Algebr Represent Theor (2015) 18:477–490
DOI 10.1007/s10468-014-9505-8

Representations of Leibniz Algebras

A. Fialowski · É. Zs. Mihálka

Received: 16 August 2014 / Accepted: 30 September 2014 / Published online: 10 October 2014
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Abstract In this paper we prove that every irredicuble representation of a Leibniz algebra can be obtained from irreducible representations of the semisimple Lie algebra from the Levi decomposition. We also prove that - in general - for (semi)simple Leibniz algebras it is not true that a representation can be decomposed to a direct sum of irreducible ones.

Keywords Leibniz algebra · Semisimple algebra · Representation · Irreducible representation

Mathematics Subject Classification (2010) Primary 17A32 · Secondary 17B10

1 Introduction

The notion of Leibniz algebra was introduced by A.M. Bloh ([7, 8] in the 1960-s, and later rediscovered and developed by J-L. Loday ([9]) in 1993. Since then it became very popular, mainly because of its applications in physics. A number of theorems for Lie algebras were generalized for Leibniz algebras, like Lie's Theorem, Engel's Theorem, Cartan's criterium, Levi's Theorem ([3–5, 12]), but some other results do not hold for Leibniz algebras.

Presented By Anne Schilling.

A. Fialowski (✉) · É. Zs. Mihálka
Institute of Mathematics, Eötvös Loránd University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary
e-mail: fialowsk@cs.elte.hu

É. Zs. Mihálka
e-mail: mihalka@coulson.chem.elte.hu

Representations of Leibniz algebras were introduced in [11]. Beside that, there is a recent result on faithful representations ([6]).

In this paper we consider representations of semisimple Leibniz algebras, and study, what can be carried over from the known Theorems in the Lie case. We prove that every irreducible representation of a Leibniz algebra L can be obtained from irreducible representations of the semisimple Lie algebra S from the decomposition $L = S \dot{+} I$, where I is the Leibniz kernel, S is a semisimple Lie algebra which is a subalgebra in L. We also prove that for (semi)simple Leibniz algebras it is not true in general that a representation can be decomposed to a direct sum of irreducible ones, by giving a counterexample.

The authors thank the referee for the useful comments.

2 Basic Definitions

For basic definitions and properties of Leibniz algebras we refer to [1, 2, 9, 11].

Definition 2.1 Let K be a field. An algebra $(L, [\cdot, \cdot])$ is called a *Leibniz algebra* over K, if for every $x, y, z \in L$ we have the Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

Obviously, every Lie algebra is a Leibniz algebra as well.

Definition 2.2 We call a K-linear map $d : L \to L$ a *derivation*, if

$$d[x, y] = [dx, y] + [x, dy] \ (x, y \in L).$$

Denote by $Der_K(L)$ the algebra of all derivations of L.

If we define the product as the bracket operation, $Der_K(L)$ becomes a Lie algebra. The Leibniz identity in L means that for every $x \in L$, $r_x := [., x] \in Der_K(L)$ is an inner derivation.

Theorem 2.3 *$Inn(L) := \{r_x | x \in L\}$ forms a Lie algebra and $Inn(L) \triangleleft Der(L)$.*

Because of this property, these Leibniz algebras are also called right Leibniz algebras. If we define the Leibniz bracket, assuming that the left multiplication should be a derivation of L, then we call such algebras left Leibniz algebras. (Of course, one can state every analogous property for left Leibniz algebras as well.)

Definition 2.4 The *Leibniz kernel* of a Leibniz algebra L is $I = span\{x^2 | x \in L\}$, where $x^2 = [x, x]$.

From $[x+y, x+y] \in I$ we get that for every $x, y \in L$ we have $[x, y] + [y, x] \in I$. If $char(K) \neq 2$, then $[x, x] = \frac{1}{2}([x, x] + [x, x])$, so $span\{[x, y] + [y, x] | x, y \in L\} = I$.

Theorem 2.5 *The Leibniz kernel I is a commutative subalgebra, it is also ideal in L, and the factor algebra L/I is a Lie algebra. This is the smallest ideal in L for which the factor is a Lie algebra.*

For Lie algebras the square of every element is 0, so $I = 0$.

Note that if $dim(L) \geq 1$, we have $I \neq L$ for the Leibniz kernel I.

If L is not a Lie algebra ($I \neq 0$), then I is a nontrivial ideal in L, so the definition of simplicity for Lie algebras can not be applied for Leibniz algebras. Instead, the definition is modified as follows:

Definition 2.6 The Leibniz algebra L is *simple*, if $[L, L] \neq I$ and it only has the following three ideals: 0, I, L. (Here 0 and I are not necessarily different.)

