THE MODULI SPACE AND VERSAL DEFORMATIONS OF THREE DIMENSIONAL LIE ALGEBRAS*

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We consider versal deformations of ordinary (non-graded) three dimensional Lie algebras as special strongly homotopy Lie algebras. They correspond precisely to the 0 even and 3 odd dimensional case. The classification of such Lie algebras is well known. As the symmetric algebra of a three dimensional odd vector space contains terms only of exterior degree less than or equal to three, the construction of versal deformations of these special L_∞ algebras can be carried out completely. We give a characterization of the moduli space of Lie algebras using L_∞ algebra deformation theory as a guide to understanding the picture.

I dedicate this paper to Izrail Moiseevich Gelfand
on the occasion of his 90-th birthday

1. Introduction

The classification of low dimensional Lie algebras has been known for a long time. For example, the classification of ordinary Lie algebras of dimension 3 appears in textbooks such as [10]. More recently, the moduli space of three dimensional Lie algebras was studied in [1, 12]. The problem of finding a versal deformation of a given object is a basic question in deformation theory because such a deformation induces all other deformations. This problem turns out to be very difficult. Versal deformation theory was first worked out for the case of Lie algebras in [2, 3, 5] and then extended to strongly homotopy Lie algebras—also called L_∞ algebras—in [6]. We apply these general results to construct versal deformations of three dimensional ordinary Lie algebras, treating them as examples of L_∞ algebras.

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L_∞ algebras are natural generalizations of Lie algebras and superalgebras if one considers $\mathbb{Z}_2$-graded vector spaces. They were discovered in [11] and have recently been the focus of attention. An ordinary 3-dimensional Lie algebra is the same as an L_∞ algebra structure on a $0|3$ (0 even and 3 odd) dimensional $\mathbb{Z}_2$-graded vector space. The advantage of considering Lie algebras as L_∞ algebras is that the deformation problem becomes simpler and we get a clearer insight into the moduli space of the variety of Lie algebras in a given dimension.

For simplicity we will suppose that the underlying vector space is defined over $\mathbb{C}$.

A detailed version of this lecture will appear in [8].

2. Definitions

2.1. *Strongly Homotopy Lie Algebras*

Let W be a $\mathbb{Z}_2$-graded vector space over a field $\mathfrak{K}$ and $T(W)$ the reduced tensor algebra $T(W) = \bigoplus_{n=1}^{\infty} W^{\otimes n}$. For an element $v = v_1 \otimes \ldots \otimes v_n$ in $T(W)$, define its parity $|v| = |v_1| + \cdots + |v_n|$, and its degree by $deg(v) = n$. With parity $T(W)$ is a $\mathbb{Z}_2$-graded space. The reduced symmetric algebra $S(W)$ is the quotient of the tensor algebra by the graded ideal generated by $u \otimes v - (-1)^{uv} v \otimes u$ for elements $u, v \in W$.

The symmetric algebra $S(W)$ has a natural coalgebra structure, which occurs as a subalgebra of the tensor coalgebra, given by the diagonal mapping

$$\Delta(w_1 \ldots w_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \epsilon(\sigma) w_{\sigma(1)} \ldots w_{\sigma(k)} \otimes w_{\sigma(k+1)} \ldots w_{\sigma(n)},$$

where we denote the induced product in $S(W)$ by juxtaposition. Here $\mathrm{Sh}(k, n-k)$ is the set of *unshuffles* of type $(k, n-k)$; that is the subset of permutations σ of n elements such that $\sigma(i) < \sigma(i+1)$ when $i \neq k$, and $\epsilon(\sigma)$ is a sign determined by σ (and $w_1 \ldots w_n$) given by

$$w_{\sigma(1)} \ldots w_{\sigma(n)} = \epsilon(\sigma) w_1 \ldots w_n.$$

Thus if σ interchanges p and $p+1$, then $\epsilon(\sigma) = (-1)^{v_p v_{p+1}}$.

Obviously the kernel of Δ is W. This mapping is clearly coassociative. The grading on $S(W)$ is compatible with the coalgebra structure, as for homogeneous $c \in S(W)$, $\Delta(c) = \sum_i u_i \otimes v_i$ where $|u_i| + |v_i| = |c|$ for all i, that is Δ has degree 0. With this coalgebra structure, and the $\mathbb{Z}_2$ grading, $S(W)$ is a cocommutatice, coassociative coalgebra (without a counit).

