

ON THE COHOMOLOGY OF NILPOTENT ALGEBRAS OF FLOWS

UDC 513.836

B. L. FEIGIN AND A. FIALOVSKI [FIALKOWSKI]

Let \mathfrak{g} be a finite-dimensional simple complex Lie algebra, \mathfrak{n} a maximal nilpotent subalgebra of it, \mathfrak{s} the subalgebra normalizing \mathfrak{n} , and $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the algebra of flows corresponding to \mathfrak{g} . Recall that the commutator in $\hat{\mathfrak{g}}$ is defined by the formula

$$[x_1 \otimes f_1, x_2 \otimes f_2] = [x_1, x_2] \otimes f_1 f_2, \quad x_i \in \mathfrak{g}, f_i \in \mathbb{C}[t, t^{-1}].$$

An element of $\hat{\mathfrak{g}}$ can be represented as a rational function $\mathbb{C} \rightarrow \mathfrak{g}$ not having poles in the set $\mathbb{C} \setminus \{0\}$; the commutator of two such functions is computed pointwise. The Lie algebra $\hat{\mathfrak{g}}$ is graded, with $\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes t^i$.

We define some subalgebras of $\hat{\mathfrak{g}}$ that will be important for us. Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ and $\hat{\mathfrak{n}} = \mathfrak{n} \otimes 1 \oplus \mathfrak{g} \otimes t \oplus \mathfrak{g} \otimes t^2 \oplus \dots$. In what follows, \mathfrak{g} will be identified with $\mathfrak{g} \otimes 1 \subset \mathfrak{g}[t] \subset \hat{\mathfrak{g}}$. The Lie algebras $\mathfrak{g}[t]$ and $\hat{\mathfrak{n}}$ inherit a grading.

Here we compute the cohomology $H^*(\hat{\mathfrak{n}}, \hat{\mathfrak{g}})$ (Theorem 1) and $H^i(\hat{\mathfrak{n}}, \hat{\mathfrak{n}})$ for $i = 1, 2$ (Theorem 2). In determining $H^2(\hat{\mathfrak{n}}, \hat{\mathfrak{n}})$ we follow Leger and Luks [1], who computed $H^2(\mathfrak{n}, \mathfrak{n})$.

Let $C^*(\mathfrak{L})$ and $C^*(\mathfrak{L}, V)$ be the standard cohomology complexes of the Lie algebra \mathfrak{L} with trivial coefficients and with coefficients in the module V , respectively. We use analogous notation for complexes of relative cochains of a Lie algebra with respect to a subalgebra (i.e., we write $C(\mathfrak{L}, \mathfrak{L}_1)$, etc.).

1. Computation of $H^*(\hat{\mathfrak{n}}, \hat{\mathfrak{g}})$. We introduce some notation. Let V be a representation of the algebra \mathfrak{g} , A some \mathbb{C} -algebra, and φ a homomorphism $\mathbb{C}[t, t^{-1}] \rightarrow A$. A representation of the algebra $\hat{\mathfrak{g}}$ in the space $V \otimes A$ is given by the formula

$$(x \otimes f)(v \otimes a) = x(v) \otimes \varphi(f)a, \quad x \in \mathfrak{g}, v \in V, f \in \mathbb{C}[t, t^{-1}], a \in A.$$

Two particular cases will be needed: $A = \mathbb{C}[t, t^{-1}]$ with φ the identity mapping of $A = \mathbb{C}$, and $\varphi(f) = f(1)$. In the first case the module $V \otimes A$ will be denoted by \hat{V} , and in the second case by V_1 . The elements of \hat{V} are rational functions $\mathbb{C} \rightarrow V$ that are regular away from the origin of coordinates. The mapping assigning to a function $\mathbb{C} \rightarrow V$ its value at 1 gives a homomorphism $\hat{V} \rightarrow V_1$. The space \hat{V} is equipped in an obvious way with the structure of a module over the algebra $\mathbb{C}[t, t^{-1}]$. It is not hard to see that multiplication by an element in $\mathbb{C}[t, t^{-1}]$ is an endomorphism of the $\hat{\mathfrak{g}}$ -module \hat{V} . Finally, note that \hat{V} is a graded $\hat{\mathfrak{g}}$ -module.

