

# MINIVERSAL DEFORMATIONS OF DIALGEBRAS

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ABSTRACT. We develop the theory of versal deformations of dialgebras and describe a method for constructing a miniversal deformation of a dialgebra.

## 1. INTRODUCTION

The notion of Leibniz algebras and dialgebras was discovered by J.-L. Loday while studying periodicity phenomena in algebraic K-theory [10]. Leibniz algebras are a non-commutative variation of Lie algebras and dialgebras are a variation of associative algebras. Recall that any associative algebra gives rise to a Lie algebra by  $[x, y] = xy - yx$ . The notion of dialgebras was invented in order to build analogue of the couple

Lie algebras  $\leftrightarrow$  associative algebras,

where Lie algebras are replaced by Leibniz algebras. Shortly, dialgebra is to Leibniz algebra, what associative algebra is to Lie algebra. A (co)homology theory associated to dialgebras was developed by J.-L. Loday, called the dialgebra cohomology where planar binary trees play a crucial role in the construction. Dialgebra cohomology with coefficients was studied by A. Frabetti [5, 6] and deformations of dialgebras were developed in [11]. In the present paper, we develop a deformation theory of dialgebras over a commutative unital algebra base, following [2], and show that dialgebra cohomology is a natural candidate for the cohomology controlling the deformations. We work out a construction of a versal deformation for dialgebras, following [3].

The paper is organized as follows. In Section 2, we recall some facts on dialgebra and its cohomology. In Section 3, we introduce the definitions of deformations of dialgebras over a commutative, unital algebra base. In Section 4, we produce an example of an infinitesimal deformation of a dialgebra  $D$  over a field  $K$ , denoted by  $\eta_D$ , and also show that this deformation is co-universal in the sense that, given any infinitesimal deformation  $\lambda$  of a dialgebra  $D$  with a finite dimensional base  $A$ , there exists a unique homomorphism  $\phi : K \oplus HY^2(D, D)' \rightarrow A$ , where  $HY^2(D, D)$  denotes the two dimensional cohomology of  $D$  with coefficients in itself, such that  $\lambda$  is equivalent to the push-out  $\phi_*\eta_D$ . Section 5 comprises results of Harrison cohomology of a commutative unital algebra  $A$  with coefficients in a  $A$ -module  $M$ , which have been used in the paper. In Section 6 we introduce obstructions to extending a deformation over a base  $A$ , to a deformation over a base  $B$ , where there exists an extension

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$0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  of  $A$ . We show that an obstruction is a cohomology class, vanishing of which is a necessary and sufficient condition for the given deformation of  $D$  over base  $A$  to be extended to a deformation of  $D$  over base  $B$ . In Section 7 we discuss extendible deformations. Let  $\lambda$  be the deformation of  $D$  over  $A$  which is extendible. We state that the two dimensional cohomology group  $HY^2(D, D)$  operates transitively on the set of equivalence classes of deformations  $\mu$  of  $D$  with base  $B$  such that  $p_*\mu = \lambda$ . We also state that the group of automorphisms of the extension  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  operates on the set of equivalence classes of deformations  $\mu$  such that  $p_*\mu = \lambda$ . These two actions are related by a map called the *differential*  $d\lambda : TA \rightarrow HY^2(D, D)$ , where  $TA$  denotes the tangent space of  $A$ . In the last section, we present a construction of a miniversal deformation of a dialgebra  $D$ .

## 2. DIALGEBRA AND ITS COHOMOLOGY

Throughout this paper,  $K$  will denote the ground field of characteristic zero. All tensor products shall be over  $K$  unless specified. In this section, we recall the definition of a dialgebra and the construction of the dialgebra cochain complex. Since we are interested in coefficients in the dialgebra itself, we shall restrict our definition to the same.

**Definition 2.1.** A dialgebra  $D$  over  $K$  is a vector space over  $K$  along with two  $K$ -linear maps  $\dashv : D \otimes D \rightarrow D$  called left and  $\vdash : D \otimes D \rightarrow D$  called right satisfying the following axioms :

$$\begin{aligned} x \dashv (y \dashv z) &\stackrel{1}{=} (x \dashv y) \dashv z \stackrel{2}{=} x \dashv (y \vdash z) \\ (x \vdash y) \dashv z &\stackrel{3}{=} x \vdash (y \dashv z) \\ (x \dashv y) \vdash z &\stackrel{4}{=} x \vdash (y \vdash z) \stackrel{5}{=} (x \vdash y) \vdash z \end{aligned} \tag{2.1.1}$$

for all  $x, y, z \in D$ .

Apart from the known algebraic examples of dialgebras, [10], we cite an interesting example of a family of dialgebras in the context of functional analysis, [1].

**Example 2.2.** Let  $\mathcal{H}$  be a Hilbert space and  $e \in \mathcal{H}$  with  $\|e\| = 1$ . Define two linear operations  $\dashv$  and  $\vdash$  by

$$a \dashv b = \langle b, e \rangle a, \quad a \vdash b = \langle a, e \rangle b,$$

for  $a, b \in \mathcal{H}$ . Then  $(\mathcal{H}, \dashv, \vdash)$  is a dialgebra, more precisely, a normed dialgebra [1].

A morphism  $\phi : D \rightarrow D'$  between two dialgebras is a  $K$ -linear map such that  $\phi(x \dashv y) = \phi(x) \dashv \phi(y)$  and  $\phi(x \vdash y) = \phi(x) \vdash \phi(y)$ .

A planar binary tree with  $n$  vertices (in short,  $n$ -tree) is a planar tree with  $(n + 1)$  leaves, one root and each vertex trivalent. Let  $Y_n$  denote the set of all  $n$ -trees. Let  $Y_0$  be the singleton set consisting of a root only. The  $n$ -trees for  $0 \leq n \leq 3$  are given by the following diagrams:

$$Y_0 = \{ \begin{array}{c} | \\ \hline \end{array} \}, \quad Y_1 = \{ \begin{array}{c} \diagup \quad \diagdown \\ \hline \end{array} \}, \quad Y_2 = \{ \begin{array}{c} \diagup \quad \diagdown \quad \diagup \\ \hline \end{array}, \begin{array}{c} \diagup \quad \diagdown \\ \hline \end{array} \},$$

$$Y_3 = \{ \begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \hline \end{array}, \begin{array}{c} \diagup \quad \diagdown \quad \diagup \\ \hline \end{array}, \begin{array}{c} \diagup \quad \diagdown \quad \diagdown \\ \hline \end{array}, \begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \hline \end{array}, \begin{array}{c} \diagup \quad \diagdown \quad \diagdown \quad \diagup \\ \hline \end{array} \}.$$

For any  $y \in Y_n$ , the  $(n+1)$  leaves are labelled by  $\{0, 1, \dots, n\}$  from left to right and the vertices are labelled  $\{1, 2, \dots, n\}$  so that the  $i$ th vertex is between the leaves  $(i-1)$  and  $i$ . The only element  $|$  of  $Y_0$  is denoted by  $[0]$  and the only element of  $Y_1$  is denoted by  $[1]$ . The grafting of a  $p$ -tree  $y_1$  and a  $q$ -tree  $y_2$  is a  $(p+q+1)$ -tree denoted by  $y_1 \vee y_2$  which is obtained by joining the roots of  $y_1$  and  $y_2$  and creating a new root from that vertex. This is denoted by  $[y_1 \ p+q+1 \ y_2]$  with the convention that all zeros are deleted except for the element in  $Y_0$ . With this notation, the trees pictured above from left to right are  $[0]$ ,  $[1]$ ,  $[12]$ ,  $[21]$ ,  $[123]$ ,  $[213]$ ,  $[131]$ ,  $[312]$ ,  $[321]$ .

For any  $i$ ,  $0 \leq i \leq n$ , there is a map, called the face map,  $d_i : Y_n \rightarrow Y_{n-1}$ ,  $y \mapsto d_i y$  where  $d_i y$  is obtained from  $y$  by deleting the  $i$ th leaf. The face maps satisfy the relations  $d_i d_j = d_{j-1} d_i$ , for all  $i < j$ .

