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MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

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Deformations and Contractions in Mathematics and Physics

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ABSTRACT. Deformations of mathematical structures are not only important in most parts of mathematics but also to a large extent in physics. Contractions are in some respect dual to deformations. The workshop brought together world experts, but also young post-docs, in these complementary topics with the goal to foster further interactions between the different scientific communities.

Mathematics Subject Classification (2000): 16xx, 17xx, 22xx, 53xx, 70xx, 81xx, 83xx.

Introduction by the Organisers

Deformations of mathematical structures are not only important in most parts of mathematics but also to a large extent in physics. Contractions are in some respect dual to deformations. The aim of the proposed workshop was to bring together world experts in these complementary topics of deformations and contractions of various algebraic structures. Deformations and contractions have been investigated by researchers who had different approaches and goals. Tools such as cohomology, gradings, etc. which are utilized in the study of one concept, are likely to be useful for the other concept as well. At this meeting there were mathematicians, mathematical physicists and physicists as well. The organizers hope that the meeting was of benefit to all groups.

Because various fields in mathematics and physics exist in which deformations are used, it was necessary to focus the topic of the workshop. The meeting mainly considered deformations of algebras (in particular, of Lie algebras), groups, and related algebraic structures, the corresponding contractions, and their applications to problems in physics. Nevertheless, other fields with strong relations to the central topic were present too. One such field, discussed in detail at the workshop,

with tight interaction was deformation quantization. But also other topics like quantum groups, deformation of Hopf algebras, q -deformed physics, fuzzy spaces, quantum systems as deformations of classical systems, etc showed up.

As the workshop had an interdisciplinary character it was considered to be useful to start with some introductory talks on

- (1) Deformations in mathematics and physics,
- (2) Contractions of Lie algebras in physics,
- (3) Cohomology and deformations,
- (4) Deformation quantization,

with the aim to introduce the necessary concepts which were not always well-known to all the different communities present. For more details on the concepts, see the corresponding extended abstracts in this Oberwolfach report.

The following is a (non-exhaustive) list of topics which were discussed at the workshop.

- (1) The concept of rigidity and deformations in its different versions; relations to cohomology, moduli spaces of algebras and existence of versal families; for formally rigid infinite dimensional algebras there exist nevertheless global deformations which are locally non-trivial; the deformations of enveloping algebras.
- (2) Contractions and its relations to deformations considered from a mathematical point of view; the different concepts of contractions, generalized Inönü - Wigner contractions, graded contractions, degenerations, orbit closure, jump deformations, expansions; invariants of Lie algebra.
- (3) Contractions and their physical implications; macroscopic quantum systems, local current algebras, supergravity, regularisations, symmetry of the hydrogen atom.
- (4) Deformation quantization; its application to field theory, algebraic varieties, superformality, unimodular vector fields.
- (5) Deformations of vector field algebras; its relation to geometric moduli spaces, algebras of Krichever-Novikov type, supergeometry
- (6) Non-commutative spaces and related algebraic objects; differential geometry of noncommutative spaces, Hopf algebras, operads, quantum groups, curvature, elliptic gamma functions and triptic curves.

The workshop was attended by 44 participants from all over the world. The official program consisted of 24 lectures. Two evening sessions of informal presentations were organised. Beside the official program, there was ample time for the participants for further activities, such as self-organised sessions and discussion groups.

Workshop: Deformations and Contractions in Mathematics and Physics

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Abstracts

Deformations in mathematics and physics

ALICE FIALOWSKI

1. The notion of deformation. The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. The fundamental idea, which should be credited to Riemann, was to introduce an analytic structure therein. The notion of local and infinitesimal deformations of a complex analytic manifold first appeared in the work of Kodaira and Spencer (1958). In particular, they proved that infinitesimal deformations can be parametrized by the corresponding cohomology group. The deformation theory of compact complex manifolds was devised by Kuranishi (1965) and Palamodov (1976). Shortly after the work of Kodaira and Spencer, algebro-geometric foundations were systematically developed by Artin (1960) and Schlessinger (1968). Formal deformations of arbitrary rings and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber, in a series of articles (1964–1968). The notion of deformation was applied to Lie algebras by Nijenhuis and Richardson (1966–68).

Because various fields in mathematics and physics exist in which deformations are used, we focused on the topic of the conference. We mainly consider here deformations of algebras (in particular, of Lie algebras), groups, and related algebraic structures and their applications to problems in physics. But beside the central topic, to open up fertile interaction, we invited also researchers from neighbouring disciplines. One such topic with tight interaction is deformation quantization. But there will also be others, like quantum groups, deformation of Hopf algebras, Leibniz and dialgebras, infinity algebras, q -deformed physics, fuzzy spaces, quantum systems as deformations of classical systems etc.

Deformation is one of the tools used to study a specific object, by deforming it into some families of “similar” structure objects. This way we get a richer picture about the original object itself. But there is also another question approached via deformation. Roughly speaking, it is the question, can we equip the set of mathematical structures under consideration (may be up to certain equivalence) with the structure of a topological or geometric space. In other words, does there exist a moduli space for these structures. If so, then for a fixed object the deformations of this object should reflect the local structure of the moduli space at the point corresponding to this object.

