

Versal Deformations of Leibniz Algebras

by

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Abstract

In this work we consider deformations of Leibniz algebras over a field of characteristic zero. The main problem in deformation theory is to describe all non-equivalent deformations of a given object. We give a method to solve this problem completely, namely work out a construction of a versal deformation for a given Leibniz algebra, which induces all non-equivalent deformations and is unique on the infinitesimal level.

Key Words: Leibniz algebra, Leibniz cohomology, infinitesimal deformation, versal deformation, obstruction

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1. Introduction

After the classical work of Gerstenhaber in the 60's [8, 9, 10, 11, 12], formal deformation theory was developed in various categories. Computations were made, but the question of obtaining all non-equivalent deformations of a given object was not properly discussed for a long time. The right approach to this is the notion of versal deformation: a deformation which induces all non-equivalent ones. The existence of such a "versal" deformation for algebraic categories follows from the work of Schlessinger[25]. For Lie algebras it was worked out in [5], and one can do this in other categories as well. It turns out that (under some minor cohomology restrictions) there exists a versal element, which is universal at the infinitesimal level.

In the current paper we consider Leibniz algebras and give a concrete construction of a versal element. It is parallel to the general constructions in deformation theory, as in [16, 24, 18, 13, 17]. The first concrete construction for Lie algebras was given in [6]. Here we are going to work out a similar construction for Leibniz algebras, suitable for explicit computations.

The rest of the paper is organized as follows: In Section 2 we give the necessary definitions for Leibniz algebras and their cohomology. In Section 3 we define the notion of deformation over a commutative local algebra base and introduce some basic definitions regarding deformations. In Section 4 we construct a specific

infinitesimal deformation of a Leibniz algebra, which turns out to be universal in the class of infinitesimal deformations. We also introduce the notion of versal deformation: a deformation which is unique on the infinitesimal level and induces all other deformations. Section 5 deals with obstructions to extending a deformation from a given base to a larger base. In Section 6 we give the construction of a versal deformation and in Section 7 we illustrate our theory by some examples.

2. Leibniz Algebra and its Cohomology

Leibniz algebras were introduced by J.-L. Loday [19, 21] and their cohomology was defined in [22, 20]. Let us recall some basic definitions.

Definition 2.1 A Leibniz algebra is a \mathbb{K} -module L , where \mathbb{K} is a field, equipped with a bracket operation that satisfies the Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \text{ for } x, y, z \in L.$$

Any Lie algebra is automatically a Leibniz algebra, as in the presence of antisymmetry, the Jacobi identity reduces to the Leibniz identity. More examples of Leibniz algebras were given in [22], and recently for instance in [1, 2].

Let L be a Leibniz algebra and M a representation of L . By definition, M is a \mathbb{K} -module equipped with two actions (left and right) of L ,

$$[-, -] : L \times M \longrightarrow M \text{ and } [-, -] : M \times L \longrightarrow M \text{ such that}$$

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds, whenever one of the variables is from M and the two others from L .

Define $CL^n(L; M) := Hom_{\mathbb{K}}(L^{\otimes n}, M)$, $n \geq 0$. Let

$$\delta^n : CL^n(L; M) \longrightarrow CL^{n+1}(L; M)$$

be a \mathbb{K} -homomorphism defined by

$$\begin{aligned} &\delta^n f(x_1, \dots, x_{n+1}) \\ &:= [x_1, f(x_2, \dots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), x_i] \\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \hat{x}_j, \dots, x_{n+1}). \end{aligned}$$

Then $(CL^*(L; M), \delta)$ is a cochain complex, whose cohomology is called the cohomology of the Leibniz algebra L with coefficients in the representation M . The n th cohomology is denoted by $HL^n(L; M)$. In particular, L is a representation of itself with the obvious action given by the bracket in L . The n th cohomology of L with coefficients in itself is denoted by $HL^n(L; L)$.