A *coderivation* on the graded coalgebra $S(W)$ is a map $\delta : S(W) \to S(W)$ satisfying

$$\Delta \circ \delta = (\delta \otimes I + I \otimes \delta) \circ \Delta.$$

Let us suppose that the even part of W has basis $e_1 \ldots e_m$, and the odd part has basis $f_1 \ldots f_n$, so that W is an $m|n$ dimensional space. Then a basis of $S(W)$ is given by all vectors of the form $e_1^{k_1} \ldots e_m^{k_m} f_1^{l_1} \ldots f_n^{l_n}$, where k_i is any nonnegative integer, and $l_i \in \mathbb{Z}_2$. An L_∞ structure on W is simply an *odd codifferential* on $S(W)$, that is to say, an odd coderivation whose square is zero. The $\mathfrak{K}$-module $\mathrm{Coder}(S(W))$ of graded coderivations has a natural structure of a graded Lie algebra with the bracket $[\varphi, \psi] = \varphi \circ \psi - (-1)^{\varphi\psi} \psi \circ \varphi$. On the other hand, this space can be identified with $\mathrm{Hom}(S(W), W)$, and the Lie superalgebra structure on $\mathrm{Coder}(S(W))$ determines a Lie bracket on $\mathrm{Hom}(S(W), W)$ as follows. Let

$$L_m = \mathrm{Hom}(S^m(W), W)$$

so that $L = \mathrm{Hom}(S(W), W)$ is the direct product of the spaces L_i. If $\alpha \in L_m$ and $\beta \in L_n$, then $[\alpha, \beta]$ is the element in L_{m+n-1} determined by

$$\begin{aligned}[\alpha, \beta](w_1 \ldots w_{m+n-1}) = \\ &\sum_{\sigma \in \mathrm{Sh}(n,m-1)} \epsilon(\sigma) \alpha(\beta(w_{\sigma(1)} \ldots w_{\sigma(n)}) w_{\sigma(n+1)} \ldots w_{\sigma(m+n-1)}) \\ &- (-1)^{\alpha\beta} \sum_{\sigma \in \mathrm{Sh}(m,n-1)} \epsilon(\sigma) \beta(\alpha(w_{\sigma(1)} \ldots w_{\sigma(m)}) w_{\sigma(m+1)} \ldots w_{\sigma(m+n-1)}).\end{aligned} \tag{1}$$

If W is completely odd, and $d \in L_2$, then d determines an ordinary Lie algebra on W, or rather on its parity reversion which is the same space with the parity of elements reversed. This is the case we consider in this talk.

Suppose that $\tilde{g} : S(W) \to S(W')$ is a coalgebra morphism, that is a map satisfying

$$\Delta' \circ \tilde{g} = (\tilde{g} \otimes \tilde{g}) \circ \Delta.$$

If d and d' are L_∞ algebra structures on W and W', resp., then $\tilde{g}$ is a homomorphism between these structures if $\tilde{g} \circ d = d' \circ \tilde{g}$. Two L_∞ structures d and d' on W are equivalent, and we write $d' \sim d$ when there is a coalgebra automorphism $\tilde{g}$ of $S(W)$ such that $d' = \tilde{g}^*(d) = \tilde{g}^{-1} \circ d \circ \tilde{g}$. Furthermore, if $d = d'$, then $\tilde{g}$ is said to be an automorphism of the L_∞ algebra.

2.2. *Versal Deformations*

For the classical theory of formal deformations we refer to [9].

Here we need a more general concept of deformation. A deformation with a base for a Lie algebra was introduced in [2] and worked out in [3]. An augmented local ring $\mathcal{A}$ with maximal ideal $\mathfrak{m}$ will be called an *infinitesimal base* if $\mathfrak{m}^2 = 0$, and a *formal base* if $\mathcal{A} = \varprojlim_n \mathcal{A}/\mathfrak{m}^n$. A deformation of an L_∞ algebra structure d on W with base given by a local ring $\mathcal{A}$ with augmentation $\epsilon : \mathcal{A} \to \mathfrak{K}$ is an $\mathcal{A}$-L_∞ structure $\tilde{d}$ on $W \hat{\otimes} \mathcal{A}$ such that the morphism of $\mathcal{A}$-L_∞ algebras $\epsilon_* = 1 \otimes \epsilon : L_\mathcal{A} = L \otimes \mathcal{A} \to L \otimes \mathfrak{K} = L$ satisfies $\epsilon_*(\tilde{d}) = d$. (Here $W \hat{\otimes} \mathcal{A}$ is an appropriate completion of $W \otimes \mathcal{A}$.) The deformation is called infinitesimal (formal) if $\mathcal{A}$ is an infinitesimal (formal) base.