Let  $D$  be a dialgebra over a field  $K$ . The cochain complex  $CY^*(D, D)$  which defines the dialgebra cohomology  $HY^*(D, D)$  is defined as follows. For any  $n \geq 0$ , let  $K[Y_n]$  denote the  $K$ -vector space spanned by  $Y_n$  and  $CY^n(D, D) := \text{Hom}_K(K[Y_n] \otimes D^{\otimes n}, D)$  be the module of  $n$ -cochains of  $D$  with coefficients in  $D$ . The coboundary operator  $\delta : CY^n(D, D) \rightarrow CY^{n+1}(D, D)$  is defined as the  $K$ -linear map  $\delta = \sum_{i=0}^{n+1} (-1)^i \delta^i$ , where

$$(\delta^i f)(y; a_1, a_2, \dots, a_{n+1}) = \begin{cases} a_1 \circ_0^y f(d_0 y; a_2, \dots, a_{n+1}), & i = 0 \\ f(d_i y; a_1, \dots, a_i \circ_i^y a_{i+1}, \dots, a_{n+1}), & 1 \leq i \leq n \\ f(d_{n+1} y; a_1, \dots, a_n) \circ_{n+1}^y a_{n+1}, & i = n+1 \end{cases}$$

for any  $y \in Y_{n+1}$ ;  $a_1, \dots, a_{n+1} \in D$  and  $f : K[Y_n] \otimes D^{\otimes n} \rightarrow D$ . Here, for any  $i$ ,  $0 \leq i \leq n+1$ . The maps  $\circ_i : Y_{n+1} \rightarrow \{-, \vdash\}$ , are defined by

$$\circ_0(y) = \circ_0^y := \begin{cases} \dashv & \text{if } y \text{ is of the form } | \vee y_1, \text{ for some } n\text{-tree } y_1 \\ \vdash & \text{otherwise} \end{cases}$$

$$\circ_i(y) = \circ_i^y := \begin{cases} \dashv & \text{if the } i^{\text{th}} \text{ leaf of } y \text{ is oriented like } \backslash \\ \vdash & \text{if the } i^{\text{th}} \text{ leaf of } y \text{ is oriented like } / \end{cases}$$

for  $1 \leq i \leq n$  and

$$\circ_{n+1}(y) = \circ_{n+1}^y := \begin{cases} \vdash & \text{if } y \text{ is of the form } y_1 \vee |, \text{ for some } n\text{-tree } y_1 \\ \dashv & \text{otherwise} \end{cases},$$

where the symbol ' $\vee$ ' stands for grafting of trees [10].

There exists a pre-Lie algebra structure on  $CY^*(D, D)$ , [11, 12], the pre-Lie product being denoted by

$$\circ : CY^n(D, D) \otimes CY^m(D, D) \longrightarrow CY^{n+m-1}.$$

Also, if we modify the coboundary map  $\delta$  by a sign, say  $dx = (-1)^{|x|}\delta(x)$ , and define a bracket product on  $CY^*(D, D)$  by  $[x, y] = x \circ y - (-1)^{|x||y|}y \circ x$ , which is the commutator of the pre-Lie product, then  $(CY^*(D, D), d)$  forms a differential graded Lie algebra, [12], where  $|x| = \deg x - 1$ .

### 3. DEFORMATIONS OF DIALGEBRAS

Let  $D$  be a dialgebra over  $K$  and let  $A$  be a commutative unital algebra over  $K$  with a fixed augmentation  $\epsilon : A \longrightarrow K$  with  $\epsilon(1) = 1$ . Let  $\mathfrak{m} = \ker \epsilon$ . We assume  $\dim(\mathfrak{m}^k/\mathfrak{m}^{k+1}) < \infty$ , for all  $k \geq 1$ .

**Definition 3.1.** A deformation  $\lambda$  of  $D$  with base  $(A, \mathfrak{m})$  is a dialgebra structure on the tensor product  $A \otimes_K D$  with the products  $\dashv_\lambda$  and  $\vdash_\lambda$  being  $A$ -linear (or simply, an  $A$ -dialgebra structure) such that  $\epsilon \otimes \text{id} : A \otimes_K D \longrightarrow K \otimes D \cong D$  is a  $A$ -linear dialgebra morphism. The left action of  $A$  on  $K \otimes D$  is given by the augmentation map.

We note that for  $x_1, x_2 \in D$ , and  $a, b \in A$ ,

$$a \otimes x_1 *_\lambda b \otimes x_2 = ab(1 \otimes x_1 * 1 \otimes x_2),$$

by  $A$ -linearity of the products, where  $*$  =  $\{\dashv, \vdash\}$ . Also, since  $\epsilon \otimes \text{id} : A \otimes D \longrightarrow K \otimes D$  is a  $A$ -linear dialgebra homomorphism,

$$\begin{aligned} (\epsilon \otimes \text{id}) \{1 \otimes x_1 *_\lambda 1 \otimes x_2\} &= (\epsilon \otimes \text{id})(1 \otimes x_1) * (\epsilon \otimes \text{id})(1 \otimes x_2) \\ &= (1 \otimes x_1) * (1 \otimes x_2) \\ &= 1 \otimes (x_1 * x_2) \\ &= (\epsilon \otimes \text{id})(1 \otimes (x_1 * x_2)). \end{aligned}$$

So,  $(1 \otimes x_1) *_\lambda (1 \otimes x_2) - 1 \otimes (x_1 * x_2) \in \ker(\epsilon \otimes \text{id})$ . Hence  $(1 \otimes x_1) *_\lambda (1 \otimes x_2) = 1 \otimes (x_1 * x_2) + \sum_i m_i \otimes d_i$ , where  $m_i \in \ker \epsilon = \mathfrak{m}$  and  $d_i \in D$  and  $\sum_i m_i \otimes d_i$  is a finite sum.

**Definition 3.2.** Two deformations of  $D$  with the same base  $A$  are called *equivalent* if there exists a  $A$ -linear dialgebra isomorphism between the two copies of  $A \otimes D$  with the two dialgebra structures, compatible with  $\epsilon \otimes \text{id}$ . A deformation of  $D$  with base  $A$  is called *local* if the algebra  $A$  is local, and will be called *infinitesimal* if, in addition,  $\mathfrak{m}^2 = 0$ , where  $\mathfrak{m}$  is the maximal ideal of  $A$ .

**Definition 3.3.** Let  $A$  be a complete local algebra, that is,  $A = \overleftarrow{\lim}_{n \rightarrow \infty} (A/\mathfrak{m}^n)$ ,  $\mathfrak{m}$  denoting the maximal ideal in  $A$ . A formal deformation of  $D$  with base  $A$  is a  $A$ -dialgebra structure on the completed tensor product  $A \widehat{\otimes} D = \overleftarrow{\lim}_{n \rightarrow \infty} (A/\mathfrak{m}^n \otimes D)$ , such that  $\epsilon \widehat{\otimes} \text{id} : A \widehat{\otimes} D \longrightarrow K \otimes D = D$  is a  $A$ -linear dialgebra morphism.

Two formal deformations of a dialgebra  $D$  with the same base  $A$  are called *equivalent* if there exists a dialgebra isomorphism between the two copies of  $A \widehat{\otimes} D$  with the two dialgebra structures compatible with  $\epsilon \widehat{\otimes} \text{id}$ .

**Example 3.4.** If  $A = K[[t]]$  then a formal deformation of  $D$  with base  $A$  is the same as a formal one-parameter deformation of  $D$ , [11].

Let  $A'$  be a commutative algebra with identity, with a fixed augmentation  $\epsilon' : A' \rightarrow K$  and let  $\phi : A \rightarrow A'$  be an algebra homomorphism, with  $\phi(1) = 1$  and  $\epsilon' \circ \phi = \epsilon$ . Then we can construct a deformation of  $D$  with base  $A'$  in the following way.

**Definition 3.5.** Let  $\lambda$  be a deformation of the dialgebra  $D$  with base  $(A, \mathfrak{m})$ . The *push-out*  $\phi_*\lambda$  is the deformation of  $D$  with base  $(A', \mathfrak{m}' = \ker \epsilon')$ , which is the dialgebra structure given by

$$\begin{aligned} a'_1 \otimes_A (a_1 \otimes x_1) \dashv_{\phi_*\lambda} a'_2 \otimes_A (a_2 \otimes x_2) &= a'_1 a'_2 \otimes_A (a_1 \otimes x_1 \dashv_{\lambda} a_2 \otimes x_2) \\ a'_1 \otimes_A (a_1 \otimes x_1) \vdash_{\phi_*\lambda} a'_2 \otimes_A (a_2 \otimes x_2) &= a'_1 a'_2 \otimes_A (a_1 \otimes x_1 \vdash_{\lambda} a_2 \otimes x_2), \end{aligned}$$

where  $a_1, a_2 \in A'$ ,  $a_1, a_2 \in A$  and  $l_1, l_2 \in D$ . Here we make use of the fact that  $A' \otimes D = (A' \otimes_A A) \otimes D = A' \otimes_A (A \otimes D)$ , where  $A'$  is regarded as an  $A$ -module by the structure  $a'a = a'\phi(a)$ .