3. Deformations

We introduce the notion of deformation of a Leibniz algebra over a commutative algebra base. For an analogous definition for Lie algebras see [5, 6]. Fix a field \mathbb{K} of characteristic zero. Let L be a Leibniz algebra over \mathbb{K} and A a commutative algebra with identity over \mathbb{K} . Let $\varepsilon : A \rightarrow \mathbb{K}$ be a fixed augmentation, that is an algebra homomorphism with $\varepsilon(1) = 1$ and $\ker(\varepsilon) = \mathfrak{M}$. Throughout the paper we shall assume that $\dim(\mathfrak{M}^k/\mathfrak{M}^{k+1}) < \infty$ for all k .

Definition 3.1 A deformation λ of L with base (A, \mathfrak{M}) , or simply with base A , is an A -Leibniz algebra structure on the tensor product $A \otimes_{\mathbb{K}} L$ with the bracket $[\cdot, \cdot]_{\lambda}$ such that

$$\varepsilon \otimes id : A \otimes L \rightarrow \mathbb{K} \otimes L$$

is an A -Leibniz algebra homomorphism (where the A -Leibniz algebra structure on $\mathbb{K} \otimes L$ is given via ε).

A deformation of the Leibniz algebra L with base A is called *local* if A is a local algebra over \mathbb{K} , and is called *infinitesimal* (or *first order*) if, in addition to this, $\mathfrak{M}^2 = 0$. Observe that for $l_1, l_2 \in L$ and $a, b \in A$ we have

$$[a \otimes l_1, b \otimes l_2]_{\lambda} = ab[1 \otimes l_1, 1 \otimes l_2]_{\lambda}$$

by A -linearity of $[\cdot, \cdot]_{\lambda}$. Thus to define a deformation λ it is enough to specify the brackets $[1 \otimes l_1, 1 \otimes l_2]_{\lambda}$ for $l_1, l_2 \in L$. Moreover, since $\varepsilon \otimes id : A \otimes L \rightarrow \mathbb{K} \otimes L$ is an A -Leibniz algebra homomorphism,

$$(\varepsilon \otimes id)[1 \otimes l_1, 1 \otimes l_2]_{\lambda} = [l_1, l_2] = (\varepsilon \otimes id)(1 \otimes [l_1, l_2])$$

which implies

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda} - 1 \otimes [l_1, l_2] \in \ker(\varepsilon \otimes id).$$

Hence we can write

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda} = 1 \otimes [l_1, l_2] + \sum_j c_j \otimes y_j,$$

where $\sum_j c_j \otimes y_j$ is a finite sum with $c_j \in \ker(\varepsilon) = \mathfrak{M}$ and $y_j \in L$.

Definition 3.2 Let A be a complete local algebra ($A = \varprojlim_{n \rightarrow \infty} (A/\mathfrak{M}^n)$), where \mathfrak{M} is the maximal ideal in A . A formal deformation of L with base A is an A -Leibniz algebra structure on the completed tensor product

$$A \hat{\otimes} L = \varprojlim_{n \rightarrow \infty} ((A/\mathfrak{M}^n) \otimes L),$$

which is the projective limit of deformations with base A/\mathfrak{M}^n such that

$$\varepsilon \otimes id : A \hat{\otimes} L \rightarrow K \otimes L = L$$

is an A -Leibniz algebra homomorphism.

Definition 3.3 Suppose λ_1 and λ_2 are two deformations of a Leibniz algebra L with base A . We call them equivalent if there exists a Leibniz algebra isomorphism

$$\phi : (A \otimes L, [\cdot, \cdot]_{\lambda_1}) \rightarrow (A \otimes L, [\cdot, \cdot]_{\lambda_2})$$

such that $(\varepsilon \otimes id) \circ \phi = \varepsilon \otimes id$.

We write $\lambda_1 \cong \lambda_2$ if λ_1 is equivalent to λ_2 .

Example 3.4 If $A = \mathbb{K}[[t]]$ then a formal deformation of a Leibniz algebra L over A is precisely a formal 1-parameter deformation of L (see [4]).