2. Definitions. For simplicity, consider the Lie algebra case.

Let \mathcal{L} be a Lie algebra with Lie bracket μ_0 over a field \mathbb{K} .

a) *Intuitive* definition: a deformation of \mathcal{L} is a one-parameter family \mathcal{L}_t of Lie algebras with the bracket

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \dots$$

where φ_i are \mathcal{L} -valued 2-cochains, i.e. elements of $\text{Hom}_{\mathbb{K}}(\Lambda^2 \mathcal{L}, \mathcal{L}) = C^2(\mathcal{L}; \mathcal{L})$, and \mathcal{L}_t is a Lie algebra for each $t \in \mathbb{K}$. Two deformations, \mathcal{L}_t and \mathcal{L}'_t are equivalent if there exists a linear automorphism $\widehat{\psi}_t = \text{id} + \psi_1 t + \psi_2 t^2 + \dots$ of \mathcal{L} where ψ_i are linear maps over \mathbb{K} , i.e. elements of $C^1(\mathcal{L}, \mathcal{L})$ such that

$$\mu'_t(x, y) = \widehat{\psi}_t^{-1}(\mu_t(\widehat{\psi}_t(x), \widehat{\psi}_t(y))) \quad \text{for } x, y \in \mathcal{L}.$$

The Jacobi identity for the algebras \mathcal{L}_t implies that the 2-cochain φ_1 is indeed a cocycle, i.e. $d_2 \varphi_1 = 0$. If φ_1 vanishes identically, the first nonvanishing φ_i will be a cocycle. If μ'_t is an equivalent deformation with cochains φ'_i , then

$$\varphi'_1 - \varphi_1 = d_1 \psi_1,$$

hence every equivalence class of deformations defines uniquely an element of $H^2(\mathcal{L}, \mathcal{L})$.

b) Consider now a deformation \mathcal{L}_t not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. The natural generalization is to allow more parameters, or to take in general a commutative algebra over \mathbb{K} with identity as base of a deformation. Let us fix an augmentation $\varepsilon : A \rightarrow \mathbb{K}$, $\varepsilon(1) = 1$, and set $\text{Ker } \varepsilon = m$, which is a maximal ideal.

Definition. A deformation λ of \mathcal{L} with base (A, m) is a Lie A -algebra structure on the tensor product $A \otimes_{\mathbb{K}} \mathcal{L}$ with bracket $[\ ,]_{\lambda}$ s.t.

$$\varepsilon \otimes \text{id} : A \otimes \mathcal{L} \rightarrow \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

A deformation with base A is called *local* if the algebra A is local, and it is called *infinitesimal* if, in addition to this, $m^2 = 0$. For general commutative algebra base, we call the deformation *global*.

c) *Formal deformations.* Let A be a complete local algebra (completeness means that $A = \overleftarrow{\lim}_{n \rightarrow \infty} (A/m^n)$, where m is the maximal ideal in A). A formal deformation of \mathcal{L} with base A is a Lie A -algebra structure on the completed tensor product $A \widehat{\otimes} \mathcal{L} = \overleftarrow{\lim}_{n \rightarrow \infty} ((A/m^n) \otimes \mathcal{L})$ s.t.

$$\varepsilon \widehat{\otimes} \text{id} : A \widehat{\otimes} \mathcal{L} \rightarrow \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$$

is a Lie algebra homomorphism.

d) *Formal versal deformations.* It is known that in the category of algebraic varieties the quotient by a group action does not always exist (Hartshorne). Specifically, there is no universal deformation in general of a Lie algebra \mathcal{L} with a commutative algebra base A with the property that for any other deformation of \mathcal{L} with base B there exists a unique homomorphism $f : B \rightarrow A$ that induces an equivalent deformation. If such a homomorphism exists (but not unique), we call the deformation of \mathcal{L} with base A *versal*.

The classical one-parameter deformation theory did not study the versal property of deformations. A more general deformation theory of Lie algebras follows from Schlessinger's work (1968). Namely, for complete local algebra base deformations, under some minor restriction, there exists a so-called *miniversal deformation*:

A formal deformation η of a Lie algebra \mathcal{L} with a complete local algebra base B is called *miniversal*, if

i) for any formal deformation λ of \mathcal{L} with any complete local base A there exists a homomorphism $f : B \rightarrow A$ s.t. the deformation λ is equivalent to the push-out of η by f ;

ii) if A satisfies $m^2 = 0$, then f is unique.

The situation is much worse for global deformations, where we lose the cohomology tool for obtaining deformations, and there is no way so far to get a versal object.

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Contractions of Lie algebras in mathematics and physics

MARC DE MONTIGNY

Hereafter I briefly review the concept of contraction of Lie algebras and some of its applications in physics. The relation with deformations is that non-trivial contractions induce inverse “jump deformations” (defined in other contributions to this workshop). First, I recall the general definition of a contraction from Ref. [1], which contains more details about contractions and deformations of Lie algebras. Consider a Lie algebra \mathfrak{g} of dimension N over an arbitrary field k (physicists work mostly with the real and the complex fields). Let us denote the basis elements of \mathfrak{g} by $\{x_1, \dots, x_N\}$ and write the Lie brackets as

$$(1) \quad [x_a, x_b] = C_{ab}^c x_c,$$

with structure constants C_{ab}^c . A contraction is defined via a non-singular linear transformation of \mathfrak{g} denoted $U_{\varepsilon} \in \text{GL}(N, k)$, which depends on a parameter ε . The