Global deformations of the Virasoro algebra

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This talk is based on a joint work with Martin Schlichenmaier (see [4]).

Introduction. Deformation is one of the tools to study a specific object, by deforming it into some families of "similar" structure objects. Another question related to deformation: Can we equip the set of nonequivalent deformations with the structure of a topological or maybe geometric space? In other words, does there exist a moduli space for these structures? If so, then for a fixed object its deformations should reflect the local structure of the moduli space at the point corresponding to this object.

There is a lot of confusion in the literature in the notion of a deformation. Several different (inequivalent) approaches exist. May aim now is to clarify the difference between deformations of geometric origin and so-called formal deformations. Formal deformation theory has the advantage of using cohomology. It is also complete in the sense that under some natural cohomology assumptions there exists a versal formal deformation which induces all other deformations. Formal deformations are deformations with a complete local algebra base. A deformation with a commutative (non-local) algebra base gives a much richer picture of deformation families, depending on the augmentation of the base algebra. If we identify the base of deformation – which is a commutative algebra of functions – with a smooth manifold, an augmentation corresponds to choosing a point on the manifold. So choosing different points should in general lead to different deformation situations. I will show in the case of the Witt and Virasoro algebra that – in the case of infinite dimensional Lie algebras – there is no tight relation between global formal deformations.

1. Deformations. Let $\mathcal{L}$ be a Lie algebra.

i) Intuitively: One-parameter family $\mathcal{L}_t$ of Lie algebras with bracket $\mu_t = \mu_0 + t\phi_1 + t^2\phi_2 + \dots$.

ii) Global deformations: Consider a deformation $\mathcal{L}_t$ not as a family of Lie algebras, but as a Lie algebra over the algebra $\mathbb{K}[[t]]$. Call it the *base* of the deformation. The natural generalization is to allow more parameters, or to take in general a commutative algebra A over $\mathbb{K}$ with identity as base of a deformation. Take such an A over $\mathbb{K}$ of char 0 with an augmentation $\varepsilon : A \to \mathbb{K}$ and $m = \operatorname{Ker}\varepsilon$ maximal ideal.

Definition. A *global deformation* λ *of* $\mathcal{L}$ *with base* (A, m) is a Lie A-algebra structure on $A \otimes_{\mathbb{K}} \mathcal{L}$ with $[\ ,\ ]_\lambda$ such that $\varepsilon \otimes \mathrm{id} : A \otimes \mathcal{L} \to \mathbb{K} \otimes \mathcal{L} = \mathcal{L}$ is a Lie algebra homomorphism.

A deformation is called *trivial* if $A \otimes_{\mathbb{K}} \mathcal{L}$ carries the trivially extended Lie structure, i.e. $[1\otimes x, 1\otimes y]_\lambda = 1\otimes[x, y]$. Two deformations of a Lie algebra $\mathcal{L}$ with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \otimes \mathcal{L}$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \mathrm{id}$. We say that the deformation is *local* if A is a local $\mathbb{K}$-algebra with unique maximal ideal $m_A = \operatorname{Ker} \varepsilon$. In case that in addition, $m_A^2 > 0$, the deformation is called *infinitesimal*.

iii) We call a deformation *formal*, if its base is a complete local algebra (with a unique maximal ideal) (see [1]).

Proposition (see [3]). *If* $\dim \mathrm{H}^2(\mathcal{L}, \mathcal{L}) < \infty$*, there exists a universal infinitesimal deformation* $\eta_{\mathcal{L}}$ *of* $\mathcal{L}$ *with base* $B = \mathbb{K} \oplus \mathrm{H}^2(\mathcal{L}, \mathcal{L})'$.

This means that for any infinitesimal deformation λ of the Lie algebra $\mathcal{L}$ with finite-dimensional (local) algebra base A there exists a unique homomorphism $\phi : \mathbb{K} \oplus \mathrm{H}^2(\mathcal{L}, \mathcal{L})' \to A$ such that λ is equivalent to the push-out $\phi_* \eta_{\mathcal{L}}$.

Definition ([1]). A formal deformation η of $\mathcal{L}$ parametrized by a complete local algebra B is called *versal* if for any deformation λ, parametrized by (A, m_A), there exists $f : B \to A$ morphism such that the push-out

1) $f_* \eta$ is equivalent to λ.

2) If A satisfies ${m_A}^2 = 0$, then f is unique.