Similarly, one can define the *push-out* of formal deformations.

**Remark 3.6.** We note that if the dialgebra structure  $\lambda$  on  $A \otimes D$  is given by

$$(1 \otimes x_1) *_{\lambda} (1 \otimes x_2) = 1 \otimes (x_1 * x_2) + \sum_{i=1}^n m_i \otimes d_i; \quad m_i \in \mathfrak{m}, d_i \in D,$$

then the dialgebra structure  $\phi_*\lambda$  on  $A' \otimes D$  is given by

$$(1 \otimes x_1) *_{\phi_*\lambda} (1 \otimes x_2) = 1 \otimes (x_1 * x_2) + \sum_{i=1}^n \phi(m_i) \otimes d_i.$$

#### 4. UNIVERSAL INFINITESIMAL AND MINIVERSAL DEFORMATIONS OF DIALGEBRAS

In [3], the authors have produced a fundamental example of an infinitesimal deformation of Lie algebras. Here we produce an example of an infinitesimal deformation of dialgebras, which is obtained from the aforesaid example, with slight modifications. Suppose  $\dim HY^2(D, D) < \infty$ . This is, in particular, true if  $\dim D < \infty$ . Consider the base of the deformation to be  $A = K \oplus HY^2(D, D)'$ , with  $'$  denoting the linear dual. Here,  $A$  is local with the maximal ideal  $\mathfrak{m} = HY^2(D, D)'$ , and  $\mathfrak{m}^2 = 0$ .

Let

$$\mu : HY^2(D, D) \rightarrow CY^2(D, D) = \text{Hom}(K[Y_2] \otimes D^{\otimes 2}, D)$$

which takes a cohomology class into a cocycle representing the class. Define a dialgebra structure on

$$\begin{aligned} A \otimes D &= (K \oplus HY^2(D, D)') \otimes D \\ &= (K \otimes D) \oplus (HY^2(D, D)' \otimes D) \\ &= D \oplus (HY^2(D, D)' \otimes D) \\ &= D \oplus \text{Hom}(HY^2(D, D), D) \end{aligned}$$

by

$$\begin{aligned} (x_1, \phi_1) \dashv (x_2, \phi_2) &= (x_1 \dashv x_2, \psi_\ell) \\ (x_1, \phi_1) \vdash (x_2, \phi_2) &= (x_1 \vdash x_2, \psi_r) \end{aligned}$$

where

$$\begin{aligned} \psi_\ell(\alpha) &= \mu(\alpha)([21]; x_1, x_2) + \phi_1(\alpha) \dashv x_2 + x_1 \dashv \phi_2(\alpha) \\ \psi_r(\alpha) &= \mu(\alpha)([12]; x_1, x_2) + \phi_1(\alpha) \vdash x_2 + x_1 \vdash \phi_2(\alpha), \end{aligned}$$

for  $\alpha \in HY^2(D, D)$ .

Using the dialgebra structure of  $D$  and the fact that  $\mu(\alpha)$  is a 2-cocycle of  $D$ , one can check that the  $\dashv$  and  $\vdash$  products defined this way satisfy the dialgebra axioms. It is to be noted that this deformation does not depend on the choice of  $\mu$ , upto an isomorphism.

Let  $\mu' : HY^2(D, D) \longrightarrow CY^2(D, D)$  be another choice of  $\mu$ . Define a homomorphism

$$\nu : HY^2(D, D) \longrightarrow CY^1(D, D) \cong \text{Hom}(D, D)$$

by  $\mu'(\alpha) - \mu(\alpha) = \delta\nu(\alpha)$ , for all  $\alpha \in HY^2(D, D)$ . We define a linear automorphism  $\rho$  of the space  $A \otimes D = D \oplus \text{Hom}(HY^2(D, D), D)$  by  $\rho(x, \phi) = (x, \psi)$  where  $\psi(\alpha) = \phi(\alpha) + \nu(\alpha)(x)$ . It is straightforward to check that  $\rho$  defines a dialgebra isomorphism between the two dialgebra structures induced by  $\mu$  and  $\mu'$  respectively. We denote the infinitesimal deformation of  $D$  as constructed above by  $\eta_D$ .

Below we will show the *couniversality* of  $\eta_D$  in the class of infinitesimal deformations: Let  $\lambda$  be an infinitesimal deformation of the dialgebra  $D$ , with a finite dimensional local algebra base  $A$ , with  $\mathfrak{m}^2 = 0$ , where  $\mathfrak{m}$  is the maximal ideal of  $A$ . Let  $\xi \in \mathfrak{m}' = \text{Hom}_K(\mathfrak{m}, K)$ . This is equivalent to  $\xi \in \text{Hom}_K(A, K)$  with  $\xi(1) = 0$ .

For  $x_1, x_2 \in D$ , let us define a 2-cochain as follows:

$$\alpha_{\lambda, \xi}([21]; x_1, x_2) = (\xi \otimes \text{id})((1 \otimes x_1) \dashv_\lambda (1 \otimes x_2))$$

and

$$\alpha_{\lambda, \xi}([12]; x_1, x_2) = (\xi \otimes \text{id})((1 \otimes x_1) \vdash_\lambda (1 \otimes x_2)).$$

We claim that  $\alpha_{\lambda, \xi} \in CY^2(D, D)$  is a 2-cocycle. This is because

$$\begin{aligned} &\delta\alpha_{\lambda, \xi}([321]; x_1, x_2, x_3) \\ &= x_1 \dashv \alpha_{\lambda, \xi}([21]; x_2, x_3) - \alpha_{\lambda, \xi}([21]; x_1 \dashv x_2, x_3) \\ &\quad + \alpha_{\lambda, \xi}([21]; x_1, x_2 \dashv x_3) - \alpha_{\lambda, \xi}([21]; x_1, x_2) \dashv x_3 \\ &= x_1 \dashv (\xi \otimes \text{id})((1 \otimes x_2) \vdash_\lambda (1 \otimes x_3)) - (\xi \otimes \text{id})(1 \otimes (x_1 \dashv x_2) \dashv_\lambda 1 \otimes x_3) \\ &\quad + (\xi \otimes \text{id})(1 \otimes x_1 \dashv_\lambda 1 \otimes x_2 \dashv x_3) - (\xi \otimes \text{id})(1 \otimes x_1 \dashv_\lambda 1 \otimes x_2) \dashv x_3. \end{aligned}$$

If  $\epsilon$  denotes the fixed augmentation of the algebra  $A$ , then

$$\epsilon \otimes \text{id} : (1 \otimes x_1 \dashv_{\lambda} 1 \otimes x_2 - 1 \otimes x_1 \dashv x_2) = 0,$$

i.e.  $1 \otimes x_1 \dashv_{\lambda} 1 \otimes x_2 - 1 \otimes x_1 \dashv x_2 \in \mathfrak{m} \otimes D$ . So,

$$\begin{aligned} & (\xi \otimes \text{id})((1 \otimes x_1 \dashv_{\lambda} 1 \otimes x_2) \dashv_{\lambda} (1 \otimes x_3)) \\ &= (\xi \otimes \text{id})(((1 \otimes x_1 \dashv x_2) + \sum_i m_i \otimes y_i) \dashv_{\lambda} (1 \otimes x_3)) \\ &= (\xi \otimes \text{id})((1 \otimes x_1 \dashv x_2) \dashv_{\lambda} (1 \otimes x_3)) + (\xi \otimes \text{id})(\sum_i (m_i \otimes y_i) \dashv_{\lambda} (1 \otimes x_3)) \\ &= (\xi \otimes \text{id})((1 \otimes x_1 \dashv x_2) \dashv_{\lambda} (1 \otimes x_3)) + (\xi \otimes \text{id})(\sum_i m_i(1 \otimes y_i) \dashv_{\lambda} (1 \otimes x_3)) \\ &= \alpha_{\lambda, \xi}([21]; x_1 \dashv x_2, x_3) + (\xi \otimes \text{id}) \sum_i m_i(1 \otimes y_i \dashv_{\lambda} 1 \otimes x_3). \end{aligned}$$

Note that in the second step from the end we make use of the action of the algebra  $A$  on  $A \otimes D$ .

