in \mathcal{L}.$$

The Jacobi identity is:

$$\begin{aligned} [[x, y]_t, z]_t &= [[x, y]_0 + t\omega_1(x, y) + t^2\omega_2(x, y) + \dots, z]_t = [[x, y]_0, z]_0 + \\ &+ t\omega_1([x, y]_0, z) + t^2\omega_2([x, y]_0, z) + t[\omega_1(x, y), z]_0 + t^2\omega_1(\omega_1(x, y), z) + \\ &+ t^2[\omega_2(x, y), z]_0 + \dots = [[x, z]_t, y]_t + [x, [y, z]_t]_t. \end{aligned}$$

From this equation it follows that  $\omega_1$  is a 2-cocycle of  $\mathcal{L}$  with coefficients in the coadjoint representation and also that the 3-cochain

$$\omega_1(\omega_1(x, y), z) - \omega_1(\omega_1(x, z), y) + \omega_1(\omega_1(y, z), x)$$

is coboundary of  $\omega_2$ . Let  $\alpha \in H^2(\mathcal{L}; \mathcal{L})$  be the cohomology class of  $\omega_1$ . We actually prove that the Lie square  $\langle \alpha, \alpha \rangle$  equals 0. For the continuation of the infinitesimal deformation  $[x, y] + t\omega_1[x, y]$  to  $\text{speck}[t]/t^3$  it is necessary that  $\langle \alpha, \alpha \rangle = 0$ .

If  $\mathcal{L} \cong L_1$  then, as easy to see,  $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 0$ , because the weights of  $\langle \alpha, \alpha \rangle$  and  $\langle \beta, \beta \rangle$  are  $-4$  and  $-6$ , and there is no such three-dimensional cohomology class. Similarly, by consideration of the dimensions we have  $\langle \alpha, \beta \rangle = \langle \alpha, \gamma \rangle = 0$ . The calculation of the remaining relations is less evident.

**PROPOSITION 2.** *The Lie products  $\langle \gamma, \gamma \rangle$ ,  $\langle \beta, \gamma \rangle$  and the Massey Lie cube  $\langle \beta, \beta, \beta \rangle$  are nontrivial, while  $\langle \alpha, \alpha, \dots, \alpha \rangle \geq 0$  for all  $i$ .*

**PROOF.** To prove  $\langle \gamma, \gamma \rangle \neq 0$  substitute  $\bar{\gamma}$  into (1). The three-dimensional cocycle we obtain is not cohomologous to 0, as its value on the cohomology class of weight  $-8$  is different from zero. Direct computation shows that  $\langle \bar{\gamma}, \bar{\gamma} \rangle$  has nonzero value on the class of weight  $-8$ .

The fact that  $\langle \beta, \gamma \rangle \neq 0$  we may prove in the same way; however, we give another proof for this.

Now  $\langle \alpha, \alpha, \dots, \alpha \rangle \geq 0$  follows from the fact that there exists a deformation with infinitesimal deformation  $\alpha$ .

The most laborious part of the proof is to show that  $\langle \beta, \beta, \beta \rangle \neq 0$ . This is equivalent to the fact that for any Lie algebra over  $k[t]/t^4$  with the basis  $e_i$  the bracket cannot be of the following form:

$$[e_i, e_j]_t = (j-i)e_{i+j} + t\bar{\beta}(e_i, e_j)e_{i+j-3} + t^2\kappa_1(e_i, e_j)e_{i+j-6} + t^3\kappa_2(e_i, e_j)e_{i+j-9}.$$

Here  $\kappa_1(e_i, e_j)$  and  $\kappa_2(e_i, e_j)$  may be defined step by step ( $i+j=1, 2, \dots$ ) from the system of equations following from the Jacobi identity. For  $i+j=12$  we get a contradiction. Jacobi identities of weight 11 define an algebra  $L$  of dimension 11.  $H^2(L; 1)$  contains no elements of weight 12. This can be explained in the following way.

The Lie algebra  $\hat{L}$  with the basis  $e_1, \dots, e_{11}$  and the bracket  $[e_i, e_j] = (j-i)e_{i+j}$  has, similarly to  $L_1$ , a three-dimensional cohomology class of weight 12 and a two-dimensional cohomology class of weight 12. But  $L$  does not have any of these. It has a filtration, and the corresponding coadjoint graded algebra is  $\hat{L}$ . Hence there exists a spectral sequence with the first term  $H^*(\hat{L}; 1)$  converging to  $H^*(L; 1)$ . In this spectral sequence the differential of the two-dimensional class of weight 12 is a three-dimensional class of weight 12.

It remains to verify that  $\langle \beta, \gamma \rangle \neq 0$ . Here equality would imply that  $L_1$  has a deformation over  $\text{spec } k[t_1, t_2]/(t_1^2, t_2^2)$  such that in the deformed algebra

$$[e_i, e_j]_t = (j-i)e_{i+j} + t_1 \bar{\beta}(e_i, e_j)e_{i+j-3} + t_2 \bar{\gamma}(e_i, e_j)e_{i+j-4} + t_1 t_2 \kappa(e_i, e_j)e_{i+j-7}.$$

Straight calculation shows that such an algebra cannot exist. Namely, the numbers  $\kappa_{i,j} = \kappa(e_i, e_j)$  can be defined step by step. At the 12-th step we get a system of equations for  $\kappa_{ij}$ , which has no solution.