The complex $C^*(\hat{\mathfrak{n}}, \hat{V})$ is graded; more precisely,

$$C^*(\hat{\mathfrak{n}}, \hat{V}) = \prod_{i \in \mathbb{Z}} C^*(\hat{\mathfrak{n}}, \hat{V})_i.$$

LEMMA 1. *The complex $C(\hat{\mathfrak{n}}, \hat{V})_i$ is isomorphic to $C(\hat{\mathfrak{n}}, V_1)$ for any i .*

1980 *Mathematics Subject Classification*. Primary 17B30, 17B56; Secondary 17B40, 17B70.

©1984 American Mathematical Society
0197-6788/84 \$1.00 + \$.25 per page

The desired isomorphism is the composition of the imbedding $C(\hat{n}, \hat{V})_i \rightarrow C(\hat{n}, \hat{V})$ and the mapping $C(\hat{n}, \hat{V}) \rightarrow C(\hat{n}, V_1)$ induced by the homomorphism $\hat{V} \rightarrow V_1$.

It will be assumed that V is a finite-dimensional irreducible \mathfrak{g} -module. If V is the adjoint representation of \mathfrak{g} , then \hat{V} is the adjoint representation of $\hat{\mathfrak{g}}$. Let us now compute $H^*(\hat{n}, V_1)$.

The Lie algebra \hat{n} is imbedded in $\mathfrak{g}[t]$ and, hence, the composition homomorphism

$$\nu: H^*(\mathfrak{g}[t], \mathfrak{g}, V_1) \rightarrow H^*(\mathfrak{g}[t], V_1) \rightarrow H^*(\hat{n}, V_1)$$

is defined. Let $\tau: H^*(\hat{n}) \otimes H^*(\mathfrak{g}[t], \mathfrak{g}, V_1) \rightarrow H^*(\hat{n}, V_1)$ be the homomorphism assigning to an element $u \otimes v$ the cohomology class $uv(v)$.

PROPOSITION 1. *The mapping τ is an isomorphism.*

The main theorem in this section follows at once from this proposition and Lemma 1.

THEOREM 1. $H^i(\hat{n}, \hat{\mathfrak{g}}) \simeq \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} H^{i-1}(\hat{n})$ for any nonnegative integer i .

Indeed, $H^i(\hat{n}, \hat{\mathfrak{g}})$ is a $\mathbb{C}[t, t^{-1}]$ -module. It follows from Lemma 1 that $H^i(\hat{n}, \mathfrak{g})$ is a free module of rank equal to $\dim H^i(\hat{n}, V_1)$. The cohomology of the Lie algebra \hat{n} is computed in [2]. By using this result it is not hard to find the cohomology H^* of the algebra $\mathfrak{g}t \oplus \mathfrak{g}t^2 \oplus \dots$. We need only the following fact. The space H^* is a \mathfrak{g} -module, and $\text{Hom}_{\mathfrak{g}}(\mathfrak{g}, H^i)$ is 0 if $i \neq 1$ and \mathbb{C} if $i = 1$. This gives us that $H^i(\mathfrak{g}[t], \mathfrak{g}, \mathfrak{g})$ is 0 for $i \neq 1$ and \mathbb{C} for $i = 1$. Applying Proposition 1, we get that

$$H^i(\hat{n}, \mathfrak{g}_1) \simeq H^{i-1}(\hat{n}).$$

The rest of this section is devoted to a proof of Proposition 1. We introduce two more subalgebras of $\hat{\mathfrak{g}}$: $\bar{\mathfrak{g}} = (t-1)\mathfrak{g} \oplus (t-1)^2\mathfrak{g} \oplus \dots$ and $\bar{\mathfrak{n}} = \hat{n} \cap \bar{\mathfrak{g}}$. Let G be the Lie group corresponding to the Lie algebra \mathfrak{g} .