In general, the cohomology $H(D)$ of d given by the operator $D : L \to L$ with $D(\varphi) = [\varphi, d]$ may not be finite dimensional. However, L has a natural filtration, which induces a filtration H^n on the cohomology. That means if W is finite dimensional, $H(D)$ is always of finite type, that is H^n/H^{n+1} is finite. A set δ_i will be called a graded *basis of the cohomology*, if any element δ of the cohomology can be expressed uniquely as a formal sum $\delta = \delta_i a^i$.

For each δ_i, let u^i be a parameter of opposite parity. Then the infinitesimal deformation $d^1 = d + \delta_i u^i$, with base $\mathcal{A} = \mathfrak{K}[u_i]/(u_i u_j)$ is universal in the sense that if $\tilde{d}$ is any infinitesimal deformation with base $\mathcal{B}$, then there is a unique homomorphism $f : \mathcal{A} \to \mathcal{B}$, such that the morphism $f_* = 1 \otimes f : L_\mathcal{A} \to L_\mathcal{B}$ satisfies $f_*(\tilde{d}) \sim d$.

For formal deformations, there is no universal object in the sense above. A *versal deformation* is a deformation d^∞ with formal base $\mathcal{A}$ such that if $\tilde{d}$ is any formal deformation with base $\mathcal{B}$, then there is some morphism $f : \mathcal{A} \to \mathcal{B}$ such that $f_*(d^\infty) \sim \tilde{d}$. If f is unique whenever $\mathcal{B}$ is infinitesimal, then the versal deformation is called *miniversal*. In [6], we constructed a miniversal deformation for L_∞ algebras with finite type cohomology.

The method of construction is as follows. Define a coboundary operator D by $D(\varphi) = [\varphi, d]$. First, one constructs the universal infinitesimal deformation $d^1 = d + \delta_i u^i$ as before. The infinitesimal assumption that the products of parameters are equal to zero gives the property that $[d^1, d^1] = 0$. Actually, we can express

$$[d^1, d^1] = (-1)^{\delta_j(\delta_i+1)}[\delta_i, \delta_j]u^i u^j = \delta_k a^k_{ij} u^i u^j + \beta_k b^k_{ij} u^i u^j,$$

where the β_i form a basis of the coboundaries, because the bracket of d^1 with itself is a cocycle. Note that the right hand side is of degree 2 in the parameters, so it is zero up to order 1 in the parameters.

If we suppose that $D(\gamma_i) = -\frac{1}{2}\beta_i$, then by replacing d^1 with

$$d^2 = d^1 + \gamma_k b^k_{ij} u^i u^j,$$

one obtains

$$[d^2, d^2] = \delta_k a^k_{ij} u^i u^j + 2[\delta_l u^l, \gamma_k b^k_{ij} u^i u^j] + [\gamma_k b^k_{ij} u^i u^j, \gamma_l b^l_{ij} u^i u^j]$$

Thus we are able to get rid of terms of degree 2 in the coboundary terms β_i, but those which involve the cohomology terms δ_i can not be eliminated. This gives rise to a set of second order relations on the parameters. One continues this process, taking the bracket of the *n-th order deformation* d^n, adding some higher order terms to cancel coboundaries, obtaining higher order relations, which extend the n-th order relations.

Either the process continues indefinitely, in which case the miniversal deformation is expressed as a formal power series in the parameters, or after a finite number of steps, the right hand side of the bracket is zero after applying the n-th order relations. In this case, the miniversal deformation is simply the n-th order deformation. We obtain a set of relations R_i on the parameters, one for each δ_i, and the algebra $A = \mathbb{C}[[u^i]]/(R_i)$ is called the base of the miniversal deformation. For details see [5, 6].

