

## Deformations of a Lie algebra of vector fields on the line

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In this note we describe all deformations of the Lie algebra  $L_1(1)$ .

We recall (see [1]) that  $L_1(1)$ , or briefly  $L_1$ , is the subalgebra of the Lie algebra of vector fields on the line with polynomial coefficients that consists of the fields such that they and their first derivatives vanish at the origin. We set  $e_i = x^{i+1} \partial/\partial x$ ,  $i \in \mathbf{Z}$ ,  $[e_i, e_j] = (j-i)e_{i+j}$ . The fields  $e_1, e_2, e_3, \dots$  form a basis of  $L_1$ . We need two other subalgebras of the Lie algebra of polynomial vector fields on the line: the algebra  $L_0$  with the basis  $e_0, e_1, e_2, \dots$  and  $L_2$  with the basis  $e_2, e_3, \dots$ ; obviously,  $L_0 \supset L_1 \supset L_2$ . The algebras  $L_i$  have a natural grading: the grade (weight) of  $e_j$  is  $j$ . We obtain the grading from the cohomology space of these algebras with coefficients in graded modules.

**Proposition 1.** *The spaces  $H^i(L_1; L_1)$  and  $H^{i-1}(L_2; 1)$  are isomorphic as graded vector spaces.*

This is a consequence of the results of [1] (see Theorems 4.1–4.3).

The cohomology groups  $H^*(L_i; 1)$  are computed in [2]. Combining the results of this computation with Proposition 1, we obtain the next result.

**Proposition 2.**  *$H^2(L_1; L_1)$  is a three-dimensional space and has a basis consisting of homogeneous elements  $\alpha, \beta$ , and  $\gamma$  of degree  $-2, -3$ , and  $-4$ , respectively.*

Proposition 2 can also be proved by a direct computation. This computation also gives explicit formulae for the cocycles  $\bar{\alpha}, \bar{\beta}$ , and  $\bar{\gamma}$  representing  $\alpha, \beta$ , and  $\gamma$  (these cocycles are uniquely determined up to proportionality by the condition  $\bar{\alpha}(e_1, e_j) = \bar{\beta}(e_1, e_j) = \bar{\gamma}(e_1, e_j) = 0$  for all  $j$ ):

$$\begin{aligned} \bar{\alpha}(e_2, e_3) &= 4e_3, \\ \bar{\alpha}(e_i, e_j) &= \begin{cases} (-1)^i (j-i+2) e_{i+j-2} & \text{if } i=2, 3, j \geq 4, \\ 0 & \text{if } i \neq 2, 3, j > i; \end{cases} \\ \bar{\beta}(e_2, e_3) &= 8e_2, \quad \bar{\beta}(e_2, e_4) = 4e_3, \quad \bar{\beta}(e_3, e_4) = -10e_4, \\ \bar{\beta}(e_i, e_j) &= \begin{cases} (-1)^i C_2^{i-2} (j-i+3) e_{i+j-3} & \text{if } i=2, 3, 4, j \geq 5, \\ 0 & \text{if } i \neq 2, 3, 4, j > i; \end{cases} \\ \bar{\gamma}(e_2, e_3) &= 14e_1, \quad \bar{\gamma}(e_2, e_4) = 0, \quad \bar{\gamma}(e_2, e_5) = 8e_3, \\ \bar{\gamma}(e_3, e_4) &= -24e_3, \quad \bar{\gamma}(e_3, e_5) = -16e_4, \quad \bar{\gamma}(e_4, e_5) = 18e_5, \\ \bar{\gamma}(e_i, e_j) &= \begin{cases} (-1)^i C_3^{i-2} (j-i+4) e_{i+j-4} & \text{if } i=2, 3, 4, 5, j \geq 6, \\ 0 & \text{if } i \neq 2, 3, 4, 5, j > i. \end{cases} \end{aligned}$$

*Remark.* A simpler formula for the class  $\alpha$  can be obtained as follows. Let  $\epsilon \in H^1(L_1; 1)$  be an element of weight  $-2$  and  $\nu \in H^1(L_1; L_1)$  the cohomology class of the cocycle  $e_i \mapsto ie_i$ . Then  $\alpha = \epsilon\nu$ . The classes  $\beta$  and  $\gamma$  do not have such a simple description.