Remark 2.7 As for Lie algebras $I = 0$, the new definition of simplicity coincides with the old one.

If L is a simple Leibniz algebra, then L/I is simple Lie algebra, but the opposite is not true.

Definition 2.8 For a Leibniz algebra $(L, [\cdot, \cdot])$, define the composition chains of ideals

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \geq 1, \qquad L^{[1]} = L,\ L^{[n+1]} = \left[L^{[n]}, L^{[n]}\right],\ n \geq 1,$$

Definition 2.9 The Leibniz algebra L is *solvable*, if there exists and integer $n \geq 1$ such that $L^{[n]} = 0$, and L is *nilpotent*, if there exists an integer $k \geq 1$ such that $L^k = 0$.

Theorem 2.10 *For positive integers i, j we have $[L^i, L^j] \subseteq L^{i+j}$. From this it follows that for every $k \geq 2$, $L^{[k]} \subseteq L^{2^{k-1}}$, so every nilpotent Leibniz algebra is solvable.*

For the Leibniz kernel I of a Leibniz algebra, $I^{[2]} = [I, I] = 0$, so I is solvable. If L is simple, then $I \neq [L, L]$. But $[L, L]$ is an ideal in L, and $I \subseteq [L, L]$ for every Leibniz algebra. We get that if L is simple, then the only possibility is $[L, L] = L$. From this it also follows that $L^k = L^{[k]} = L$ $(k \geq 1)$, so L is neither nilpotent nor solvable.

Corollary 2.11 *If L is finite dimensional, it has a maximal solvable ideal R, which we call the radical of L. Also, there exists a maximal nilpotent ideal, containing every nilpotent ideal. We call this the nilradical of L, and denote it by N. Clearly, $N \leq R$.*

Definition 2.12 A Leibniz algebra L is *semisimple*, if its maximal solvable ideal is I.

Obviously, in every case $I \triangleleft R$. So if L is simple, $R = I$ or $R = L$. Because of the solvability of R, $[R, R] \neq R$, which means $[R, R] \subset R$ is a proper ideal. If $R = L$, then $I \subseteq [L, L] = [R, R] \subset R = L$. In L, except $0, I, L$ there are no other ideals, so we would get $[L, L] = I$, which contradicts the simplicity of L. So for L we have $R = I$, which means that from simplicity it follows semisimplicity.

If L is a Lie algebra, again we get that the two definitions of semisimplicity coincide.

A Leibniz algebra L is semisimple if and only if the factor algebra L/I is a semisimple Lie algebra.

We know the following:

Theorem 2.13 (Theorem Levi) [4] *Let L be a finite dimensional Leibniz algebra over K, where $char(K) = 0$. Let $R \triangleleft L$ its solvable radical. Then there exists a semisimple subalgebra $S \leq L$ such that $L = S + R$ and $S \cap R = 0$, so $L = S \dot{+} R$. This subalgebra S is a semisimple Lie algebra.*

Using this Theorem we get the following:

Corollary 2.14 *Let L be a semisimple Leibniz algebra. Then $L = S \dot{+} I$, where I is the Leibniz kernel, $S \leq L$ and S is a semisimple Lie algebra.*

Indeed, as L is semisimple, $R = I$. Then $L = S \dot{+} R$ and $S \cong L/I$, which means S is a semisimple Lie algebra.

Definition 2.15 Let L be a Leibniz algebra, M a vector space over the field K. Assume we have two K-linear functions:

$\lambda : L \to \mathfrak{gl}(M)$ and

$\rho : L \to \mathfrak{gl}(M)$.

Denote $\lambda(x)$ and $\rho(y)$ by λ_x and ρ_y for every $x, y \in L$. We say that M is a *representation* of L, if the following properties are satisfied:

(1) $\rho_{[x,y]} = \rho_y \rho_x - \rho_x \rho_y$,
(2) $\lambda_{[x,y]} = \rho_y \lambda_x - \lambda_x \rho_y$,
(3) $\lambda_{[x,y]} = \rho_y \lambda_x + \lambda_x \lambda_y$ for every $x, y \in L$.

If M is a representation of L, then M becomes an L-module with the following $[\cdot, \cdot] : M \times L \to M$ and $[\cdot, \cdot] : L \times M \to M$ with the products: $[m, x] := \rho_x(m)$ and $[x, m] := \lambda_x(m)$ for every $x \in L, m \in M$.