3. Equivalence Classes of 3-Dimensional Lie Algebras

Suppose that $W = \langle f_1, f_2, f_3 \rangle$. Then $S(W)$ decomposes into three pieces.

$$\begin{aligned}
S^1(W) &= \langle f_1, f_2, f_3 \rangle, & \dim(S^1(W)) &= 0|3 \\
S^2(W) &= \langle f_1 f_2, f_1 f_3, f_2 f_3 \rangle, & \dim(S^2(W)) &= 3|0 \\
S^3(W) &= \langle f_1 f_2 f_3 \rangle, & \dim(S^3(W)) &= 0|1.
\end{aligned}$$

Let $L = \mathrm{Hom}(S(W), W)$ and $L_n = \mathrm{Hom}(S^n(W), W)$. Then

$$\begin{aligned}
L_1(W) &= \{\varphi^I_j \mid I \in \{100, 010, 001\}, j = 1 \ldots 3\}, & \dim(L_1) &= 9|0 \\
L_2(W) &= \{\varphi^I_j \mid I \in \{110, 101, 011\}, j = 1 \ldots 3\}, & \dim(L_2) &= 0|9 \\
L_3(W) &= \{\varphi^{111}_j \mid j = 1 \ldots 3\}, & \dim(L_3) &= 3|0
\end{aligned}$$

It follows that the only candidate for an odd codifferential is of the form

$$\begin{aligned}
d = \varphi^{110}_1 a_1 + \varphi^{110}_2 a_2 + \varphi^{110}_3 a_3 + \\
\varphi^{101}_1 a_4 + \varphi^{101}_2 a_5 + \varphi^{101}_3 a_6 + \\
\varphi^{011}_1 a_7 + \varphi^{011}_2 a_8 + \varphi^{011}_3 a_9 \quad (2)
\end{aligned}$$

Being a quadratic codifferential, we see that d gives an L_∞ structure precisely when it determines a Lie algebra structure. It is natural to consider the derived subalgebra $W' = d(S^2(W))$. Let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}.$$

It is easy to see that the rank of A is precisely equal to the dimension of the derived subalgebra. In particular, when $\det(A) = 0$, the derived subalgebra has dimension less than three.

The codifferential condition $[d, d] = 0$ is equivalent to a system of three quadratic equations.

If $\det A \neq 0$ then the only possibility is $a_6 = -a_2$, $a_9 = a_1$, and $a_8 = -a_4$. Thus

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & -a_2 \\ a_7 & -a_4 & a_1 \end{pmatrix}, \tag{3}$$

whose determinant does not vanish in general. Consequently we have only this one pattern to consider for when the derived subalgebra has dimension three.

When the derived subalgebra has dimension one, we can choose a basis such that the codifferential d has the simple form

$$d = \varphi_1^{110} a_1 + \varphi_1^{101} a_4 + \varphi_1^{011} a_7. \tag{4}$$

Moreover, it is easy to check that any coderivation of this form is a codifferential.

Finally, suppose that the derived subalgebra has dimension two. Then we can express d in the form

$$d = \varphi_1^{101} a_4 + \varphi_2^{101} a_5 + \varphi_1^{011} a_7 + \varphi_2^{011} a_8. \tag{5}$$

Each of these three cases can be reduced to a much simpler form, up to equivalence. Let us begin with the one dimensional derived subalgebra case first. Let us suppose that g is a linear automorphism of the symmetric coalgebra of W.

Two codifferentials d' and d are equivalent precisely when there is some automorphism g such that $d' = gdg^{-1}$, in other words, $d'g = gd$. We get that all one dimensional solutions are equivalent to one of two codifferentials $d = \varphi_1^{110}$ which is the Lie algebra $\mathfrak{n}_2 \otimes \mathbb{C}$ or $d = \varphi_1^{011}$ which is the 3-dimensional nilpotent Lie algebra $\mathfrak{n}_3$.

Next, let us consider the two dimensional solutions.

We get that the nonequivalent solutions have the form $d(\lambda) = \varphi_1^{101} + \lambda\varphi_2^{011}$ parameterized by the punctured unit disc in $\mathbb{C}$, with the boundary glued together. Moreover, if we consider the case $\lambda = 0$, which is a one dimensional solution, it is then easy to see that it is equivalent to the solution $d = \varphi_1^{110}$. We therefore add this one-dimensional solution as a limit point to the family of two dimensional solutions. (For details see [8].)

Now, if the matrix associated to d is not diagonalizable, then they all arise from the single codifferential $d = \varphi_1^{101} + \varphi_2^{101} + \varphi_2^{011}$ which is the solvable 3-dimensional Lie algebra.