LEMMA 2. $H^*(\bar{\mathfrak{n}}) \simeq H^*(\hat{n}) \otimes H^*(\bar{\mathfrak{g}}) \otimes H^*(\Omega G)$. Here ΩG is the loop space of G .

We remark that $\mathfrak{g}[t] = \hat{n} + \bar{\mathfrak{g}}$ and $\bar{\mathfrak{n}} = \hat{n} \cap \bar{\mathfrak{g}}$. This means that

$$C^*(\bar{\mathfrak{n}}) = C^*(\hat{n}) \otimes_{C^*(\mathfrak{g}[t])} C^*(\bar{\mathfrak{g}}).$$

The tensor product is taken in the category of differential algebras. Let us now consider an Eilenberg-Moore spectral sequence whose second term E_2 is $\text{Tor}_A(H^*(\hat{n}), H^*(\bar{\mathfrak{g}}))$, where $A = H^*(\bar{\mathfrak{g}}[t])$, and whose limit term is $H^*(\bar{\mathfrak{n}})$. We remark that $H^*(\mathfrak{g}[t]) \simeq H^*(\mathfrak{g})$ (see, for example, [3]) and that $H^*(\mathfrak{g})$ acts trivially on $H^*(\hat{n})$ and $H^*(\bar{\mathfrak{g}})$. Hence, $E_2 = H^*(\hat{n}) \otimes H^*(\bar{\mathfrak{g}}) \otimes \text{Tor}_A(\mathbb{C}, \mathbb{C})$. Finally, note that $\text{Tor}_A(\mathbb{C}, \mathbb{C}) \simeq H^*(\Omega G)$. It is then not hard to show that the second term of the spectral sequence coincides with the limit term (it is possible, for example, to explicitly determine the cycles of the complex $C^*(\bar{\mathfrak{n}})$ that represent the generators of E_2). The lemma is proved.

The algebra \mathfrak{g} acts on $H^*(\bar{\mathfrak{n}})$, because $\bar{\mathfrak{n}}$ is an ideal in \hat{n} and $\hat{n}/\bar{\mathfrak{n}} \simeq \mathfrak{g}$. On $H^*(\hat{n})$ and $H^*(\Omega G)$ the action of \mathfrak{g} is trivial, but on $H^*(\bar{\mathfrak{g}})$ the action is standard ($\bar{\mathfrak{g}}$ is an ideal of $\mathfrak{g}[t]$, and $\mathfrak{g}[t]/\bar{\mathfrak{g}} \simeq \mathfrak{g}$). It is easy to see that the action of \mathfrak{g} on $H^*(\bar{\mathfrak{g}})$ is semisimple.

To prove Proposition 1 we now write the Hochschild-Serre spectral sequence associated with the algebra \hat{n} , the ideal $\bar{\mathfrak{n}}$ of it, and the module V_1 . The second term in this spectral sequence is

$$\begin{aligned} H^*(\mathfrak{g}, H^*(\bar{\mathfrak{n}}) \otimes V_1) &\simeq H^*(\mathfrak{g}, H^*(\hat{n}) \otimes H^*(\bar{\mathfrak{g}}) \otimes H^*(\Omega G) \otimes V_1) \\ &\simeq H^*(\hat{n}) \otimes H^*(\Omega G) \otimes H^*(\mathfrak{g}, H^*(\bar{\mathfrak{g}}) \otimes V_1) \\ &\simeq H^*(\hat{n}) \otimes H^*(\Omega G) \otimes H^*(\mathfrak{g}) \otimes H^*(\mathfrak{g}[t], \mathfrak{g}, V_1). \end{aligned}$$

The differentials are constructed as follows. They carry the generators of the algebra $H^*(\Omega G)$ into the generators of $H^*(\mathfrak{g})$ and are trivial on $H^*(\hat{\mathfrak{n}}) \otimes H^*(\mathfrak{g}[t], \mathfrak{g}, V_1)$. This means that the spectral sequence converges to $H^*(\hat{\mathfrak{n}}) \otimes H^*(\mathfrak{g}[t], \mathfrak{g}, V_1)$. In other words, our spectral sequence is the product of $H^*(\hat{\mathfrak{n}}) \otimes H^*(\mathfrak{g}[t], \mathfrak{g}, V_1)$ by the Leray spectral sequence of the Serre fibration $* \rightarrow G$. This completes the proof of Proposition 1.