Definition 3.5 Suppose λ is a given deformation of L with base (A, \mathfrak{M}) and augmentation $\varepsilon : A \rightarrow \mathbb{K}$. Let A' be another commutative algebra with identity and a fixed augmentation $\varepsilon' : A' \rightarrow \mathbb{K}$. Suppose $\phi : A \rightarrow A'$ is an algebra homomorphism with $\phi(1) = 1$ and $\varepsilon' \circ \phi = \varepsilon$. Let $ker(\varepsilon') = \mathfrak{M}'$. Then the push-out $\phi_*\lambda$ is the deformation of L with base (A', \mathfrak{M}') and bracket

$$[a'_1 \otimes_A (a_1 \otimes l_1), a'_2 \otimes_A (a_2 \otimes l_2)]_{\phi_*\lambda} = a'_1 a'_2 \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_{\lambda}$$

where $a'_1, a'_2 \in A'$, $a_1, a_2 \in A$ and $l_1, l_2 \in L$. Here A' is considered as an A -module by the map $a' \cdot a = a' \phi(a)$ so that

$$A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L).$$

Remark 3.6 If the bracket $[\cdot, \cdot]_{\lambda}$ is given by

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda} = 1 \otimes [l_1, l_2] + \sum_j c_j \otimes y_j \text{ for } c_j \in \mathfrak{M} \text{ and } y_j \in L \tag{1}$$

then the bracket $[\cdot, \cdot]_{\phi_*\lambda}$ can be written as

$$[1 \otimes l_1, 1 \otimes l_2]_{\phi_*\lambda} = 1 \otimes [l_1, l_2] + \sum_j \phi(c_j) \otimes y_j. \tag{2}$$

4. Universal Infinitesimal Deformation

In this section we construct a specific infinitesimal deformation of a Leibniz algebra L , which turns out to be universal in the class of all infinitesimal deformations of

L . Let L be a Leibniz algebra which satisfies the condition $dim(HL^2(L;L)) < \infty$. This is true for example, if L is finite dimensional. Let us denote the space $HL^2(L;L)$ by \mathbb{H} throughout the paper. Consider the algebra $C_1 = \mathbb{K} \oplus \mathbb{H}'$ by setting $(k_1, h_1) \cdot (k_2, h_2) = (k_1 k_2, k_1 h_2 + k_2 h_1)$ where \mathbb{H}' is the dual of \mathbb{H} . Observe that the second summand is an ideal of C_1 with zero multiplication. Fix a homomorphism

$$\mu : \mathbb{H} \longrightarrow CL^2(L;L) = Hom(L^{\otimes 2};L)$$

which takes a cohomology class into a cocycle representing it. Notice that there is an isomorphism $\mathbb{H}' \otimes L \cong Hom(\mathbb{H};L)$, so we have

$$C_1 \otimes L = (\mathbb{K} \oplus \mathbb{H}') \otimes L = (\mathbb{K} \otimes L) \oplus (\mathbb{H}' \otimes L) = L \oplus Hom(\mathbb{H};L).$$

Using the above identification, define a Leibniz bracket on $C_1 \otimes L$ as follows. For $(l_1, \phi_1), (l_2, \phi_2) \in L \oplus Hom(\mathbb{H};L)$ let

$$[(l_1, \phi_1), (l_2, \phi_2)] = ([l_1, l_2], \psi)$$

where the map $\psi : \mathbb{H} \longrightarrow L$ is given by

$$\psi(\alpha) = \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)] \text{ for } \alpha \in \mathbb{H}.$$

It is straightforward to check that $C_1 \otimes L$ along with the above bracket is a Leibniz algebra over C_1 . The Leibniz identity is a consequence of the fact that $\delta\mu(\alpha) = 0$ for $\alpha \in \mathbb{H}$. Hence we get an infinitesimal deformation of L with base $C_1 = \mathbb{K} \oplus \mathbb{H}'$.

Proposition 4.1 *Up to an isomorphism, this deformation does not depend on the choice of μ .*

Proof: Let

$$\mu' : \mathbb{H} \longrightarrow CL^2(L;L)$$

be another choice for μ . Then for $\alpha \in \mathbb{H}$, the cochains $\mu(\alpha)$ and $\mu'(\alpha)$ in $CL^2(L;L)$ represent the same class α . So $\mu(\alpha) - \mu'(\alpha)$ is a coboundary. Hence we can define a homomorphism

$$\gamma : \mathbb{H} \longrightarrow CL^1(L;L)$$

on a basis $\{h_i\}_{1 \leq i \leq n}$ of \mathbb{H} by $\gamma(h_i) = \gamma_i$ with $\delta\gamma_i = \mu(h_i) - \mu'(h_i)$. Clearly, $\mu - \mu' = \delta\gamma$.