Now we have

$$\begin{aligned} 1 \otimes y_i \dashv_{\lambda} 1 \otimes x_3 - 1 \otimes y_i \dashv x_3 &\in \mathfrak{m} \otimes D, \\ 1 \otimes y_i \dashv_{\lambda} 1 \otimes x_3 &= 1 \otimes y_i \dashv x_3 + h, \end{aligned}$$

where  $h \in \mathfrak{m} \otimes D$ . Hence,

$$m_i(1 \otimes y_i \dashv_{\lambda} 1 \otimes x_3) = m_i(1 \otimes y_i \dashv x_3 + h).$$

Since  $\mathfrak{m}^2 = 0$ , we have  $m_i h = 0$ . So,  $m_i(1 \otimes y_i \dashv_{\lambda} 1 \otimes x_3) = m_i \otimes (y_i \dashv x_3)$ , making use of the action of  $A$  on  $A \otimes D$ . Next

$$\begin{aligned} (\xi \otimes \text{id}) \sum_i m_i(1 \otimes y_i \dashv_{\lambda} 1 \otimes x_3) &= \sum_i (\xi \otimes \text{id})(m_i \otimes y_i \dashv x_3) \\ &= \sum_i \xi(m_i)(y_i \dashv x_3) \\ &= \sum_i (\xi(m_i)y_i \dashv x_3) \\ &= (\xi \otimes \text{id})(\sum_i m_i \otimes y_i) \dashv x_3 \\ &= (\xi \otimes \text{id})\{(1 \otimes x_1 \dashv_{\lambda} 1 \otimes x_2) - 1 \otimes x_1 \dashv x_2\} \dashv x_3 \\ &= ((\xi \otimes \text{id})(1 \otimes x_1 \dashv_{\lambda} 1 \otimes x_2) \dashv x_3 \quad [\text{using } \xi(1) = 0] \\ &= \alpha_{\lambda, \xi}([12]; x_1, x_2) \dashv x_3. \end{aligned}$$

Thus,

$$\xi \otimes \text{id} ((1 \otimes x_1 \dashv_{\lambda} 1 \otimes x_2) \dashv_{\lambda} (1 \otimes x_3)) = \alpha_{\lambda, \xi}([21]; x_1 \dashv x_2, x_3) + \alpha_{\lambda, \xi}([21]; x_1, x_2) \dashv x_3.$$

In the same way,

$$\xi \otimes \text{id} (1 \otimes x_1 \dashv_{\lambda} (1 \otimes x_2 \dashv_{\lambda} (1 \otimes x_3))) = x_1 \dashv \alpha_{\lambda, \xi}([21]; x_2, x_3) + \alpha_{\lambda, \xi}([21]; x_1, x_2 \dashv x_3).$$

Since

$$\xi \otimes \text{id}((1 \otimes x_1 \dashv_{\lambda} 1 \otimes x_2) \dashv_{\lambda} (1 \otimes x_3)) - \xi \otimes \text{id}(1 \otimes x_1 \dashv_{\lambda} (1 \otimes x_2 \dashv_{\lambda} 1 \otimes x_3)) = 0,$$

we have

$$\delta\alpha_{\lambda,\xi}([321]; x_1, x_2, x_3) = 0,$$

and we can also show that  $\delta\alpha_{\lambda,\xi}(y; x_1, x_2, x_3) = 0$  for all  $y \in \{[312], [131], [213], [123]\}$ .

The following proposition classifies all infinitesimal deformations of  $D$  over finite dimensional bases.

**Proposition 4.1.** *For any infinitesimal deformation  $\lambda$  of a dialgebra  $D$  with a finite dimensional base  $A$  there exists a unique homomorphism  $\phi : K \oplus HY^2(D, D)' \longrightarrow A$  such that  $\lambda$  is equivalent to the push-out  $\phi_*\eta_D$ .*

**Proof.** Let  $a_{\lambda,\xi} \in HY^2(D, D)$  be the cohomology class of the cocycle  $\alpha_{\lambda,\xi}$ , corresponding to  $\xi \in \mathfrak{m}'$ . Thus we have the following homomorphisms:

$$\begin{aligned} \alpha_{\lambda} : \mathfrak{m}' &\longrightarrow CY^2(D, D) \\ a_{\lambda} : \mathfrak{m}' &\longrightarrow HY^2(D, D). \end{aligned}$$

**Step 1.** We show that the deformations  $\lambda, \lambda'$  are equivalent if and only if  $a_{\lambda} = a_{\lambda'}$ . Let  $\lambda_1$  and  $\lambda_2$  be two equivalent deformations of the dialgebra  $D$ , with base  $A$ . By definition, there exists a  $A$ -linear dialgebra isomorphism

$$\rho : A \otimes D \longrightarrow A \otimes D, \text{ such that } (\epsilon \otimes \text{id}) \circ \rho = \epsilon \otimes \text{id}. \quad (4.1.1)$$

Since  $A \otimes D = D \oplus (\mathfrak{m} \otimes D)$ , the isomorphism  $\rho$  can be written as  $\rho = \rho_1 + \rho_2$  where  $\rho_1 : D \longrightarrow D$  and  $\rho_2 : D \longrightarrow \mathfrak{m} \otimes D$ .

By using equation (4.1.1), we get  $\rho_1 = \text{id}$ . Note that by the adjunction property of tensor products,

$$\text{Hom}(D; \mathfrak{m} \otimes D) \cong \mathfrak{m} \otimes \text{Hom}(D, D) \cong \text{Hom}(\mathfrak{m}'; \text{Hom}(D, D)),$$

where the isomorphisms are given by

$$\rho_2 \longmapsto \sum_1^k m_i \otimes \phi_i \longmapsto \sum_i^k \chi_i. \quad (4.1.2)$$

Here  $\phi_i = (\xi_i \otimes \text{id}) \circ \rho_2$  and  $\chi_i(\xi_j) = \delta_{i,j} \phi_i$ , where  $\{m_i\}_{1 \leq i \leq k}$  is a basis of  $\mathfrak{m}$  and  $\{\xi_j\}_{1 \leq j \leq k}$  is a basis of  $\mathfrak{m}'$ .

We have by equation (4.1.2),

$$\begin{aligned} \rho(1 \otimes x) &= \rho_1(1 \otimes x) + \rho_2(1 \otimes x) \\ &= 1 \otimes x + \sum_1^k m_i \otimes \phi_i(x). \end{aligned}$$

Using the notation  $* = \{\dashv, \vdash\}$ , the map  $\rho$  is a dialgebra homomorphism iff

$$\rho(1 \otimes x_1 *_{\lambda_1} 1 \otimes x_2) = \rho(1 \otimes x_1) *_{\lambda_2} \rho(1 \otimes x_2),$$

Let us set  $\psi_i^r = \alpha_{\lambda_r, \xi_i}$ ,  $i = 1, 2, \dots, k$  and  $r = 1, 2$ . Then we have

$$1 \otimes x_1 \dashv_{\lambda_r} 1 \otimes x_2 = 1 \otimes x_1 \dashv x_2 + \sum_i^k m_i \otimes \psi_i^r([21]; x_1, x_2) \quad (4.1.3)$$

and

$$1 \otimes x_1 \vdash_{\lambda_r} 1 \otimes x_2 = 1 \otimes x_1 \vdash x_2 + \sum_i^k m_i \otimes \psi_i^r([12]; x_1, x_2). \quad (4.1.4)$$

Therefore, using the fact that  $m_i.m_j = 0$  for elements  $m_i, m_j \in \mathfrak{m}$ ,

$$\rho(1 \otimes x_1 \dashv_{\lambda_1} 1 \otimes x_2) = 1 \otimes x_1 \dashv x_2 + \sum_{i=1}^k m_i \otimes \phi_i(x_1 \dashv x_2) + \sum_{i=1}^k m_i (1 \otimes \psi_i^1([21]; x_1, x_2)).$$