2. The cohomology $H^i(\hat{\mathfrak{n}}, \hat{\mathfrak{n}})$ for $i = 1, 2$. Consider the following two exact sequences:

$$0 \rightarrow \hat{\mathfrak{n}} \rightarrow \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}/\hat{\mathfrak{n}} \rightarrow 0, \quad 0 \rightarrow \mathfrak{f} \rightarrow \hat{\mathfrak{g}}/\hat{\mathfrak{n}} \rightarrow \hat{\mathfrak{n}}^* \rightarrow 0$$

($\hat{\mathfrak{g}}/\hat{\mathfrak{n}}$ is identified with $\hat{\mathfrak{n}}^*$ by means of the Killing form). We write the corresponding exact cohomology sequences:

$$H^0(\hat{\mathfrak{n}}, \hat{\mathfrak{g}}/\hat{\mathfrak{n}}) \rightarrow H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{n}}) \rightarrow H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{g}}) \rightarrow H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{g}}/\hat{\mathfrak{n}}) \rightarrow H^2(\hat{\mathfrak{n}}, \hat{\mathfrak{n}}) \rightarrow H^2(\hat{\mathfrak{n}}, \hat{\mathfrak{g}}),$$

$$H^0(\hat{\mathfrak{n}}, \hat{\mathfrak{n}}^*) \rightarrow H^1(\hat{\mathfrak{n}}, \mathfrak{f}) \rightarrow H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{g}}/\hat{\mathfrak{n}}) \rightarrow H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{n}}^*).$$

The first sequence enables us at once to compute $H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{n}})$ in practice. We formulate the result.

An element $u \in \mathfrak{f}$ determines the cohomology class in $H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{n}})$ containing the cocycle $f \rightarrow uf$, $(uf)(t) = [u, f(t)]$, where $f: \mathbb{C} \rightarrow \mathfrak{g}$, $f(0) \in \mathfrak{n}$. Further, to a vector field $P(\partial/\partial t)$, where P is a polynomial in t , we assign the following cocycle

$$\hat{\mathfrak{n}} \rightarrow \hat{\mathfrak{n}}: (P(\partial/\partial t))f(t) = tP(\partial f(t)/\partial t), f: \mathbb{C} \rightarrow \mathfrak{g}.$$

Recall that the cocycles $\hat{\mathfrak{n}} \rightarrow \hat{\mathfrak{n}}$ are derivations of the Lie algebra $\hat{\mathfrak{n}}$.

PROPOSITION 2. *The mappings assigning to elements of the spaces \mathfrak{f} and $\mathbb{C}[t](\partial/\partial t)$ the cohomology classes of the constructed cocycles make up an isomorphism $\mathfrak{f} \oplus \mathbb{C}[t](\partial/\partial t) \cong H^1(\hat{\mathfrak{n}}, \hat{\mathfrak{n}})$. In other words, an arbitrary derivation of the algebra $\hat{\mathfrak{n}}$ can be uniquely represented in the form $u + P(\partial/\partial t) + q$, where $u \in \mathfrak{f}$, $P \in \mathbb{C}[t]$, and q is an inner derivation of $\hat{\mathfrak{n}}$.*