Finally, in the three dimensional case we obtain one equivalence class with codifferential of the form $d' = \varphi_3^{110} + \varphi_2^{101} + \varphi_1^{011}$ which gives the simple Lie algebra $\mathrm{sl}(2, \mathbb{C})$.

Thus, up to equivalence, we obtain precisely the codifferentials

$$\begin{aligned} d_1 &= \varphi_1^{011} \\ d(\lambda) &= \varphi_1^{101} + \varphi_2^{011}\lambda \\ d_2 &= \varphi_1^{101} + \varphi_2^{101} + \varphi_2^{011} \\ d_3 &= \varphi_3^{110} + \varphi_2^{101} + \varphi_1^{011}, \end{aligned}$$

where $d(\lambda)$ is identified with $d(\lambda^{-1})$.

4. Versal Deformations and the Moduli Space

First, we compute the cohomology of the codifferential, use it to write the universal infinitesimal deformation, and then apply the bracket process above to determine a miniversal deformation and the relations on the parameters. Along the way, we will discover that the cohomology of the differentials reveals a lot of information about the moduli space of three dimensional Lie algebras.

4.1. *The Codifferential* $d_3 = \varphi_3^{110} + \varphi_2^{101} + \varphi_1^{011}$

Here the space H^1 is $3|0$-dimensional, and all higher cohomology vanishes.

Nevertheless, since we have three cocycles, all even, we do have a nontrivial infinitesimal deformation. Let us adopt the convention to use the Greek letter θ for an odd parameter, and the Roman letter t for an even one. For an odd parameter θ, we have $\theta^2 = 0$ as a consequence of the graded commutativity, so we do not consider this to be a relation on our parameter algebra. Thus, we have

$$d_3^1 = \varphi_3^{110} + \varphi_2^{101} + \varphi_1^{011} + (\varphi_3^{010} + \varphi_2^{001})\theta_1 + (\varphi_3^{100} - \varphi_1^{001})\theta_2 + (\varphi_2^{100} + \varphi_1^{010})\theta_3.$$

In computing $[d_3^1, d_3^1]$, we note that the brackets of the cocycles with d_3 vanish, so we only need to compute the brackets of the cocycles with each other. We obtain

$$[d_3^1, d_3^1] = (\varphi_2^{100} + \varphi_1^{010})\theta_1\theta_2 + (\varphi_3^{100} - \varphi_1^{001})\theta_1\theta_3 + (\varphi_3^{010} + \varphi_2^{001})\theta_2\theta_3.$$

Of course, these are all cocycles which are not coboundaries, so we obtain the relations $\theta_i\theta_j = 0$ for all i, j. Thus the infinitesimal deformation is miniversal, and the base of the miniversal deformation is given by $\mathbb{C}[\theta_1, \theta_2, \theta_3]/(\theta_1\theta_2, \theta_1\theta_3, \theta_2\theta_3)$.

Note that the vanishing of H^2 is consistent with the observation that any small change in the codifferential d_3 will give rise to a codifferential d' which will still have a 3-dimensional derived subalgebra. Thus any small change in d_3 gives rise to the same codifferential, and we see that d_3 does not deform into any of the other codifferentials.

4.2. *The Codifferential* $d_2 = \varphi_1^{101} + \varphi_2^{101} + \varphi_2^{011}$

Let us first determine its cohomology. We get three obvious 1-cocycles, φ_2^{100}, φ_1^{001} and φ_2^{001}, and in addition also $\varphi_1^{100} + \varphi_2^{010}$. The space H^2 is 1-dimensional with a basis φ_2^{011} and the higher cohomology is zero. The universal infinitesimal deformation is given by

$$d_2^1 = \varphi_1^{101} + \varphi_2^{101} + (1+t)\varphi_2^{011} + \varphi_2^{100}\theta_1 + \varphi_1^{001}\theta_2 + \varphi_2^{001}\theta_4 + (\varphi_1^{100} + \varphi_2^{010})\theta_4.$$

We have

$$[d_2^1, d_2^1] = 2\varphi_2^{101}t\theta_1 + \varphi_2^{001}(\theta_1\theta_2 - \theta_3\theta_4) - \varphi_1^{001}\theta_2\theta_4.$$