Using the identification $C_1 \otimes L \cong L \oplus Hom(\mathbb{H};L)$, define

$$\rho : C_1 \otimes L \longrightarrow C_1 \otimes L \text{ by } \rho((l, \phi)) = (l, \psi),$$

where $\psi(\alpha) = \phi(\alpha) + \gamma(\alpha)(l)$, $l \in L$ and $\phi \in Hom(\mathbb{H};L)$.

It is routine to check that ρ is a C_1 -linear automorphism of $C_1 \otimes L$, where $\rho^{-1}(l, \psi) = (l, \phi)$ with $\phi(\alpha) = \psi(\alpha) - \gamma(\alpha)(l)$ for $\alpha \in \mathbb{H}$.

It remains to show that ρ preserves the bracket. Let $(l_1, \phi_1), (l_2, \phi_2) \in C_1 \otimes L$ with $\rho(l_1, \phi_1) = (l_1, \psi_1)$ and $\rho(l_2, \phi_2) = (l_2, \psi_2)$. Suppose

$$\begin{aligned} [(l_1, \phi_1), (l_2, \phi_2)] &= ([l_1, l_2], \phi_3) \\ \text{where } \phi_3(\alpha) &= \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)], \text{ and} \\ [(l_1, \psi_1), (l_2, \psi_2)] &= ([l_1, l_2], \psi_3) \\ \text{where } \psi_3(\alpha) &= \mu'(\alpha)(l_1, l_2) + [\psi_1(\alpha), l_2] + [l_1, \psi_2(\alpha)]. \end{aligned}$$

$$\begin{aligned} \text{Then } \psi_3(\alpha) &= \mu'(\alpha)(l_1, l_2) + [\psi_1(\alpha), l_2] + [l_1, \psi_2(\alpha)] \\ &= \mu(\alpha)(l_1, l_2) - \delta\gamma(\alpha)(l_1, l_2) + [\phi_1(\alpha) + \gamma(\alpha)(l_1), l_2] \\ &\quad + [l_1, \phi_2(\alpha) + \gamma(\alpha)(l_2)] \\ &= \mu(\alpha)(l_1, l_2) - [l_1, \gamma(\alpha)(l_2)] - [\gamma(\alpha)(l_1), l_2] + \gamma(\alpha)([l_1, l_2]) \\ &\quad + [\phi_1(\alpha), l_2] + [\gamma(\alpha)(l_1), l_2] + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)] + [l_1, \gamma(\alpha)(l_2)] \\ &= \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)] + \gamma(\alpha)([l_1, l_2]) \\ &= \phi_3(\alpha) + \gamma(\alpha)([l_1, l_2]). \end{aligned}$$

$$\text{Hence } \rho([l_1, l_2], \phi_3) = ([l_1, l_2], \psi_3) = [(l_1, \psi_1), (l_2, \psi_2)] = [\rho(l_1, \phi_1), \rho(l_2, \phi_2)].$$

Therefore, up to an isomorphism, the infinitesimal deformation obtained above is independent of the choice of μ . □

We shall denote this deformation of L by η_1 .

Remark 4.2 Suppose $\{h_i\}_{1 \leq i \leq n}$ is a basis of \mathbb{H} and $\{g_i\}_{1 \leq i \leq n}$ is the dual basis. Let $\mu(h_i) = \mu_i \in CL^2(L; L)$. Under the identification $C_1 \otimes L = L \oplus Hom(\mathbb{H}; L)$, an element $(l, \phi) \in L \oplus Hom(\mathbb{H}; L)$ corresponds to $1 \otimes l + \sum_{i=1}^n g_i \otimes \phi(h_i)$. Then for $(l_1, \phi_1), (l_2, \phi_2) \in L \oplus Hom(\mathbb{H}; L)$ their bracket $([l_1, l_2], \psi)$ corresponds to

$$1 \otimes [l_1, l_2] + \sum_{i=1}^n g_i \otimes (\mu_i(l_1, l_2) + [\phi_1(h_i), l_2] + [l_1, \phi_2(h_i)]).$$

In particular, for $l_1, l_2 \in L$ we have

$$[1 \otimes l_1, 1 \otimes l_2]_{\eta_1} = 1 \otimes [l_1, l_2] + \sum_{i=1}^n g_i \otimes \mu_i(l_1, l_2).$$

The main property of η_1 is the universality in the class of infinitesimal deformations with a finite dimensional base (Proposition 4.4).