Conversely, for a given L-module M, we get the representation $\rho : L \to \mathfrak{gl}(M)$, $\lambda : L \to \mathfrak{gl}(M)$ with

$$\rho_x := [\cdot, x]\ \forall x \in L, \lambda_x := [x, \cdot]\ \forall x \in L.$$

Let L be a Leibniz-algebra, M a vector space over K, and $\rho, \lambda : L \to \mathfrak{gl}(M)$ a representation of L (M is an L-bimodule). Denote for $x \in L$ $\rho(x) = [., x] \in \mathfrak{gl}(M)$ and $\lambda(x) = [x, .] \in \mathfrak{gl}(M)$ the right and left multiplication by x.

Definition 2.16 Let $\rho_1, \lambda_1 : L \to \mathfrak{gl}(M_1)$ and $\rho_2, \lambda_2 : L \to \mathfrak{gl}(M_2)$ be two Leibniz representations. These two representations are *equivalent*, if there exists an isomorphism $\varphi : M_1 \to M_2$ such that $\varphi \circ \rho_1(x) = \rho_2(x) \circ \varphi$ ($\forall x \in L$), and $\varphi \circ \lambda_1(x) = \lambda_2(x) \circ \varphi$ ($\forall x \in L$).

Define now the map $\psi : \mathfrak{gl}(M_1) \to \mathfrak{gl}(M_2)$ as follows: for $f \in \mathfrak{gl}(M_1)$ let $\psi(f) := \varphi \circ f \circ \varphi^{-1}$. We get the isomorphism $\mathfrak{gl}(M_1) \to \mathfrak{gl}(M_2)$ with the property that the two diagrams commute:

$$\begin{tikzcd} L \arrow[r, "\rho_1"] \arrow[dr, "\rho_2"'] & \mathfrak{gl}(M_1) \arrow[d, "\psi"] \\ & \mathfrak{gl}(M_2) \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "\lambda_1"] \arrow[dr, "\lambda_2"'] & \mathfrak{gl}(M_1) \arrow[d, "\psi"] \\ & \mathfrak{gl}(M_2) \end{tikzcd} \tag{1}$$

If L is a Lie algebra, then its Lie representation $\varphi : L \to \mathfrak{gl}(M)$ becomes a Leibniz-representation with $\lambda := \varphi$ and $\rho := -\varphi$. (One can also take the choice $\lambda := 0$ and $\rho := -\varphi$.)

If a Lie algebra L has two equivalent Lie representations φ_1, φ_2 (with the isomorphism $\varphi : M_1 \to M_2$), then, using the above construction to form Leibniz representations, they will be equivalent as well (with the isomorphism φ).

Let L be a Leibniz algebra and $\rho_i, \lambda_i : L \to \mathfrak{gl}(M_i)$ ($i = 1, 2$) two Leibniz-representations of L. Assume that these representations are isomorphic. For a given $x \in L$ assume that $0 \neq v \in M_1$ is an eigenvector of the map $\rho_1(x)$ for the eigenvalue α, $\rho_1(x)(v) = \alpha v$. Then $\varphi(v) \neq 0$ and $\rho_2(x)(\varphi(v)) = \varphi(\rho_1(x)(v)) = \varphi(\alpha v) = \alpha\varphi(v)$, so α is an eigenvalue of $\rho_2(x)$, and $\varphi(v)$ is an eigenvector for α.

Definition 2.17 Let K be a field, L a Leibniz algebra over K, and V a vector space over K. We say that the Leibniz representation $\rho, \lambda : L \to \mathfrak{gl}(V)$ is *irreducible*, if ρ and λ are irreducible. In other words, if $U \leq V$ is an invariant subspace of ρ and λ ($\forall x \in L$, $\rho_x(U) \subseteq U$ and $\lambda_x(U) \subseteq U$), then $U = 0$ or $U = V$.

Remark 2.18 It is clear that every Leibniz algebra has irreducible representations, e.g. the one-dimensional ones (which always exist, e.g. the trivial one $\lambda = \rho = 0$). On the other hand, if V_n is an n-dimensional module, there are two possibilities: either it is simple, or

there exists a submodule of smaller dimension. As the dimension is finite, we get a simple module in finite steps.

Definition 2.19 We say that the representation $\rho, \lambda : L \to \mathfrak{gl}(V)$ is the *direct sum* of lower dimensional ones, if V can be written as the direct sum of ρ- and λ-invariant subspaces $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ such that for every $x \in L$, $\rho_x(V_i) \subseteq V_i$ and $\lambda_x(V_i) \subseteq V_i$ $(i = 1, \ldots, k)$.

If a Leibniz algebra is not semisimple, then already in the 2-dimensional case the study of its representations is very complicated.

Let us now deal with the semisimple case, and study what can be carried over from the statements for Lie algebras to Leibniz algebras.