Of the three cocycles appearing on the right hand side of this equation, only the first is a coboundary. Thus we obtain two second order relations, $\theta_1\theta_2 - \theta_3\theta_4 = 0$ and $\theta_2\theta_4 = 0$, and we need to add something to d_2^1 to obtain a second order deformation. Since $D(\varphi_1^{100}) = -\varphi_2^{101}$, we can express $d_2^2 = d_2^1 + \varphi_1^{100}t\theta_1$. We compute

$$[d_2^2, d_2^2] = \varphi_2^{001}(\theta_1\theta_2 - \theta_3\theta_4) - \varphi_1^{001}(\theta_2\theta_4 + t\theta_1\theta_2).$$

Since no coboundary terms occur, we obtain that d_2^2 is a miniversal deformation, and the base is given by

$$A = \mathbb{C}[[t, \theta_1, \theta_2, \theta_3, \theta_4]]/(\theta_1\theta_2 - \theta_3\theta_4, \theta_2\theta_4 + t\theta_1\theta_2).$$

Let us study the induced topology on the moduli space of equivalence classes of codifferentials. This topology is not Hausdorff. If every neighborhood of a point a contains the point b, then we note that a is in the closure of b. In this case, we shall say that a is *infinitesimally close* to b.

Since there is a nontrivial deformation in the Lie algebra direction itself, we can explore how the deformation moves our codifferential in the moduli space. Note that d_2^1 is a codifferential which has two dimensional derived algebra, and it has eigenvalues 1 and $1+t$, so it lies in the family $d(\lambda)$, and is near to $d(1)$. In fact, a punctured neighborhood of d_2 looks exactly like a neighborhood of $d(1)$. This phenomena shows up in the classical deformation theory as jump deformation. However, $d(1)$ is not close to d_2, in the sense that a small neighborhood of d_2 does not contain $d(1)$. When we study $d(1)$, we shall see that the opposite statement is not true.

4.3. *The Family of Codifferentials* $d(\lambda) = \varphi_1^{101} + \varphi_2^{011}\lambda$

The cohomology will depend to some extent on the value of λ.

We shall see that the only thing special about the case $\lambda = 0$ is that the dimension of the derived algebra drops to 1, but as far as deformations go, it will behave like a generic element of the family. The cases $\lambda = \pm 1$, however, are not generic in terms of their deformations. This makes sense, because in the identification of $td\lambda$ with $\tilde{d}\lambda^{-1}$, we see that every point in the unit disc has a neighborhood that is like a usual disc in $\mathbb{C}$, with the exception of ± 1, which are orbifold points. So it is not surprising to find some kind of exceptional behavior for these codifferentials.

4.3.1. *Generic case of* $d(\lambda)$

First, we treat the generic case. Clearly, $H^1 = \langle \varphi_1^{100}, \varphi_2^{010}, \varphi_1^{001}, \varphi_2^{001} \rangle$. The space H^2 is one-dimensional and we can choose φ_2^{011} as a basis. All the higher cohomology classes are zero. Thus, the universal infinitesimal deformation is given by

$$d(\lambda)^1 = \varphi_1^{101} + \varphi_2^{011}(\lambda + t) + \varphi_1^{100}\theta_1 + \varphi_2^{010}\theta_2 + \varphi_1^{001}\theta_3 + \varphi_2^{001}\theta_4.$$

It is easy to verify that

$$[d(\lambda)^1, d(\lambda)^1] = 2\varphi_1^{001}\theta_1\theta_3 + 2\varphi_2^{001}\theta_2\theta_4.$$

Thus $d(\lambda)^1$ is miniversal and the base of the miniversal deformation is

$$A = \mathbb{C}[[t, \theta_1, \theta_2, \theta_3, \theta_4]]/(\theta_1\theta_3, \theta_2\theta_4).$$

Looking at the deformation in the Lie algebra direction, we see that the deformation simply moves along the $d(\lambda)$ family.