Let λ be an infinitesimal deformation of the Leibniz algebra L with a finite dimensional base A . Let $\{m_i\}_{1 \leq i \leq r}$ be a basis of $\mathfrak{M} = ker(\varepsilon)$ and $\{\xi_i\}_{1 \leq i \leq r}$ be the

dual basis. Note that any element $\xi \in \mathfrak{M}'$ can be viewed as an element in the dual space A' with $\xi(1) = 0$. For any such ξ set

$$\alpha_{\lambda,\xi}(l_1, l_2) = \xi \otimes id([1 \otimes l_1, 1 \otimes l_2]_\lambda) \quad \text{for } l_1, l_2 \in L. \tag{3}$$

This defines a cochain $\alpha_{\lambda,\xi} \in Hom(L^{\otimes 2}; L) = CL^2(L; L)$.

If we set $\psi_i = \alpha_{\lambda,\xi_i}$ for $1 \leq i \leq r$, the Leibniz bracket (1) in terms of the basis of \mathfrak{M} takes the form

$$\begin{aligned} [1 \otimes l_1, 1 \otimes l_2]_\lambda &= 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes x_i \\ &= 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes \psi_i(l_1, l_2). \end{aligned} \tag{4}$$

Lemma 4.3 *The cochain $\alpha_{\lambda,\xi} \in CL^2(L; L)$ is a cocycle.*

Proof: By definition,

$$\begin{aligned} &\delta\alpha_{\lambda,\xi}(l_1, l_2, l_3) \\ &= [l_1, \alpha_{\lambda,\xi}(l_2, l_3)] + [\alpha_{\lambda,\xi}(l_1, l_3), l_2] - [\alpha_{\lambda,\xi}(l_1, l_2), l_3] \\ &\quad - \alpha_{\lambda,\xi}([l_1, l_2], l_3) + \alpha_{\lambda,\xi}([l_1, l_3], l_2) + \alpha_{\lambda,\xi}(l_1, [l_2, l_3]) \text{ for } l_1, l_2, l_3 \in L. \end{aligned}$$

Observe that

$$\begin{aligned} &(\xi \otimes id)([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda) \\ &= (\xi \otimes id)([1 \otimes l_1, 1 \otimes [l_2, l_3]]_\lambda + [1 \otimes l_1, \sum_{j=1}^r m_j \otimes x_j]_\lambda) \text{ (using (4))} \\ &= \alpha_{\lambda,\xi}(l_1, [l_2, l_3]) + \sum_{j=1}^r (\xi \otimes id)[1 \otimes l_1, m_j \otimes x_j]_\lambda. \end{aligned}$$

Moreover,

$$\begin{aligned} &(\xi \otimes id)[1 \otimes l_1, m_j \otimes x_j]_\lambda = (\xi \otimes id)m_j[1 \otimes l_1, 1 \otimes x_j]_\lambda \\ &= (\xi \otimes id)m_j(1 \otimes [l_1, x_j] + \sum_{i=1}^r m_i \otimes x_{ji}) \\ &= (\xi \otimes id)(m_j \otimes [l_1, x_j]) \quad (\mathfrak{M}^2 = 0) \\ &= [l_1, (\xi \otimes id)(m_j \otimes x_j)]. \end{aligned}$$