Similarly,

$$\rho(1 \otimes x_1 \vdash_{\lambda_2} 1 \otimes x_2) = 1 \otimes x_1 \vdash x_2 + \sum_{i=1}^k m_i \otimes \phi_i(x_1 \vdash x_2) + \sum_{i=1}^k m_i (1 \otimes \psi_i^1([12]; x_1, x_2)).$$

Again,

$$\begin{aligned} & \rho(1 \otimes x_1) \dashv_{\lambda_2} \rho(1 \otimes x_2) \\ &= 1 \otimes (x_1 \dashv x_2) + \sum_{i=1}^k m_i \otimes (\psi_i^2([21]; x_1, x_2)) + \sum_{i=1}^k m_i \otimes (x_1 \dashv \phi_i(x_2)) \\ & \quad + \sum_{i=1}^k m_i \otimes (\phi_i(x_1) \dashv x_2), \end{aligned}$$

and

$$\begin{aligned} & \rho(1 \otimes x_1) \vdash_{\lambda_2} \rho(1 \otimes x_2) \\ &= 1 \otimes (x_1 \vdash x_2) + \sum_{i=1}^k m_i \otimes (\psi_i^2([21]; x_1, x_2)) + \sum_{i=1}^k m_i \otimes (x_1 \vdash \phi_i(x_2)) \\ & \quad + \sum_{i=1}^k m_i \otimes (\phi_i(x_1) \vdash x_2). \end{aligned}$$

Thus, the following are equivalent:

- a)  $\rho(1 \otimes x_1 \dashv_{\lambda_1} 1 \otimes x_2) = \rho(1 \otimes x_1) \dashv_{\lambda_2} \rho(1 \otimes x_2)$
- b)  $\sum_{i=1}^k m_i \otimes (\psi_i^2([21]; x_1, x_2) - \psi_i^1([21]; x_1, x_2)) + \sum_{i=1}^k m_i \otimes \delta\phi_i([21]; x_1, x_2) = 0$
- c)  $\psi_i^1([21]; x-1, x_2) - \psi_i^2([21]; x_1, x_2) = \delta\phi_i([21]; x_1, x_2)$

and similarly these are equivalent, too:

$$\begin{aligned} \text{a')} \quad & \rho(1 \otimes x_1 \vdash_{\lambda_1} 1 \otimes x_2) = \rho(1 \otimes x_1) \vdash_{\lambda_2} \rho(1 \otimes x_2), \\ \text{b')} \quad & \sum_{i=1}^k m_i \otimes (\psi_i^2([12]; x_1, x_2) - \psi_i^1([12]; x_1, x_2)) + \sum_{i=1}^k m_i \otimes \delta\phi_i([12]; x_1, x_2) = 0 \\ \text{c')} \quad & \psi_i^1([12]; x_1 - 1, x_2) - \psi_i^2([12]; x_1, x_2) = \delta\phi_i([12]; x_1, x_2). \end{aligned}$$

Hence,

$$\alpha_{\lambda_1, \xi_i} - \alpha_{\lambda_2, \xi_i} = \delta\phi_i \quad \text{for } i \in \{1, 2, \dots, k\} \quad \text{if and only if} \quad a_{\lambda_1} = a_{\lambda_2}.$$

This proves step 1.

**Step 2.** Let

$$\phi = \text{id} \oplus a'_\lambda : K \oplus HY^2(D, D)' \longrightarrow K \oplus \mathfrak{m} = A.$$

Claim:  $\phi_*\eta_D$  is equivalent to  $\lambda$ . It follows from definitions that  $\alpha_{\phi_*\eta_D} = \mu \circ a_\lambda$ . Thus,  $a_{\phi_*\eta_D} = a_\lambda$ . Hence by step 1,  $\phi_*\eta_D$  and  $\lambda$  are isomorphic. This completes the proof of Proposition 4.1.  $\square$

Let  $A$  be a local algebra with  $\dim(A/\mathfrak{m}^2) < \infty$ . Then,  $A/\mathfrak{m}^2$  is also local with the maximal ideal  $\mathfrak{m}/\mathfrak{m}^2$ , and  $(\mathfrak{m}/\mathfrak{m}^2)^2 = 0$ .

**Definition 4.2.** The linear dual space  $\text{Hom}(\mathfrak{m}/\mathfrak{m}^2, K)$  is called the *tangent space* of  $A$ , and is denoted by  $TA$ .

**Definition 4.3.** Let  $\lambda$  be a deformation of  $D$  with base  $A$ . Then the mapping

$$a_{\pi_*\lambda} : TA = (\mathfrak{m}/\mathfrak{m}^2)' \longrightarrow HY^2(D, D),$$

where  $\pi$  is the projection  $A \longrightarrow A/\mathfrak{m}^2$ , is called the *differential* of  $\lambda$  and is denoted by  $d\lambda$ .

**Definition 4.4.** A formal deformation  $\eta$  of a dialgebra  $D$  with base  $B$  is called *miniversal* if

- (1) for any formal deformation  $\lambda$  of a dialgebra  $D$  with any local base  $A$  there exists a homomorphism  $f : B \rightarrow A$  such that the deformation  $\lambda$  is equivalent to  $f_*\eta$ ;
- (2) with the above notations if  $A$  satisfies the condition  $\mathfrak{m}^2 = 0$ , then  $f$  is unique.

If  $\eta$  satisfies only condition (1), then it is called *versal*.

The following proposition takes its shape from the general results of Schlessinger [13]. It was first shown for the case of Lie algebras in [2], and stated for Leibniz algebras in [4]. It is straightforward to see that it is true for the case of dialgebras, too.

**Proposition 4.5.** *If the dimension of  $HY^2(D, D)$  is finite, then there exists a miniversal deformation of the dialgebra  $D$ .*

$\square$

## 5. SOME FACTS ABOUT HARRISON COHOMOLOGY

Let  $A$  denote a commutative algebra over  $K$ . In this section we shall state a few results, without proof [9], about Harrison cohomology groups of  $A$  with coefficients in a  $A$ -module  $M$ . Let  $Ch(A) = \{Ch_q(A), \delta\}$  denote the Harrison complex of  $A$ .

**Definition 5.1.** For an  $A$ -module  $M$ , the Harrison homology and cohomology of  $A$  with coefficients in  $M$  are defined as follows:

$$\begin{aligned} H_q^{Harr}(A; M) &= H_q(Ch(A) \otimes M), \\ H_{Harr}^q(A; M) &= H^q(\text{Hom}(Ch(A), M)); \end{aligned}$$

**Proposition 5.2.** (1)  $H_{Harr}^1(A; M)$  is the space of derivations  $A \rightarrow M$ .  
 (2) Elements of  $H_{Harr}^2(A; M)$  correspond bijectively to isomorphism classes of extensions  $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$  of the algebra  $A$  by means of  $M$ .

□

**Corollary 5.3.** If  $A$  is a local algebra with the maximal ideal  $\mathfrak{m}$ , then

$$H_{Harr}^1(A; K) = (\mathfrak{m}/\mathfrak{m}^2)' = TA.$$

**Proposition 5.4.** Suppose  $0 \rightarrow M_r \xrightarrow{\bar{i}} B_{r-1} \xrightarrow{\bar{p}} A \rightarrow 0$  is an  $r$ -dimensional extension of  $A$ . Then there is a  $(r-1)$ -dimensional extension  $0 \rightarrow M_{r-1} \xrightarrow{i} B_r \xrightarrow{p} A \rightarrow 0$  of  $A$  and a 1-dimensional extension  $0 \rightarrow K \xrightarrow{i'} B_r \xrightarrow{p'} B_{r-1} \rightarrow 0$ .

□

**Proposition 5.5.** Let  $0 \rightarrow M \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  be an extension of an algebra  $A$  by  $M$ .

- (1) If  $A$  has an identity then so does  $B$ .
- (2) If  $A$  is local with the maximal ideal  $\mathfrak{m}$ , then  $B$  is local with the maximal ideal  $p^{-1}(\mathfrak{m})$ .