To find a versal deformation of  $L_1$  we use the methods of [3]. That paper describes Massey's Lie operations and proposes the following procedure for finding the tangent cone to the base of a versal deformation. In  $H^2(L_1; L_1)$  (we recall that  $H^2(L_1; L_1)$  is the tangent space to the base of a versal deformation), we select the cone  $K_1$  consisting of those  $\xi$  for which the square  $\langle \xi, \xi \rangle \in H^3(L_1; L_1)$  is zero. Next, let  $K_2$  be the cone consisting of those  $\xi \in K_1$  such that  $\langle \xi, \xi, \xi \rangle = 0$ , and in general,  $K_i = \{\xi \in K_{i-1} \mid \langle \xi, \dots, \xi \rangle_{i+1} = 0\}$ . The tangent cone to the base of a versal deformation is the intersection of all the  $K_i$ .

**Proposition 3.** *The products  $\langle \gamma, \gamma \rangle$ ,  $\langle \beta, \gamma \rangle$  and the Massey cube  $\langle \beta, \beta, \beta \rangle$  are non-trivial;  $\langle \alpha, \dots, \alpha \rangle_i = 0$  for all  $i$ .*

The proof consists in a direct check, based on the explicit formulae for  $\bar{\alpha}, \bar{\beta}$ , and  $\bar{\gamma}$  presented above, and information on  $H^3(L_1; L_1)$  contained in [1] and [2] (this space is five-dimensional and has a basis consisting of homogeneous elements of degrees  $-7, -8, -9, -10$ , and  $-11$ ).

It follows from Proposition 3 that the tangent cone to the base of a versal deformation of  $L_1$  is a triple line generated by  $\alpha$ . It turns out that the base itself of a versal deformation consists of three curves touching  $\alpha$ . Namely, we define three deformations,  $[ ]_t^1$ ,  $[ ]_t^2$ ,  $[ ]_t^3$  of the Lie algebra structure of  $L_1$ :

$$\begin{aligned} [e_i, e_j]_t^1 &= (j-i)(e_{i+j} + te_{i+j-1}); \\ [e_i, e_j]_t^2 &= \begin{cases} (j-i)e_{i+j} & \text{if } i, j > 1, \\ (j-i)e_{i+j} + tje_j & \text{if } i=1, j > 1; \end{cases} \\ [e_i, e_j]_t^3 &= \begin{cases} (j-i)e_{i+j} & \text{if } i, j \neq 2, \\ (j-2)e_{2+j} + tje_j & \text{if } i=2, j \neq 2. \end{cases} \end{aligned}$$

These three deformations can be represented as three families of subalgebras of  $L_0$ . Namely, the Lie algebras with the brackets  $[ ]_t^1$ ,  $[ ]_t^2$ , and  $[ ]_t^3$  are isomorphic to the subalgebras of  $L_0$  with the bases  $\{e_1 + te_0, e_2 + te_1, e_3 + te_2, \dots\}$ ,  $\{e_1 + te_0, e_2, e_3, \dots\}$ , and  $\{e_1, e_2 + te_0, e_3, e_4, \dots\}$ , respectively. We remark that the Lie algebra with the bracket  $[ ]_t^1$  consists of the vector fields on the line that vanish at 0 and  $t$ .

**Theorem.** Every deformation of the Lie algebra structure of  $L_1$  is obtained by a change of the variable in one of the families  $[ ]_t^1$ ,  $[ ]_t^2$ , and  $[ ]_t^3$ .

In view of Proposition 3, to prove the theorem it is enough to verify that the three families we have constructed are non-trivial and pairwise non-isomorphic, but in fact, as is easy to prove, even the algebras with the commutators  $[ ]_t^{i'}$  and  $[ ]_t^{j''}$ , where  $i \neq j$  and  $t', t'' \neq 0$ , are mutually non-isomorphic and are not isomorphic to  $L_1$ .

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#### References

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