4.3.2. *The special case* $d(-1)$

Now, let us consider the special case $\lambda = -1$. Then we have an extra cocycle φ_3^{110} in H^2, and correspondingly, an extra cocycle φ_3^{111} in H^3. Thus, the universal infinitesimal deformation becomes

$$\begin{aligned} d(-1)^1 =& \varphi_1^{101} + \varphi_2^{011}(-1+t_1) + \varphi_1^{100}\theta_1 + \varphi_2^{010}\theta_2 + \varphi_1^{001}\theta_3 + \varphi_2^{001}\theta_4 \\ &+ \varphi_3^{110}t_2 + \varphi_3^{111}\theta_5. \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2}[d(-1)^1, d(-1)^1] = &\varphi_1^{001}\theta_1\theta_3 + \varphi_2^{001}\theta_2\theta_4 + \varphi_3^{110}(t_2\theta_1 + t_2\theta_2) \\ &+ \varphi_3^{111}(\theta_5\theta_1 + \theta_5\theta_2 - t_1t_2) - \varphi_1^{111}\theta_5\theta_3 - \varphi_2^{111}\theta_5\theta_4 \\ &+ (-\varphi_1^{110} - \varphi_3^{011})t_2\theta_3 + (-\varphi_2^{110} + \varphi_3^{101})t_2\theta_4. \end{aligned}$$

Note that the first four terms are cocycles, so they give rise to second order relations, while the last four terms are coboundaries, so we need to add corresponding terms to obtain the second order deformation

$$d(-1)^2 = d(-1)^1 + \varphi_3^{011}\theta_3\theta_5 + \varphi_2^{110}\theta_4\theta_5 - \varphi_3^{010}t_2\theta_3 - \varphi_3^{100}t_2\theta_4.$$

Finally, let us compute the bracket of the second order deformation with itself. We obtain

$$\begin{aligned} \frac{1}{2}[d(-1)^2, d(-1)^2] = &\varphi_1^{001}\theta_1\theta_3 + \varphi_2^{001}\theta_2\theta_4 + \varphi_3^{110}(t_2\theta_1 + t_2\theta_2) + \\ &\varphi_3^{111}(\theta_5\theta_1 + \theta_5\theta_2 - t_1t_2) + \varphi_3^{011}(\theta_3\theta_5\theta_2 - t_1t_2\theta_3) + \\ &\varphi_2^{110}(\theta_4\theta_5\theta_1 - t_1t_2\theta_4) - \varphi_3^{010}t_2\theta_3\theta_2 + \varphi_2^{010}t_2\theta_3\theta_4 - \\ &\varphi_3^{100}t_2\theta_4\theta_1 + \varphi_1^{100}t_2\theta_4\theta_3. \end{aligned}$$

All the terms except φ_3^{011}, φ_2^{110}, φ_3^{010}, and φ_3^{100} are cocycles, and these exceptional terms are not coboundaries. Thus, by the general theory, their coefficients must be zero.

Note that $d(-1)$ is infinitesimally close to d_3, but not the other way around. In some sense, this explains the extra infinitesimal deformation in the Lie algebra direction.

Now at first it may seem strange that this is the first case where one of our codifferentials deforms into d_3. After all, generically, we would expect the matrix of a codifferential to be invertible. But looking carefully at the codifferential with matrix (3) and the form of a solution in the family, it becomes clear that only when $\lambda = -1$ can a small change in the codifferential give a solution satisfying (3).

4.3.3. *The special case* $d(1)$

In this case, there are two additional 1-cocycles, φ_2^{100} and φ_1^{010}. Thus $\dim H^1 = 6$, $\dim H^2 = 3$ and $\dim H^3 = 0$. So we pick up two extra 2-cocycles, φ_2^{101} and φ_1^{011}. It is convenient to replace the cocycle φ_2^{011}, which we used as a basis of the cohomology in the generic case, with $\varphi_1^{101} - \varphi_2^{011}$, because it simplifies the interpretation of the bracket of the universal infinitesimal deformation with itself. Thus,

$$\begin{aligned} d(1)^1 = &\varphi_1^{101} + \varphi_2^{011}\lambda + (\varphi_1^{101} - \varphi_2^{011})t_1 + \varphi_2^{101}t_2 + \varphi_1^{011}t_3 \\ &+ \varphi_1^{100}\theta_1 + \varphi_2^{010}\theta_2 + \varphi_1^{001}\theta_3 + \varphi_2^{001}\theta_4 + \varphi_2^{100}\theta_5 + \varphi_1^{010}\theta_6, \end{aligned}$$