□

**Definition 5.6.** Two extensions  $B$  and  $B'$  of the algebra  $A$  by  $M$  are said to be equivalent if there exists a  $K$ -algebra isomorphism  $f : B \rightarrow B'$  such that the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{i_1} & B & \xrightarrow{p_1} & A & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow f & & \downarrow \text{id} & & \\ 0 & \longrightarrow & M & \xrightarrow{i_2} & B' & \xrightarrow{p_2} & A & \longrightarrow & 0. \end{array}$$

An equivalence from  $B$  to  $B$  is said to be an automorphism of  $B$  over  $A$ .

**Proposition 5.7.**  $H_{Harr}^1(A; M)$  is isomorphic to the set of automorphisms of any given extension  $0 \rightarrow M \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  of  $A$  by  $M$ .

□

## 6. OBSTRUCTIONS TO EXTENDING DEFORMATIONS

Let  $A$  be a finite dimensional commutative, unital, local algebra with a fixed augmentation  $\epsilon$ , and let  $\lambda$  be a deformation of a dialgebra  $D$  with base  $A$ . Let  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  be an extension of  $A$ , corresponding to a cohomology class  $f \in H_{Harr}^2(A; K)$ . Let  $q : A \rightarrow B$  be a splitting. Let  $\hat{\epsilon} : B \rightarrow K$  be the augmentation of  $B$ . Let  $I = i \otimes \text{id} : D = K \otimes D \rightarrow B \otimes D$  and  $P = p \otimes \text{id} : B \otimes D \rightarrow A \otimes D$ . Let  $E = \hat{\epsilon} \otimes \text{id} : B \otimes D \rightarrow K \otimes D = D$  and let  $Q = q \otimes \text{id} : A \otimes D \rightarrow B \otimes D$ . We define two  $B$ -bilinear operations  $\{ , \}_-, \{ , \}_+$  on  $B \otimes D$  as follows:

Let  $l_1, l_2 \in B \otimes D$ . Define

$$\{l_1, l_2\}_- = Q\{P(l_1) \dashv_\lambda P(l_2)\} + I[I^{-1}(l_1 - Q \circ P(l_1)) \dashv I^{-1}(l_2 - Q \circ P(l_2))],$$

$$\{l_1, l_2\}_+ = Q\{P(l_1) \vdash_\lambda P(l_2)\} + I[I^{-1}(l_1 - Q \circ P(l_1)) \vdash I^{-1}(l_2 - Q \circ P(l_2))].$$

It is easy to verify that the two operations thus defined satisfy the following properties:

$$(i) P\{l_1, l_2\}_* = P(l_1) * P(l_2), \quad \text{where } * \in \{\dashv, \vdash\}, l_1, l_2 \in B \otimes D, \quad (6.0.1)$$

$$(ii) \{I(l), l_1\}_* = I[l * E(l_1)], \quad \text{where } * \in \{\dashv, \vdash\}, l \in D, l_1 \in B \otimes D. \quad (6.0.2)$$

Using the above two properties, one can show that

$$\begin{aligned} E\{l_1, l_2\}_- &= E(l_1) \dashv E(l_2) \\ E\{l_1, l_2\}_+ &= E(l_1) \vdash E(l_2). \end{aligned}$$

We define

$$\phi([321]; l_1, l_2, l_3) = \{l_1, \{l_2, l_3\}_-\}_- - \{\{l_1, l_2\}_-, l_3\}_-, \quad (6.0.3)$$

$$\phi([312]; l_1, l_2, l_3) = \{\{l_1, l_2\}_-, l_3\}_- - \{l_1, \{l_2, l_3\}_+\}_-, \quad (6.0.4)$$

$$\phi([131]; l_1, l_2, l_3) = \{l_1, \{l_2, l_3\}_+\}_- - \{\{l_1, l_2\}_+, l_3\}_-, \quad (6.0.5)$$

$$\phi([213]; l_1, l_2, l_3) = \{\{l_1, l_2\}_-, l_3\}_+ - \{l_1, \{l_2, l_3\}_+\}_+, \quad (6.0.6)$$

$$\phi([213]; l_1, l_2, l_3) = \{l_1, \{l_2, l_3\}_+\}_+ - \{\{l_1, l_2\}_+, l_3\}_+. \quad (6.0.7)$$

It is easy to see that  $\phi(y; l_1, l_2, l_3) \in \ker P$  for all  $y \in Y_3$ . Also, note that if any  $l_i \in \ker E, i \in \{1, 2, 3\}$ , then  $\phi(l_1, l_2, l_3) = 0$ . This defines the map

$$\bar{\phi} : K[Y_3] \otimes D^{\otimes 3} = K[Y_3] \otimes ((B \otimes D)/\ker E)^{\otimes 3} \rightarrow \ker P = D. \quad (6.0.8)$$

Thus  $\bar{\phi} \in CY^3(D, D)$ . One can check that  $\delta\bar{\phi} = 0$ .

Let  $f'$  be cohomologous to  $f$ , and let  $0 \rightarrow K \xrightarrow{i'} B' \xrightarrow{p'} A \rightarrow 0$  be the extension corresponding to  $f'$ , which is isomorphic to the extension corresponding to  $f$ . Since  $B$  and  $B'$  are isomorphic, without loss of generality, we shall work with  $B$ .

Let  $\{ , \}'_*, * \in \{\dashv, \vdash\}$  be another set of  $B$ -bilinear operations on  $B \otimes D$ , satisfying (1) and (2) above. Then  $\{l_1, l_2\}'_* - \{l_1, l_2\}_* \in \ker P, * \in \{\dashv, \vdash\}$  for all  $l_1, l_2 \in B \otimes D$ . Also,  $\{l_1, l_2\}'_* - \{l_1, l_2\}_* = 0, * \in \{\dashv, \vdash\}$  if  $l_i \in \ker E, i \in \{1, 2\}$ . This determines a map  $\psi : K[Y_2] \otimes D^{\otimes 2} = K[Y_2] \otimes ((B \otimes D)/\ker E)^{\otimes 2} \rightarrow \ker P = D$ . Thus,  $\psi$  defines a 2-cochain. Also, given an arbitrary  $\psi \in CY^2(D, D)$ , there exists an appropriate  $\{ , \}'_*$  such that  $\psi$  can be obtained as  $\{ , \}'_* - \{ , \}_*$ , where  $* \in \{\dashv, \vdash\}$ .

We remark here that if  $\bar{\phi}, \bar{\phi}' \in CY^3(D, D)$  are the cochains corresponding to  $\{, \}_*, \{, \}'_*$  in the sense of the construction above, then

$$\bar{\phi}' - \bar{\phi} = \delta\psi.$$

Let  $\mathbb{O}_\lambda(f) \in HY^3(D, D)$  be the cohomology class of the cochain  $\bar{\phi}$ . We define the following linear map.

$$\mathbb{O}_\lambda : H_{Harr}^2(A, K) \longrightarrow HY^3(D, D), f \mapsto \mathbb{O}_\lambda(f).$$

We thus make the following proposition.

**Proposition 6.1.** *The deformation  $\lambda$  with base  $A$  can be extended to a deformation of the dialgebra  $D$  with base  $B$  if and only if  $\mathbb{O}_\lambda(f) = 0$ .*

□

The cohomology class  $\mathbb{O}_\lambda(f)$  is called the *obstruction* to the extension of the deformation  $\lambda$  from  $A$  to  $B$ .

## 7. EXTENDIBLE DEFORMATIONS

Let  $A$  be a finite dimensional commutative, unital, local algebra with a fixed augmentation  $\epsilon$ , and let  $\lambda$  be a deformation of a dialgebra  $D$  with base  $A$ . Let  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  be an extension of  $A$ , corresponding to a cohomology class  $f \in H_{Harr}^2(A; K)$ . Following the same arguments as in [3], we can state the following proposition.

**Proposition 7.1.**  *$HY^2(D, D)$  operates transitively on the set of equivalence classes of deformations  $\mu$  of the dialgebra  $D$  with base  $B$  such that  $p_*\mu = \lambda$ .*

□

We remark here that the group of automorphisms of the extension  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  is  $H_{Harr}^1(A; K)$ , (5.7) and  $H_{Harr}^1(A; K) = (\mathfrak{m}/\mathfrak{m}^2)' = TA$ , (5.3). Note that by 4.3, there exists a map  $d\lambda : TA \rightarrow HY^2(D, D)$ . The group of automorphisms of the extension  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  operates on the set of equivalence classes of deformations  $\mu$  such that  $p_*\mu = \lambda$ .