3 Irreducible Representations of Leibniz Algebras

3.1 Irreducible Representations as Lie Representations

Theorem 3.1 *Let L be a Leibniz algebra, M a vector space and $\lambda, \rho : L \to \mathfrak{gl}(M)$ an irreducible representation of L. Then the Leibniz representation can be essentially viewed as Lie representation. We mean that either $\rho+\lambda = 0$ (multiplication by x is anticommutative on M), or $\lambda = 0$ (and so defining $\varphi := -\rho$ we get a Lie representation).*

Proof Let $V := span\{[y, m] + [m, y] | y \in L, m \in M\}$. Then

$$0 \leq V \leq \bigcap_{x \in L} Ker([x, .]) \leq M,$$

and V is invariant subspace for ρ and λ. The first inclusion is obvious, for the second let $x, y \in L, m \in M$ be arbitrary, then using the properties of Leibniz representation, we have $[x, [y, m] + [m, y]] = [[x, y], m] - [[x, m], y] + [[x, m], y] - [[x, y], m] = 0$ and we proved the second inclusion. This also proves the invariance for λ.

The invariant property for ρ is also clear: let again $x, y \in L, m \in M$ arbitrary. $[[y, m] + [m, y], x] = [[y, x], m] + [y, [m, x]] + [[m, x], y] + [m, [y, x]] = ([[y, x], m] + [m, [y, x]]) + ([y, [m, x]] + [[m, x], y]) \in V$, because $[y, x] \in L$ and $[m, x] \in M$.

We showed that indeed V is an invariant subspace for ρ and λ. On the other hand, we assumed that the representation is irreducible, so there are two possible cases:

(1) $V = 0$. Then for a given $y \in L$, $[y, m] + [m, y] = 0 \; \forall m \in M$, so $\lambda(y) + \rho(y) = 0$. As y was arbitrary, we get $\lambda + \rho = 0$.

(2) $V = M$. Then $M = V = \bigcap_{x \in L} Ker([x, .]) = M$, so $\forall x \in L$, $Ker([x, .]) = M$. But this exactly means that for every $x \in L$, $\lambda(x) = 0$, in other words, $\lambda = 0$. Then by the properties of a Leibniz representation

$\rho([x, y]) = \rho(y)\rho(x) - \rho(x)\rho(y)$, for the linear map $\varphi := -\rho$, $\varphi([x, y]) = [\varphi(x), \varphi(y)]$, so φ is a Lie representation.

□

Remark 3.2 Using the terminology of [10], Theorem 3.1 states that for Leibniz algebras only symmetric or antisymmetric representations can be irreducible.

Corollary 3.3 *The existence of an invariant subspace depends on the irreducibility of ρ, so a Leibniz representation (ρ, λ) is irreducible exactly when ρ is irreducible.*

As a consequence, for a Leibniz representation not being a Lie representation in the above sense, it is necessary that ρ is not irreducible.

On the other hand, it is easy to show that for a representation not being Lie, it is not sufficient that ρ is not irreducible. If $dim(M) \geq 2$, then with the choice $\rho = \lambda = 0$ we clearly get a Lie representation, but because of $\rho = 0$, every subspace is invariant, and because of $dim(M) \geq 2$, there exists a proper nontrivial subspace.

3.2 Leibniz Representations of the Algebra $\mathfrak{sl}_2$

Computing Leibniz representations of the Lie algebra $\mathfrak{sl}_2$ is straightforward.

Let V be a finite dimensional complex vector space and denote $m + 1 := dim_{\mathbb{C}}(V)$. Let $\rho, \lambda : \mathfrak{sl}_2 \to \mathfrak{gl}(V)$ a Leibniz representation of the algebra $\mathfrak{sl}_2$. By the multiplication table the necessary conditions are the following:

(1) $\rho_h = \rho_f\rho_e - \rho_e\rho_f$
(2) $2\rho_e = \rho_h\rho_e - \rho_e\rho_h$
(3) $2\rho_f = \rho_f\rho_h - \rho_h\rho_f$
(4) $\lambda_h = \rho_f\lambda_e - \lambda_e\rho_f = \lambda_f\rho_e - \rho_e\lambda_f$
(5) $\lambda_h = \rho_f\lambda_e + \lambda_e\lambda_f = -\lambda_f\lambda_e - \rho_e\lambda_f$
(6) $0 = \rho_h\lambda_h - \lambda_h\rho_h = \rho_h\lambda_h + \lambda_h^2$
(7) $2\lambda_e = \rho_h\lambda_e - \lambda_e\rho_h = \lambda_h\rho_e - \rho_e\lambda_h$
(8) $2\lambda_e = \rho_h\lambda_e + \lambda_e\lambda_h = -\lambda_h\lambda_e - \rho_e\lambda_h$
(9) $0 = \rho_e\lambda_e - \lambda_e\rho_e = \rho_e\lambda_e + \lambda_e^2$
(10) $2\lambda_f = \rho_f\lambda_h - \lambda_h\rho_f = \lambda_f\rho_h - \rho_h\lambda_f$
(11) $2\lambda_f = \rho_f\lambda_h + \lambda_h\lambda_f = -\lambda_f\lambda_h - \rho_h\lambda_f$
(12) $0 = \rho_f\lambda_f - \lambda_f\rho_f = \rho_f\lambda_f + \lambda_f^2$