To compute $H^2(\hat{\mathfrak{n}}, \hat{\mathfrak{n}})$ in the same way it is necessary to know $H^1(\mathfrak{n}, \mathfrak{n}^*)$. For this we use the general lemma below (cf. Theorem 4.1 in [1]). Let \mathcal{L} be a Lie algebra, and T a derivation of it that acts on \mathcal{L} semisimply and has only positive eigenvalues (these requirements on T can be weakened considerably). Clearly, such a derivation is certainly outer. It is not hard to see that $\hat{\mathfrak{n}}$ has such a derivation. Let $W(\mathcal{L})$ be the Weyl algebra associated with \mathcal{L} . Recall that $W(\mathcal{L})$ is the standard complex of a differential Lie superalgebra $\bar{\mathcal{L}} = \mathcal{L}_0 \oplus \mathcal{L}_1$, where $\mathcal{L}_0 \cong \mathcal{L}$, \mathcal{L}_1 is the adjoint representation as an \mathcal{L}_0 -module, and $[x, x] = 0$ if $x \in \mathcal{L}_1$. The differential δ is constructed as follows: $\delta(\mathcal{L}_0) = 0$ and $\delta: \mathcal{L}_1 \rightarrow \mathcal{L}_0$ is an isomorphism of \mathcal{L} -modules. In $W(\mathcal{L})$ we give the filtration $W_i = \bigoplus_{j \geq i} \Lambda^* \mathcal{L}_0^* \otimes S^j \mathcal{L}_1^*$ and consider the spectral sequence E corresponding to it.

LEMMA 3. *The spectral sequence E reduces to zero; what is more, its third term is already equal to zero.*

This lemma implies, in particular, that the sequence

$$(*) \quad 0 \rightarrow H^2(\mathcal{L}) \rightarrow H^1(\mathcal{L}, \mathcal{L}^*) \rightarrow H^0(\mathcal{L}, S^2 \mathcal{L}^*) \rightarrow 0$$

is exact. The arrows here are the differentials in the second term of the spectral sequence. We remark that $H^0(\mathcal{L}, S^2 \mathcal{L}^*)$ is precisely the space of invariant bilinear forms on the algebra \mathcal{L} .

PROPOSITION 3. *Each invariant form on $\hat{\mathfrak{n}}$ is a linear combination of forms in the following two disjoint spaces:*

a) *The first space consists of the forms with kernel containing $[\hat{\mathfrak{n}}, \hat{\mathfrak{n}}]$, and this is a space of dimension $(l+1)(l+2)/2$, where l is the rank of \mathfrak{g} .*

b) The second space consists of forms like

$$(x, y) \rightsquigarrow \langle Q(t^{-1})(\partial x / \partial t), y \rangle + \langle Q(t^{-1})(\partial y / \partial t), x \rangle, \quad x, y \in \hat{n},$$

where $Q(t^{-1})(\partial / \partial t)$ is a vector field with Q a polynomial without a free term, and $\langle \cdot, \cdot \rangle$ is the Killing form on \hat{g} .

The proof of this proposition is based on Theorem 5.1 in [1], and on the following easily verified fact.

LEMMA 4. $\dim \text{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathfrak{n}^*) = 1 + l$.

Proposition 3 and the exact sequence (*) enable us to find $H^1(\hat{n}, \hat{n}^*)$, after which we can then find $H^2(\hat{n}, \hat{n})$ without difficulty.

THEOREM 2. a) The kernel of the natural mapping

$$\psi: H^1(\hat{n}) \times H^1(\hat{n}, \hat{n}) \rightarrow H^2(\hat{n}, \hat{n})$$

consists of $l + 1$ elements, where l is the rank of the algebra \mathfrak{g} .

b) If $\text{rk } \mathfrak{g} > 1$, then $\dim \text{coker } \psi = l + 1 + p$, where p is the number of positive roots of \hat{g} that can be represented as a sum of two simple roots.

c) If $\mathfrak{g} = \mathfrak{sl}_2$, then $\dim \text{coker } \psi = 4$.

Institute of Solid-State Physics

Academy of Sciences of the USSR

Moscow State University

Received 8/DEC/82

BIBLIOGRAPHY

1. George F. Leger and Eugene M. Luks, *Trans. Amer. Math. Soc.* **195** (1974), 305-316.
2. J. Lepowsky, *Ann. Sci. Ecole Norm. Sup.* (4) **12** (1979), 169-235.
3. B. L. Feigin, *Uspekhi Mat. Nauk* **35** (1980), no. 2 (212), 225-226; English transl. in *Russian Math. Surveys* **35** (1980).

Translated by H. H. McFADEN