We have the next proposition, the proof of which is straightforward.

**Proposition 7.2.** *The operation of  $HY^2(D, D)$  on the set of equivalence classes of deformations  $\mu$  such that  $p_*\mu = \lambda$  and the operation of the group of automorphisms of the extension  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  are related by the differential  $d\lambda : TA \rightarrow HY^2(D, D)$ . In other words, if  $r : B \rightarrow B$  determines an automorphism of the extension  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  which corresponds to an element  $h \in H_{Harr}^1(A; K) = TA$ , then for any deformation  $\mu$  of  $D$  with base  $B$  such that  $p_*\mu = \lambda$ , the difference between the push-out  $r_*\mu$  and  $\mu$  is a cocycle of the cohomology class  $d\lambda(h)$ .*

□

**Corollary 7.3.** *Suppose that the differential map  $d\lambda : TA \rightarrow HY^2(D, D)$  is onto. Then the group of automorphisms of the extension  $0 \rightarrow K \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$  operates transitively on the set of equivalence classes of deformations  $\mu$  of  $D$  with base  $B$  such that  $p_*\mu = \lambda$ .*

The proof of the following proposition is an imitation of the proof presented in [4], for Leibniz algebras.

**Proposition 7.4.** *Let  $A_1$  and  $A_2$  be two finite dimensional local algebras with augmentations  $\epsilon_1$  and  $\epsilon_2$ , respectively. Let  $\phi : A_2 \rightarrow A_1$  be an algebra homomorphism with  $\phi(1) = 1$  and  $\epsilon_1 \circ \phi = \epsilon_2$ . Suppose  $\lambda_2$  is a deformation of a dialgebra  $D$  with base  $A_2$  and  $\lambda_1 = \phi_*\lambda_2$  is the push-out via  $\phi$ . Then the following diagram commutes:*

$$\begin{array}{ccc} H_{Harr}^2(A_1; K) & \xrightarrow{\phi^*} & H_{Harr}^2(A_2; K) \\ \theta_{\lambda_1} \searrow & & \swarrow \theta_{\lambda_2} \\ & HY^3(D; D) & \end{array} .$$

**Proof.** Let  $[f_{A_1}] \in H_{Harr}^2(A_1; K)$  correspond to the extension

$$0 \rightarrow K \xrightarrow{i_1} A'_1 \xrightarrow{p_1} A_1 \rightarrow 0.$$

Also, let  $[f_{A_2}] = \phi^*([f_{A_1}]) \in H_{Harr}^2(A_2; K)$  correspond to the extension

$$0 \rightarrow K \xrightarrow{i_2} A'_2 \xrightarrow{p_2} A_2 \rightarrow 0.$$

Let  $q_k : A_k \rightarrow A'_k$  be sections of  $p_k$  for  $k = 1, 2$ . There exist  $K$ -module isomorphisms  $A'_k \cong A_k \oplus K$ . Let  $(b, x)_{q_k}$  denote the inverse of  $(b, x) \in A_k \oplus K$  under the isomorphisms. Define a linear map  $\psi : A'_2 \cong (A_2 \oplus K) \rightarrow A'_1 \cong (A_1 \oplus K)$  by  $\psi((a, x)_{q_2}) = (\phi(a), x)_{q_1}$  for  $(a, x)_{q_2} \in A'_2$ . Thus we have a morphism of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{i_2} & A'_2 & \xrightarrow{p_2} & A_2 & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \psi & & \downarrow \phi & & \\ 0 & \longrightarrow & K & \xrightarrow{i_1} & A'_1 & \xrightarrow{p_1} & A_1 & \longrightarrow & 0. \end{array}$$

Let  $I_k = i_k \otimes id$ ,  $P_k = p_k \otimes id$  and  $E_k = \widehat{\epsilon}_k \otimes id$ , where  $\widehat{\epsilon}_k = \epsilon_k \circ p_k$  for  $k = 1, 2$ . If  $m_{A_k}$  denote the unique maximal ideal of  $A_k$  then  $m_{A'_k} = p_k^{-1}(m_{A_k})$  is the unique maximal ideal of  $A'_k$ . Let the basis of  $m_{A_k}$  and  $m_{A'_k}$  be  $\{m_{k_i}\}_{1 \leq i \leq r_k}$  and  $\{n_{k_i}\}_{1 \leq i \leq r_k+1}$  respectively, for  $k = 1, 2$ . Note that,  $n_{k_j} = (m_{k_j}, 0)_{q_k}$  for  $1 \leq j \leq r_k$  and  $n_{k_{r_k+1}} = (0, 1)_{q_k}$ . The dialgebra products on  $A_2 \otimes D$  is given by

$$\begin{aligned} (1 \otimes x_1) \dashv_{\lambda_2} (1 \otimes x_2) &= 1 \otimes (x_1 \dashv x_2) + \sum_{i=1}^{r_2} m_{2_i} \otimes \psi_i^2([21]; x_1, x_2) \\ (1 \otimes x_1) \vdash_{\lambda_2} (1 \otimes x_2) &= 1 \otimes (x_1 \vdash x_2) + \sum_{i=1}^{r_2} m_{2_i} \otimes \psi_i^2([12]; x_1, x_2) \end{aligned}$$

for  $x_1, x_2 \in D$  and  $\psi_i^2 = \alpha_{\lambda_2, \xi_{2_i}}$ , where  $\{\xi_{2_i}\}$  is the dual basis of  $\{m_{2_i}\}$ .

Let  $\phi(m_{2_i}) = \sum_{j=1}^{r_1} c_{i,j} m_{1_j}$ ,  $c_{i,j} \in K$  for  $1 \leq i \leq r_2$  and  $1 \leq j \leq r_1$ . Then the push-out  $\lambda_1 = \phi_* \lambda_2$  on  $A_1 \otimes D$  is defined by

$$\begin{aligned} (1 \otimes x_1) \dashv_{\lambda_1} (1 \otimes x_2) &= 1 \otimes (x_1 \dashv x_2) + \sum_{i=1}^{r_2} (\sum_{j=1}^{r_1} c_{i,j} m_{1_j}) \otimes \psi_i^2([21]; x_1, x_2) \\ &= 1 \otimes (x_1 \dashv x_2) + \sum_{i=1}^{r_1} m_{1_j} \otimes \psi_j^1([21]; x_1, x_2) \\ (1 \otimes x_1) \vdash_{\lambda_1} (1 \otimes x_2) &= 1 \otimes (x_1 \vdash x_2) + \sum_{i=1}^{r_2} (\sum_{j=1}^{r_1} c_{i,j} m_{1_j}) \otimes \psi_i^2([12]; x_1, x_2) \\ &= 1 \otimes (x_1 \vdash x_2) + \sum_{i=1}^{r_1} m_{1_j} \otimes \psi_j^1([12]; x_1, x_2) \end{aligned}$$

where  $\psi_j^1 \in CY^2(D, D)$  id defined by

$$\begin{aligned} \psi_j^1([21]; x_1, x_2) &= \sum_{i=1}^{r_2} c_{i,j} \psi_i^2([21]; x_1, x_2) \\ \psi_j^1([12]; x_1, x_2) &= \sum_{i=1}^{r_2} c_{i,j} \psi_i^2([12]; x_1, x_2) \end{aligned}$$

for  $x_1, x_2 \in D$ .

For any 2-cochain  $\chi \in CY^2(D, D)$ , let us define  $A'_k$  bilinear operations  $\{, \}_\dashv, k, \{, \}_\vdash, k : (A'_k \otimes D)^{\otimes 2} \longrightarrow A'_k \otimes D$  by lifting  $\lambda_k$ ,

$$\begin{aligned} (1 \otimes x_1) \dashv_k (1 \otimes x_2) &= 1 \otimes (x_1 \dashv x_2) + \sum_{j=1}^{r_k} n_{k_j} \otimes \psi_j^k([21]; x_1, x_2) \\ &\quad + n_{k_{r_k+1}} \chi([21]; x_1, x_2) \\ (1 \otimes x_1) \vdash_k (1 \otimes x_2) &= 1 \otimes (x_1 \vdash x_2) + \sum_{j=1}^{r_k} n_{k_j} \otimes \psi_j^k([12]; x_1, x_2) \\ &\quad + n_{k_{r_k+1}} \chi([12]; x_1, x_2) \end{aligned}$$

for  $k = 1, 2$  and  $x_1, x_2 \in D$ . The operations  $\{, \}_\dashv, k, \{, \}_\vdash, k$ , for  $k = 1, 2$  satisfy the conditions (i) and (ii) of 6.0.1.