Let us restrict ourselves to the irreducible case, assuming that ρ is irreducible. If we concentrate on the first three conditions, we see that ρ by itself gives a Lie representation of $\mathfrak{sl}_2$, or more precisely its (-1) multiple. On the other hand, we know that for every m, $\mathfrak{sl}_2$ has a unique (up to equivalence) $(m + 1)$-dimensional irreducible representation, and in an appropriate basis, we know the matrices corresponding to the elements e, f, h. From this we

get that in an appropriate basis of V, ρ_e, ρ_f, ρ_h has the following form ($1 \le i, j \le m+1$):

$$(\rho_e)_{i,j} = \begin{cases} i(m+1-i), & \text{if } j = i+1 \\ 0, & \text{if } j \neq i+1 \end{cases} \qquad (\rho_f)_{i,j} = \begin{cases} -1, & \text{if } j = i-1 \\ 0, & \text{if } j \neq i-1 \end{cases}$$

$$(\rho_h)_{i,j} = \begin{cases} m+2-2i, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$$

In matrix form:

$$\rho_e = \begin{pmatrix} 0 & m & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2(m-1) & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 3(m-2) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 2(m-1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & m \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$\rho_f = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

$$\rho_h = \begin{pmatrix} m & 0 & 0 & \cdots & 0 \\ 0 & m-2 & 0 & \cdots & 0 \\ 0 & 0 & m-4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -m \end{pmatrix}$$

Let $(\lambda_f)_{i,j}$ be the element of the i-th row and j-th column of λ_f. Easy to compute that $(\lambda_f\rho_h - \rho_h\lambda_f)_{i,j} = 2(i-j)(\lambda_f)_{i,j} \overset{10.}{=} 2(\lambda_f)_{i,j}$, from where we get $(\lambda_f)_{i,j} = 0$, if $j \neq i-1$. Also

$$(\lambda_f\rho_f)_{i,j} = \begin{cases} -(\lambda_f)_{i,i-1}, & \text{if } 3 \le i \le m+1 \text{ and } j = i-2 \\ 0 & \text{otherwise} \end{cases}$$

$$(\rho_f\lambda_f)_{i,j} = \begin{cases} -(\lambda_f)_{i-1,i-2}, & \text{if } 3 \le i \le m+1 \text{ and } j = i-2 \\ 0 & \text{otherwise} \end{cases}$$

By equation (12):
$(\lambda_f\rho_f)_{i,j} = (\rho_f\lambda_f)_{i,j}$, from where $(\lambda_f)_{i,i-1} = (\lambda_f)_{j,j-1}$ $(2 \le i, j \le m+1)$. That means $\lambda_f = a\rho_f$. By the other term of equation (12), $0 = \rho_f\lambda_f + \lambda_f^2 = \left(a + a^2\right)\rho_f$, and as $\rho_f \neq 0$, we get $a + a^2 = 0$. That means $a = 0$ or $a = -1$.

(1) If $a = 0$, then $\lambda_f = 0$, so by equation (4) we get $\lambda_h = 0$, and by (7) we get $\lambda_e = 0$.
(2) If $a = -1$, so $\lambda_f = -\rho_f$, then $\lambda_h \overset{4.}{=} \lambda_f\rho_e - \rho_e\lambda_f = \rho_e\rho_f - \rho_f\rho_e \overset{1.}{=} -\rho_h$. Also $\lambda_e \overset{7.}{=} \frac{1}{2}(\lambda_h\rho_e - \rho_e\lambda_h) = \frac{1}{2}(\rho_e\rho_h - \rho_h\rho_e) \overset{2.}{=} -\rho_e$. We get that $\lambda = -\rho$.

It follows that these two representations are irreducible, as $\lambda = -\rho$ where ρ was chosen to be an irreducible representation of $\mathfrak{sl}_2$.

Summarizing: for every $m \in \mathbb{N}$, the algebra $\mathfrak{sl}_2$ has - up to equivalence - exactly two irreducible Leibniz representations of dimension $m+1$, each of these are Lie representations in the previous sense. The two Leibniz-representations are as follows.