Therefore

$$\begin{aligned} & (\xi \otimes id)([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda) \\ &= \alpha_{\lambda, \xi}(l_1, [l_2, l_3]) + [l_1, (\xi \otimes id) \sum_{j=1}^r m_j \otimes x_j] \\ &= \alpha_{\lambda, \xi}(l_1, [l_2, l_3]) + [l_1, (\xi \otimes id)([1 \otimes l_2, 1 \otimes l_3]_\lambda - 1 \otimes [l_2, l_3])] \text{ (by using (4))} \\ &= \alpha_{\lambda, \xi}(l_1, [l_2, l_3]) + [l_1, \alpha_{\lambda, \xi}(l_2, l_3)] \text{ } (\xi(1) = 0). \end{aligned}$$

Similarly,

$$\begin{aligned} & (\xi \otimes id)([[1 \otimes l_1, 1 \otimes l_2]_\lambda, 1 \otimes l_3]_\lambda) = \alpha_{\lambda, \xi}([l_1, l_2], l_3) + [\alpha_{\lambda, \xi}(l_1, l_2), l_3], \\ & (\xi \otimes id)([[1 \otimes l_1, 1 \otimes l_3]_\lambda, 1 \otimes l_2]_\lambda) = \alpha_{\lambda, \xi}([l_1, l_3], l_2) + [\alpha_{\lambda, \xi}(l_1, l_3), l_2]. \end{aligned}$$

It follows that

$$\begin{aligned} \delta\alpha_{\lambda, \xi}(l_1, l_2, l_3) &= (\xi \otimes id)([1 \otimes l_1, [1 \otimes l_2, 1 \otimes l_3]_\lambda]_\lambda - [[1 \otimes l_1, 1 \otimes l_2]_\lambda, 1 \otimes l_3]_\lambda \\ &\quad + [[1 \otimes l_1, 1 \otimes l_3]_\lambda, 1 \otimes l_2]_\lambda) \\ &= 0 \text{ (by the Leibniz relation)}. \end{aligned}$$

□

Proposition 4.4 *For any infinitesimal deformation λ of a Leibniz algebra L with a finite dimensional base A there exists a unique homomorphism $\phi : C_1 = (\mathbb{K} \oplus \mathbb{H}') \rightarrow A$ such that λ is equivalent to the push-out $\phi_*\eta_1$.*

Proof: Let λ be an infinitesimal deformation of a Leibniz algebra L with base A , where A is a finite dimensional local algebra over \mathbb{K} and \mathfrak{M} is the maximal ideal in A . Let $\dim(\mathfrak{M}) = r$. Suppose $\{m_i\}_{1 \leq i \leq r}$ is a basis of \mathfrak{M} and $\{\xi_i\}_{1 \leq i \leq r}$ be the corresponding dual basis of \mathfrak{M}' . For $\xi_i \in \mathfrak{M}'$ let $a_{\lambda, \xi_i} \in \mathbb{H}$ be the cohomology class of the cocycle α_{λ, ξ_i} . The correspondences

$$\xi_i \mapsto \alpha_{\lambda, \xi_i} \text{ and } \xi_i \mapsto a_{\lambda, \xi_i}$$

for $1 \leq i \leq r$ define homomorphisms

$$\alpha_\lambda : \mathfrak{M}' \rightarrow CL^2(L; L) \text{ with } \delta \circ \alpha_\lambda = 0 \text{ and } a_\lambda : \mathfrak{M}' \rightarrow \mathbb{H}.$$

We claim that

- (a) Two deformations λ_1 and λ_2 are equivalent if and only if $a_{\lambda_1} = a_{\lambda_2}$.
- (b) If $\phi = id \oplus a'_\lambda : C_1 \rightarrow \mathbb{K} \oplus \mathfrak{M} = A$ then $\phi_*\eta_1$ is equivalent to λ .

Let λ_1 and λ_2 be two equivalent deformations of the Leibniz algebra L with base A . Then there exists an A -Leibniz algebra isomorphism

$$\rho : (A \otimes L, [\cdot, \cdot]_{\lambda_1}) \rightarrow (A \otimes L, [\cdot, \cdot]_{\lambda_2}) \text{ with } (\varepsilon \otimes id) \circ \rho = \varepsilon \otimes id.$$