Let us define now three deformations of the Lie algebra structure in  $L_1$ ;  $[\ , \ ]_t^1$ ,  $[\ , \ ]_t^2$ ,  $[\ , \ ]_t^3$ .

$$(2) \quad \begin{aligned} [e_i, e_j]_t^1 &= (j-i)(e_{i+j} + te_{i+j-1}); \\ [e_i, e_j]_t^2 &= \begin{cases} (j-i)e_{i+j}, & \text{if } i, j > 1, \\ (j-1)e_{j+1} + te_j, & \text{if } i = 1, j > 1; \end{cases} \\ [e_i, e_j]_t^3 &= \begin{cases} (j-i)e_{i+j}, & \text{if } i, j \neq 2, \\ (j-2)e_{j+2} + te_j, & \text{if } i = 2, j \neq 2. \end{cases} \end{aligned}$$

These three Lie algebra families may be realized inside the Lie algebra  $L_0$ . By the first deformation  $e_i$  deforms into  $e_i + te_{i-1}$ ,  $i > 0$ . In other words,  $L_1$  deforms into the Lie algebra of vector fields on the line, vanishing at the origin and  $t$ . By the second deformation the  $e_i$ 's,  $i > 1$  are unchanged and  $e_1$  deforms into  $e_1 + te_0$ . Analogously, by the third deformation  $e_2$  turns into  $e_2 + te_0$  and the remaining elements are unchanged. We show now that no other deformations exist.

From Proposition 2 it follows that the tangent cone to the base of versal deformation of  $L_1$  is a curve of degree 3, generated by  $\alpha$ . The versal deformation consists of three curves, tangential to  $\alpha$ . These three curves correspond to the three Lie algebra families.

It follows then

**THEOREM.** *Each deformation of the Lie algebra structure in  $L_1$  may be obtained by substitution of the variable from one of the three families  $[\ , \ ]_t^1$ ,  $[\ , \ ]_t^2$ ,  $[\ , \ ]_t^3$  (see formulas (2)).*

PROOF. It is sufficient to verify that the three families are nontrivial and pairwise nonisomorphic. Indeed, the Lie algebras with the brackets  $[\cdot, \cdot]_t, [\cdot, \cdot]_{t'}, [\cdot, \cdot]_{t''}$ ,  $i \neq j, t', t'' \neq 0$  are not isomorphic neither with each other nor with  $L_1$ . (Notice that the algebras with  $[\cdot, \cdot]_t, [\cdot, \cdot]_{t'}$  are isomorphic for  $t', t'' \neq 0$ .) Let  $L_1(i)$  be an algebra from the  $i$ -th family. First,  $L_1(i) \not\cong L_1$  as  $L_1$  is a nilpotent algebra, while  $L_1(i)$ , for all  $i$ , is only solvable. Further, the maximal nilpotent subalgebra in  $L_1(1)$  has codimension two, and in  $L_1(2)$  and  $L_1(3)$  codimension one. Finally, the algebra  $[L_1(3), L_1(3)]$  has two generators, while  $[L_1(2), L_1(2)]$  has three. So  $L_1(1), L_1(2)$  and  $L_1(3)$  are neither isomorphic to each other, nor to  $L_1$ .

REMARKS. 1. From the Theorem it follows that all the nontrivial deformations of  $L_1$  lose the grading. In the class of graded Lie algebras  $L_1$  has no nontrivial deformations.

2. Let us study the change of grading by deformations of  $L_1$  in detail. Let  $L$  be a Lie algebra, lying in one of the three families. Choose in  $L$  the following basis:  $\{\bar{e}_1, \bar{e}_2, \dots\}$ ,  $\bar{e}_1 = e_1, \bar{e}_2 = e_2 + \alpha e_1; \bar{e}_3, \bar{e}_4, \dots$  are defined so that  $[\bar{e}_1, \bar{e}_i] = (i-1)\bar{e}_{i+1}$  holds. In the new basis we have:

$$[\bar{e}_i, \bar{e}_j] = (j-i)\bar{e}_{i+j} + \omega_1(i, j)\bar{e}_{i+j-1} + \omega_2(i, j)\bar{e}_{i+j-2} + \dots$$

Here  $\omega_1$  is a 2-cocycle of  $L_1$  with coefficients in  $L_1$  of weight  $-1$  such that  $\omega_1(e_1, e_j) = 0$ . Such a cocycle is represented as a differential  $\omega = dv$ . It is easy to see that  $v(e_i) = 0, i \neq 2$  and  $v(e_2)$  is proportional to  $e_1$ . Hence the parameter  $\alpha$  may be chosen so that  $\omega_1 = 0$ . Let us call a basis in  $L$  canonical if  $\omega_1 = 0$ .

Notice that  $\omega_2, \omega_3$  are cocycles of  $L_1$  and  $d\omega_4 = [\omega_1, \omega_1]$ . Cocycles  $\omega_2, \omega_3$  are proportional to those in (1). Choose in the three families three algebras —  $L^1, L^2, L^3$  — such that  $\omega_2 = \omega_2(L^i)$  exactly coincides with the first cocycle in (1). For the first and third family we have  $\omega_3 = 0$ , but for the second one  $\omega_3 \neq 0$ . Further

$$d(\omega_4(L^2) - \omega_4(L^1)) = 0 \quad \text{and} \quad d(\omega_4(L^3) - \omega_4(L^1)) = 0.$$

Cocycles  $\bar{u}_1 = \omega_4(L^2) - \omega_4(L^1)$  and  $\bar{u}_2 = \omega_4(L^3) - \omega_4(L^1)$  represent nontrivial cohomology classes of weight  $-4$  from  $H^2(L_1; L_1)$ . These cocycles are proportional to the third one in (1). Consequently, the quotient of these cocycles with coefficients  $\mu_1$  and  $\mu_2$ , respectively,  $\mu = \mu_1/\mu_2$  is a number, which may be easily computed. This number occurs as one of the numerical characteristics of the versal deformation of  $L_1$ . It would be interesting to connect this number with other characteristics of the deformation.

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