(1) In appropriate basis, $\lambda = 0$, and for $1 \le i, j \le m+1$,

$$(\rho_e)_{i,j} = \begin{cases} i(m+1-i), & \text{if } j = i+1 \\ 0, & \text{if } j \neq i+1 \end{cases},$$

$$(\rho_f)_{i,j} = \begin{cases} -1, & \text{if } j = i-1 \\ 0, & \text{if } j \neq i-1 \end{cases},$$

$$(\rho_h)_{i,j} = \begin{cases} m+2-2i, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}.$$

(2) In appropriate basis, $\lambda = -\rho$, and for $(1 \le i, j \le m+1)$,

$$(\rho_e)_{i,j} = \begin{cases} i(m+1-i), & \text{if } j = i+1 \\ 0, & \text{if } j \neq i+1 \end{cases}$$

$$(\rho_f)_{i,j} = \begin{cases} -1, & \text{if } j = i-1 \\ 0, & \text{if } j \neq i-1 \end{cases}$$

$$(\rho_h)_{i,j} = \begin{cases} m+2-2i, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$$

These two irreducible representations are obviously not equaivalent to each other.

We know that the extensions of dimension at least 5 of $\mathfrak{sl}_2$ can be obtained as follows.

Let L be a simple n-dimensional Leibniz algebra ($n \ge 5$), for which the Lie factor L/I is isomorphic to $\mathfrak{sl}_2$. We know that in this case there exists a basis $\{e, f, h, x_0, x_1, \ldots, x_{n-4}\}$ of L, in which the multiplication table is as follows: ([13])

$$\begin{aligned}
&[x_k, h] = (n-4-2k)x_k, \ (0 \le k \le n-4)\\
&[x_k, f] = x_{k+1}, \ (0 \le k \le n-5)\\
&[x_k, e] = k(k+3-n)x_{k-1}, \ (1 \le k \le n-4)\\
&[e, h] = -[h, e] = 2e, \ [h, f] = -[f, h] = 2f,\\
&[e, f] = -[f, e] = h
\end{aligned}$$

It is easy to compute its irreducible representations. Let $dim_{\mathbb{C}}(V) = m+1$, and search for possible homomorphisms $\rho, \lambda :\to \mathfrak{gl}(V)$. We know that a representation can be restricted to $\mathfrak{sl}_2$, and the restriction of ρ is irreducible. This way ρ_e, ρ_f, ρ_h and $\lambda_e, \lambda_f, \lambda_h$ are given by the previous rules. We would like to determine the images of the elements x_i. We can distinguish two cases:

(1) n is odd. By the equality $(n-4)\rho_{x_0} = \rho_{[x_0,h]} = \rho_h\rho_{x_0} - \rho_{x_0}\rho_h$ we get

$$\begin{aligned}(n-2)(\rho_{x_0})_{i,j} &= 2(j-i)(\rho_{x_0})_{i,j}\\ (n-2+2j-2i)(\rho_{x_0})_{i,j} &= 0.\end{aligned}$$

As n is odd, $(n-2+2i-2j) \neq 0$, so $(\rho_{x_0})_{i,j} = 0$ for every pair i, j. We get $\rho_{x_0} = 0$.
Using induction, we can see that $\rho_{x_k} = 0$. For $k = 0$ we have this, and using the inductional assumption we get

$$\rho_{x_{k+1}} = \rho_{[x_k,f]} = \rho_f\rho_{x_k} - \rho_{x_k}\rho_f = 0 - 0 = 0.$$

Apply now the above argument, replacing ρ_{x_k} by λ_{x_k}, and use $\lambda_{[x,y]} = \rho_y\lambda_x - \lambda_x\rho_y$. We get that in this case $\rho_{x_k} = \lambda_{x_k} = 0$, if $0 \leq k \leq n-4$.

(2) n is even. Then let $n-4 = 2s$. In this case $[x_s, h] = (n-4-2s)x_s = 0$.
From here by $0 = \rho_{[x_s,h]} = \rho_h\rho_{x_s} - \rho_{x_s}\rho_h$ we get that ρ_{x_s} must be diagonal. Denote

$$(\rho_{x_s})_{i,i} = b_i,\ (1 \leq i \leq m+1).$$

On the other hand,

$$-s(s+1)\rho_{x_{s-1}} = \rho_{[x_s,e]} = \rho_e\rho_{x_s} - \rho_{x_s}\rho_e,$$

which gives $(\rho_{x_{s-1}})_{i,j} = 0$, if $j \neq i+1$, and

$$a_{i,i+1} = (\rho_{x_{s-1}})_{i,i+1} = \frac{(b_{i+1}-b_i)i(m+1-i)}{s(s+1)},$$

if $1 \leq i \leq m$. As $[x_{s-1}, x_l] = 0$, we get

$$0 = \rho_{x_s}\rho_{x_{s-1}} - \rho_{x_{s-1}}\rho_{x_s},$$

which means $(b_{i+1}-b_i)a_{i,i+1} = 0$, if $1 \leq i \leq m$.
So

$$0 = (b_{i+1}-b_i)a_{i,i+1} = (b_{i+1}-b_i)\frac{(b_{i+1}-b_i)i(m+1-i)}{s(s+1)},$$