Theorem. *Assume* $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$ *is finite dimensional.*

a) ([1]) *There exists a versal formal deformation of* $\mathcal{L}$.

b) ([3]) *The base of the versal deformation is formally embedded into* $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$*, i.e. it can be described in* $\mathrm{H}^2(\mathcal{L}, \mathcal{L})$ *by a finite system of formal equation.*

Corollary. $\mathrm{H}^2(\mathcal{L}, \mathcal{L}) = \{0\}$ *implies that* $\mathcal{L}$ *is formally rigid.*

Theorem ([2]). *The Witt and Virasoro algebra is formally rigid.*

2. Krichever–Novikov algebras. They are generalizations of the Virasoro and all its related algebras. Let M be a compact Riemann surface of genus g, or a smooth projective curve over $\mathbb{C}$. Let $I = \{P\}$ and $O = \{Q\}$ be distinct points ("marked points") on the curve. Denote $A = I \cup O$ as a set. Denote by $\mathcal{L}$ the Lie algebra consisting of those meromorphic sections of the holomorphic tangent line bundle which are holomorphic outside of A, equipped with the Lie bracket of vector field. Call them Krichever–Novikov algebras. For the Riemann sphere ($g = 0$) with quasi-global coordinate z, $I = \{0\}$, $O = \{\infty\}$, the introduced algebra is the Witt algebra. The Witt and Virasoro algebras are graded, but these Krichever–Novikov algebras are only almost graded, as was observed by Krichever–Novikov in the two-point case [5] and generalized by Schlichenmaier [6].

We consider the genus one case, i.e., the case of one-dimensional complex tori, or, equivalently the elliptic curve case. Consider now two marked points. One marking we always put to $\infty = (0 : 1 : 0)$, and the other one to the affine coordinate $(e, 0)$. Set

$$B := \{(e_1, e_2, e_3) \in \mathbb{C}^3 \mid e_1 + e_2 + e_3 = 0,\ e_i \neq e_j \text{ for } i \neq j\}.$$

In $B \times \mathbb{P}^2$ we consider the family of elliptic curves $\mathcal{E}$ over B defined via $Y^2 Z = 4(X - e_1 Z)(X - e_2 Z)(X - e_3 Z)$. Consider the complex lines in $\mathbb{C}^2$:

$$D_s := \{(e_1, e_2) \in \mathbb{C}^2 \mid e_2 = s \cdot e_1\},\ s \in \mathbb{C}, \quad D_\infty := \{(0, e_2) \in \mathbb{C}^2\}.$$

Then B is isomorphic to $\mathbb{C}^2 \setminus (D_1 \cup D_{-\frac{1}{2}} \cup D_{-2})$.

Theorem ([7]). *For any elliptic curve $E_{(e_1,e_2)}$ over $(e_1,e_2) \in \mathbb{C}^2 \setminus (D_1 \cup D_{-1/2} \cup D_{-2})$ the Lie algebra $\mathcal{L}^{(e_1,e_2)}$ of vector fields on $E_{e_1,e_2)}$ has a basis $\{V_n, n \in \mathbb{Z}\}$ such that the Lie algebra structure is given as*

$$(*)\quad [V_n,V_m] = \begin{cases} (m-n)V_{n+m}, & n,m \text{ odd}, \\ (m-n)(V_{n+m} + 3e_1 V_{n+m-2} & \\ \quad +(e_1-e_2)(e_1-e_3)V_{n+m-4}), & n,m \text{ even}, \\ (m-n)V_{n+m} + (m-n-1)3e_1V_{n+m-2} & \\ \quad +(m-n-2)(e_1-e_2)(e_1-e_3)V_{n+m-4}, & n \text{ odd, } m \text{ even}. \end{cases}$$

These algebras make sense also for the points $(e_1,e_2) \in D_1 \cup D_{-\frac{1}{2}} \cup D_{-2}$. Altogether this defines a 2-dimensional family of Lie algebras parametrized over $\mathbb{C}^2$. In particular, for $(e_1,e_2) = 0$ we get the Witt algebra.

Now consider the family of algebras obtained by taking as base variety the line D_s (for an s). We get that for fixed s in all cases the algebras will be isomorphic above every point in D_s as long as we are not above $(0,0)$.

Proposition. *For $(e_1,e_2) \neq (0,0)$ the algebras $\mathcal{L}^{(e_1,e_2)}$ are* not *isomorphic to $\mathcal{W}$.*

In particular, we obtain a family of algebras over the base D_s, which is always the affine line. In this family, the algebra over the point $(0,0)$ is the Witt algebra and the isomorphy type above all other points will be the same but different from this special Witt element. We obtain the following

Theorem. *For every $s \in \mathbb{C} \cup \{\infty\}$ the families of Lie algebras defined by $(*)$ define global deformations $\mathcal{W}_t^{(s)}$ of $\mathcal{W}$ over the affine line $\mathbb{C}[t]$. Here t corresponds to the parameter e_1 and e_2 respectively. The Lie algebra over $t=0$ corresponds always to the Witt algebra, the algebras above $t \neq 0$ belong (if s is fixed) to the same isomorphy class, but are not isomorphic to $\mathcal{W}$.*

Remark. It is easy to incorporate a central term defined by a local cocycle and easy to show that the centrally extended algebras have the same properties.

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