and we compute that

$$\begin{aligned} \frac{1}{2}[d(1)^1, d(1)^1] = &(\varphi_1^{101} - \varphi_2^{011})(t_3\theta_5 - t_2\theta_6) - \varphi_2^{101}(2t_1\theta_5 + t_2(\theta_2 - \theta_1)) \\ &+ \varphi_1^{011}(2t_1\theta_6 + t_3(\theta_2 - \theta_1)) - \varphi_1^{100}\theta_5\theta_6 + \varphi_2^{010}\theta_5\theta_6 \\ &+ \varphi_1^{001}(\theta_1\theta_3 - \theta_4\theta_6) + \varphi_2^{001}(\theta_2\theta_4 - \theta_3\theta_5) \\ &+ \varphi_2^{100}(\theta_2\theta_5 - \theta_1\theta_5) + \varphi_1^{010}(\theta_1\theta_6 - \theta_2\theta_6), \end{aligned}$$

which is precisely the set of cocycles appearing in the universal infinitesimal deformation, multiplied by the second order relations. Thus the infinitesimal deformation is miniversal.

Now let us interpret how $d(1)$ fits into the moduli space. Note that there is a deformation along the family, given by the cocycle $\varphi_1^{101} - \varphi_2^{011}$, and two other directions of deformation, each of which corresponds to a deformation of $d(1)$ into the special codifferential d_2. In fact, if we consider the three dimensional deformation space parameterized by (t_1, t_2, t_3), we see that precisely two curves correspond to a deformation in the d_2 direction. Thus we see that $d(1)$ is infinitesimally close to d_2, although the converse is not true.

So far, we have been able to associate two of the three special codifferentials with the family in some manner.

4.4. *The Codifferential* $d_1 = \varphi_1^{011}$

This gives a nilpotent Lie algebra structure, so that we expect to find a lot of deformations.

In this case $\dim H^1 = 6$, $\dim H^2 = 5$ and $\dim H^3 = 2$. Let us analyze how this codifferential sits in the moduli space. Clearly there are a lot of directions one can deform. As $\varphi_1^{101} - \varphi_2^{011}$ and $\varphi_1^{110} + \varphi_3^{011}$ are coboundaries, there remain three ways to deform d_1 into d_3, via the cocycles φ_2^{110},

$\varphi_3^{101} + \varphi_2^{101}$ and $\varphi_3^{101} - \varphi_2^{101}$. Generically, if we add small multiples of these cocycles, we will obtain a codifferential which is equivalent to d_3. Thus d_1 is infinitesimally close to d_3.

Next, if we add an appropriate multiple of $\varphi_1^{101} + a\varphi_2^{101}$, then we are constructing an element in the family $d(\lambda)$ with a matrix B given by $B = \begin{pmatrix} t & at \\ 1 & 0 \end{pmatrix}$. (The fraction of its eigenvalues determines the element of the family, see [8].) This way we get an element of the family for any value of λ except $\lambda = \pm 1$. The reason that we do not obtain an element of the family for $\lambda = 1$ is that the resulting matrix is defective, so we obtain d_2 instead. There is a solution for $\lambda = -1$, however, given by the cocycle φ_2^{101}. Thus we see that d_1 is infinitesimally close to d_2 and to every member of the family except for $\lambda = 1$. Moreover, it is not hard to check that if we take d to be of the form

$$d = \varphi_1^{011} + \varphi_1^{110} t_1 + \varphi_1^{101} t_2 + (\varphi_2^{110} - \varphi_3^{101}) t_3 + \varphi_3^{110} t_4 + \varphi_2^{101} t_5,$$

then there is no automorphism which takes it to $d(1)$.

Thus we conclude that d_1 is infinitesimally close to every codifferential except $d(1)$. From the behavior of the elements d_2 and $d(1)$, it is more natural to consider d_2 as a member of the family, because it has the same cohomology as the other members of the family except $d(1)$. Note, for example, that $\dim H^2(d_2) = 1$, the same as the generic elements of the family, while $\dim H^2(d(1)) = 3$.

For constructing a miniversal deformation for this Lie algebra, we have to do three steps which we will not discuss now in this talk.

4.5. *Deformations of the Trivial Codifferential* $d_0 = 0$

This codifferential evidently must deform into every possible type, so we know that it is infinitesimally close to every point in the moduli space. Moreover, since there are no coboundaries in the bracket, the infinitesimal deformation is obviously miniversal.

On the other hand, we do obtain some relations, and these relations carry some information about how the moduli space is put together. In addition, all second order relations can be determined from the relations on the zero codifferential, by using appropriate values for the coefficients.

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