from where $b_1 = b_2 = \cdots = b_{m+1}$, and so $\rho_{x_{s-1}} = 0$. For $k \geq s$, using the product $[x_{k-1}, f] = x_k$, by induction we get $\rho_{x_k} = 0$.
For $k \leq s-1$ using the product $[x_k, e] = k(k+3-n)x_{k-1}$ we get by induction, that $k(k+3-n)\rho_{x_{k-1}} = 0$, and as $1 \leq k \leq s-1 \leq n-4$, so $k(k+3-n) \neq 0$, which means $\rho_{x_{k-1}} = 0$.

We showed for every k $(0 \leq k \leq n-4)$ that $\rho_{x_k} = 0$.

For defining the values of λ, consider the expression $0 = \lambda_{[x_s,h]} = \rho_h\lambda_{x_s} - \lambda_{x_s}\rho_h$. We get that λ_{x_s} is diagonal. Denote

$$(\lambda_{x_s})_{i,i} = c_i,\ (1 \leq i \leq m+1).$$

On the other hand,

$$-s(s+1)\lambda_{x_{s-1}} = \lambda_{[x_s,e]} = \rho_e\lambda_{x_s} - \lambda_{x_s}\rho_e,$$

from which $(\lambda_{x_{s-1}})_{i,j} = 0$, if $j \neq i+1$, and

$$d_{i,i+1} = (\lambda_{x_{s-1}})_{i,i+1} = \frac{(c_{i+1}-c_i)i(m+1-i)}{s(s+1)},$$

if $1 \leq i \leq m$. Using $[x_{s-1}, x_l] = 0$, we get

$$\begin{aligned} 0 &= \rho_{x_s}\lambda_{x_{s-1}} + \lambda_{x_{s-1}}\lambda_{x_s} = \lambda_{x_{s-1}}\lambda_{x_s} = \\ &= \rho_{x_{s-1}}\lambda_{x_s} + \lambda_{x_s}\lambda_{x_{s-1}} = \lambda_{x_s}\lambda_{x_{s-1}}. \end{aligned}$$

This means $0 = \lambda_{x_{s-1}}\lambda_{x_s} - \lambda_{x_s}\lambda_{x_{s-1}}$ and

$$0 = (c_{i+1}-c_i)d_{i,i+1} = (c_{i+1}-c_i)\frac{(c_{i+1}-c_i)i(m+1-i)}{s(s+1)}.$$

With the previous argument we get $\lambda_{x_k} = 0$, if $0 \leq k \leq n-4$.

In both cases the final result is that for the elements $\{x_0, \ldots, x_{n-4}\}$, ρ and λ is zero. This means that the Leibniz algebra L has for every m two types of $(m+1)$-dimensional Leibniz representations, such that ρ, restricted to the subalgebra $\mathfrak{sl}_2$ is irreducible. We get these two representations from representations of $\mathfrak{sl}_2$ by chosing $\rho|_I = \lambda|_I = 0$ (here I denotes the Leibniz kernel (generated by the elements x_k^2).

3.3 General Case

Using the results from the previous section, consider irreducible representations of a semisimple Leibniz algebra L. We know that $L = S + I$ as a vector space. As I is generated by the squares of elements in L, for every $x \in I$ there exists $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in K$, and $x_1, \ldots, x_n \in L$ such that $x = \alpha_1[x_1, x_1] + \cdots + \alpha_n[x_n, x_n]$. Using the linearity of ρ and the identity $\rho_{[x,y]} = \rho_y\rho_x - \rho_x\rho_y$ we get:

$\rho_x = \alpha_1\rho_{[x_1,x_1]} + \cdots + \alpha_n\rho_{[x_n,x_n]}$

$\rho_{[x_i,x_i]} = \rho_{x_i}\rho_{x_i} - \rho_{x_i}\rho_{x_i} = 0$, so $\rho_x = 0$ for every $x \in I$.

We have $L = S + I$ as a vector space, so every $y \in L$ can be uniquely written in the form $y = s + x$, where $s \in S$ and $x \in I$. We get $\rho_y = \rho_s + \rho_x = \rho_s$, which means $\rho(L) = \rho(S)$.