We shall show that  $\psi \otimes id : A'_2 \otimes D \longrightarrow A'_1 \otimes D$  preserves the liftings. It is enough to show that

$$(\psi \otimes id)(1 \otimes x_1 *_2 1 \otimes x_2) = \psi \otimes id(1 \otimes x_1) *_1 \psi \otimes id(1 \otimes x_2),$$

where  $* \in \{\dashv, \vdash\}$  and  $x_1, x_2 \in D$ .

Now

$$\begin{aligned} (\psi \otimes id)(1 \otimes x_1 \dashv_2 1 \otimes x_2) &= \psi(1) \otimes (x_1 \dashv x_2) \\ &\quad + \sum_{j=1}^{r_2} \psi(1) \psi(n_{2_j}) \otimes \psi_j^2([21]; x_1, x_2) + \psi(1) \psi(n_{2_{r_2+1}}) \otimes \chi([21]; x_1, x_2) \\ &= 1 \otimes (x_1 \dashv x_2) + \sum_{j=1}^{r_2} \left( \sum_{i=1}^{r_1} c_{j,i} m_{1_i} \right) \otimes \psi_j^2([21]; x_1, x_2) + n_{1_{r_1+1}} \otimes \chi([21]; x_1, x_1), \end{aligned}$$

where we used that  $\phi(m_{2_j}) = \sum_{i=1}^{r_1} c_{j,i} m_{1_i}$  and

$$\psi(n_{2_{r_2+1}}) = \psi((0, 1)_{q_2}) = (\phi(0), 1)_{q_1} = n_{1_{r_1+1}}.$$

Simplifying, we conclude that

$$\begin{aligned}
& (\psi \otimes id)(1 \otimes x_1 \dashv_2 1 \otimes x_2) \\
&= \psi(1) \otimes (x_1 \dashv x_2) + \sum_{i=1}^{r_1} \psi(1)m_{1_i} \otimes \psi_i^1([21]; x_1, x_2) + \psi(1)n_{1_{r_1+1}} \otimes \chi([21]; x_1, x_2) \\
&= (\psi(1) \otimes x_1) \dashv_1 (\psi(1) \otimes x_2) \\
&= \psi \otimes id(1 \otimes x_1) \dashv_1 \psi \otimes id(1 \otimes x_2).
\end{aligned}$$

Similarly, we can show that

$$(\psi \otimes id)(1 \otimes x_1 \vdash_2 1 \otimes x_2) = \psi \otimes id(1 \otimes x_1) \vdash_1 \psi \otimes id(1 \otimes x_2).$$

Let  $\phi_k$  be defined by the operations  $\{, \}_-, \{, \}_+, k$  as has been defined in 6.0.3 and  $\overline{\phi}_k$  the corresponding cocycle as in 6.0.8. Since,  $\psi(n_{2(r_2+1)}) = n_{1(r_1+1)}$ , we have  $[\overline{\phi}_2] = [\overline{\phi}_1]$ . Therefore,

$$\theta_{\lambda_1}([f_{A_1}]) = [\overline{\phi}_1] = [\overline{\phi}_2] = \theta_{\lambda_2}([f_{A_2}]) = \theta_{\lambda_2} \circ \phi^*([f_{A_1}]).$$

Hence,  $\theta_{\lambda_1} = \theta_{\lambda_2} \circ \phi^*$ .  $\square$

## 8. CONSTRUCTION OF A MINIVERSAL DEFORMATION OF A DIALGEBRA

An explicit description of the construction of a versal deformation of a Lie algebra is given in [3], and of a Leibniz algebra is given in [4]. Here we sketch the construction, for the case of a dialgebra, following the same techniques developed in [3], [4].

Start with a dialgebra  $D$  with  $\dim(HY^2(D, D)) < \infty$ . Consider the extension

$$0 \longrightarrow HY^2(D, D)' \xrightarrow{i} C_1 \xrightarrow{p} C_0 \longrightarrow 0,$$

where  $C_0 = K$ ,  $C_1 = K \oplus HY^2(D, D)'$ . Let  $\eta_1$  denote the universal infinitesimal deformation with base  $C_1$  as described in Section 4.

Suppose for some  $k \geq 1$ , we have constructed a finite dimensional local algebra  $C_k$ , and a deformation  $\eta_k$  of  $D$  with base  $C_k$ . Let

$$\mu : H_{Harr}^2(C_k; K) \longrightarrow (Ch_2(C_k))'$$

be a homomorphism mapping a cohomology class into a cocycle representing the class. The dual map of  $\mu$

$$f_{C_k} : Ch_2(C_k) \longrightarrow H_{Harr}^2(C_k; K)'$$

corresponds to the following extension of  $C_k$ :

$$0 \longrightarrow H_{Harr}^2(C_k; K)' \xrightarrow{\bar{i}_{k+1}} \overline{C}_{k+1} \xrightarrow{\bar{p}_{k+1}} C_k \longrightarrow 0.$$

The obstruction  $\theta([f_{C_k}]) \in H_{Harr}^2(C_k; K)' \otimes HY^3(D, D)$  yields a map  $\omega_k : H_{Harr}^2(C_k; K) \longrightarrow HY^3(D, D)$ , by adjunction property of tensor products, with the dual map

$$\omega'_k : HY^3(D, D)' \longrightarrow H_{Harr}^2(C_k; K)'.$$

This induces the following extension

$$0 \longrightarrow \text{coker}(\omega'_k) \longrightarrow \overline{C}_{k+1} / \bar{i}_{k+1} \circ \omega'_k(HY^3(D, D)') \longrightarrow C_k \longrightarrow 0.$$

This yields the extension

$$0 \longrightarrow (\ker(\omega_k))' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p_{k+1}} C_k \longrightarrow 0$$

where  $C_{k+1} = \overline{C}_{k+1}/\overline{i}_{k+1} \circ \omega'_k(HY^3(D, D)')$  and  $i_{k+1}, p_{k+1}$  are the mappings induced by  $\overline{i}_{k+1}$  and  $\overline{p}_{k+1}$  respectively. Along the same lines as in [3], [4] we have the following proposition:

**Proposition 8.1.** *The deformation  $\eta_k$  with base  $C_k$  of a dialgebra  $D$  admits an extension to a deformation with base  $C_{k+1}$  which is unique up to an isomorphism and an automorphism of the extension*

$$0 \longrightarrow (\ker(\omega_k))' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p_{k+1}} C_k \longrightarrow 0.$$

□

This process gives rise to a sequence of finite dimensional local algebras  $C_k$  and deformations  $\eta_k$  of the dialgebra  $D$  with base  $C_k$

$$K \xleftarrow{p_1} C_1 \xleftarrow{p_2} C_2 \xleftarrow{p_3} \dots \xleftarrow{p_k} C_k \xleftarrow{p_{k+1}} C_{k+1} \dots$$

such that  $p_{k+1} * \eta_{k+1} = \eta_k$ . By taking the projective limit we obtain a formal deformation  $\eta$  of  $D$  with base  $C = \varprojlim_{k \rightarrow \infty} C_k$ .

Let  $\dim(HY^2(D, D)) = n$  and  $K[[HY^2(D, D)']]$  denote the formal power series ring in  $n$  variables. Also let  $m$  denote the unique maximal ideal in  $K[[HY^2(D, D)']]$ , consisting of all elements with constant term zero. We have the following proposition, whose proof can be found in [4].

**Proposition 8.2.** *The complete local algebra  $C = \varprojlim_{k \rightarrow \infty} C_k$  can be described as*

$$C \cong K[[HY^2(D, D)']]/I,$$

where  $I$  is an ideal contained in  $m^2$ .

□

Along the same lines as in [3], [4], we state the following theorem, proof of which obeys the same techniques as developed in [3].

**Theorem 8.3.** *Let  $D$  be a dialgebra with  $\dim(HY^2(D, D)) < \infty$ . Then the formal deformation  $\eta$  with base  $C$  as described above is a miniversal deformation of  $D$ .*

□

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