As ρ is irreducible, by Theorem 3.1 we get $\lambda = -\rho$ or $\lambda = 0$. In any case, as $\rho|_I = 0$, we must have $\lambda|_I = 0$. Here $(-\rho)|_S$ is an irreducible representation of the semisimple Lie algebra S.

Let us summarize the results.

Theorem 3.4 *Let L be a semisimple Leibniz algebra over the field K, with $char(K) = 0$. Let V be a vectorspace over K. Then L can be written in the form $L = S\dot{+}I$, where I is the Leibniz kernel, S is a semisimple Lie algebra which is a subalgebra in L. Then every irreducible representation $\rho, \lambda : L \to \mathfrak{gl}(V)$ of L can be written in the following form:*

$\rho|_I = \lambda|_I = 0$, $\rho|_S = -\varphi$, where $\varphi : S \to \mathfrak{gl}(V)$ is an irreducible representation of the semisimple Lie algebra S, and $\lambda|_S = (-\rho)|_S$ or $\lambda|_S = 0$.

That means that every irreducible representation of L can be obtained from the irreducible representations of S, and we do not get new, not Lie type representations.

4 Reducible Representations of Semisimple Leibniz Algebras

For semisimple Lie algebras we know that every representation is completely reducible, that means it can be written as a direct sum of irreducible ones. The question is the following. Can this statement be carried over to Leibniz algebras?

If a Leibniz representation splits into the sum of irreducible ones, i.e. $V = V_1 \oplus \cdots \oplus V_k$, then for every i, $(1 \leq i \leq k)$ $\lambda|_{V_i} = 0$ or $\lambda|_{V_i} = (-\rho)|_{V_i}$. As for $x \in I$ (I is the Leibniz kernel) $\rho_x = 0$, then in both cases, $\lambda_x|_{V_i}$=0. As $V = V_1 \oplus \cdots \oplus V_k$, we get $\lambda_x = 0$.

That means if a representation decomposes into a direct sum of irreducible ones, then we must have $\rho|_I = \lambda|_I = 0$.

Example 4.1 Consider the following simple Leibniz algebra: $L = span\{e, f, h, x, y\}$, the nontrivial Leibniz brackets are:

$$[e, f] = -[f, e] = h, [e, h] = -[h, e] = 2e, [f, h] = -[h, f] = -2f,$$
$$[x, h] = -[y, e] = x, [x, f] = -[y, h] = y.$$

For this algebra $I = span\{x, y\}$. Let $V := L$, so consider L as an L-module: for $v, z \in L$, $\rho_z(v) = [v, z]$ and $\lambda_z(v) = [z, v]$.

It is easy to see from the bracket table that $\lambda|_I \neq 0$ and the representation can not be decomposed into irreducible ones. Even more, in this case the only nontrivial invariant subspace is $U = I$, so we indeed did not find any appropriate decomposition. The fact that I is the only nontrivial invariant subspace, follows from the simplicity: the invariant subspaces are in this case exactly the ideals.

We get the following

Theorem 4.2 *For semisimple Leibniz algebras it is not true that a representation of a (semi)simple Leibniz algebra always decomposes into the direct sum of irreducible ones.*

Remark 4.3 Of course, there are cases when the representation can be decomposed into irreducible ones.

Example 4.4 Consider a five-dimensional, *not* irreducible representation of $\mathfrak{sl}_2$:

$$\rho_e = \left(\begin{array}{ccc|cc} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \rho_f = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

$$\rho_h = \left(\begin{array}{ccc|cc} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

Again, starting with the identity (10), we get

$$\lambda_f = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 \end{array}\right), \text{ so } \lambda_f \text{ is block diagonal.}$$

From the identity (4), λ_h, from (7), λ_e are also block diagonal, which means this representation can be written as a direct sum of irreducible ones. The whole point was, that λ_f had block diagonal form. This happened because $(\lambda_f)_{i,j}$=0, if $(\rho_h)_{j,j} - (\rho_h)_{i,i} \neq 2$.

For λ_f not (necessarily) being block diagonal, one needs $(\rho_h)_{j,j} - (\rho_h)_{i,i} = 2$ for such pairs $\{i, j\}$, where i and j are indices of rows which do not belong to the same block along the diagonal.

If for instance the representation has dimension $(2n + 1)$, and we try to decompose it to the sum of two irreducible ones, then one of them has dimension $2m$, in the appropriate block there are odd numbers along the main diagonal of ρ_h, while the other one has dimension $(2n - 2m + 1)$, and in the appropriate block there are even numbers along the main diagonal of ρ_h. So the condition $(\rho_h)_{j,j} - (\rho_h)_{i,i} = 2$ can only be satisfied, if i and j are indices of such rows, which belong to the same block. In this case everything is diagonal, and so the representation can be decomposed into the sum of irreducible ones.

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