Now $A \otimes L = (\mathbb{K} \oplus \mathfrak{M}) \otimes L = (\mathbb{K} \otimes L) \oplus (\mathfrak{M} \otimes L) = L \oplus (\mathfrak{M} \otimes L)$. Thus any element of $A \otimes L$ is of the form $(l, \sum_{i=1}^r m_i \otimes l_i)$ where $l_i \in L$ for $1 \leq i \leq r$. By A -linearity, ρ is determined by the values $\rho(1 \otimes l)$ for $l \in L$ and hence ρ is of the form $\rho = \rho_1 + \rho_2$ where $\rho_1 : L \rightarrow L$ and $\rho_2 : L \rightarrow \mathfrak{M} \otimes L$. The map ρ_1 must be the identity map $id : L \rightarrow L$ by the compatibility $(\varepsilon \otimes id) \circ \rho = \varepsilon \otimes id$. We shall use the following standard identifications.

$$Hom(L; \mathfrak{M} \otimes L) \cong \mathfrak{M} \otimes Hom(L; L) \cong Hom(\mathfrak{M}'; Hom(L; L)).$$

In terms of bases of \mathfrak{M} and \mathfrak{M}' , the above isomorphisms are given as follows.

$$\rho_2 \mapsto \sum_{i=1}^r m_i \otimes \phi_i \mapsto \sum_{i=1}^r \chi_i$$

where $\phi_i = (\xi_i \otimes id) \circ \rho_2$ and $\chi_i(\xi_j) = \delta_{ij} \phi_i$. We have

$$\rho(1 \otimes l) = \rho_1(1 \otimes l) + \rho_2(1 \otimes l) = 1 \otimes l + \sum_{i=1}^r m_i \otimes \phi_i(l) \text{ for } l \in L.$$

The map ρ is a Leibniz algebra homomorphism if and only if

$$\rho([1 \otimes l_1, 1 \otimes l_2]_{\lambda_1}) = [\rho(1 \otimes l_1), \rho(1 \otimes l_2)]_{\lambda_2}.$$

If we take $\psi_i^k = \alpha_{\lambda_k, \xi_i}$ $1 \leq i \leq r$ for $k = 1$ and 2 , we have

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda_k} = 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes \psi_i^k(l_1, l_2).$$

Therefore $\rho([1 \otimes l_1, 1 \otimes l_2]_{\lambda_1})$

$$\begin{aligned}
 &= 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes \phi_i([l_1, l_2]) \\
 &\quad + \sum_{i=1}^r m_i (1 \otimes \psi_i^1(l_1, l_2) + \sum_{j=1}^r m_j \otimes \phi_j(\psi_i^1(l_1, l_2))) \\
 &= 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes \phi_i([l_1, l_2]) + \sum_{i=1}^r m_i (1 \otimes \psi_i^1(l_1, l_2)) \\
 &\quad (m_i m_j = 0).
 \end{aligned}$$

Similarly, $[\rho(1 \otimes l_1), \rho(1 \otimes l_2)]_{\lambda_2}$

$$= 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes \psi_i^2(l_1, l_2) + \sum_{i=1}^r m_i \otimes [l_1, \phi_i(l_2)]$$

$$+ \sum_{i=1}^r m_i \otimes [\phi_i(l_1), l_2].$$

Thus, $[\rho(1 \otimes l_1), \rho(1 \otimes l_2)]_{\lambda_2} - \rho([1 \otimes l_1, 1 \otimes l_2]_{\lambda_1}) = 0,$

equivalently, $\sum_{i=1}^r m_i \otimes (\psi_i^2(l_1, l_2) - \psi_i^1(l_1, l_2)) + \sum_{i=1}^r m_i \otimes \delta\phi_i(l_1, l_2) = 0,$

or equivalently, $\psi_i^1(l_1, l_2) - \psi_i^2(l_1, l_2) = \delta\phi_i(l_1, l_2),$

that is, $\alpha_{\lambda_1, \xi_i} - \alpha_{\lambda_2, \xi_i} = \delta\phi_i$ for $1 \leq i \leq r.$

Therefore, ρ is a Leibniz algebra homomorphism if and only if $a_{\lambda_1} = a_{\lambda_2}.$ This proves (a).

Now consider the map

$$\phi = id \oplus a_{\lambda'} : C_1 \longrightarrow \mathbb{K} \oplus \mathfrak{M} = A.$$

By (a) it is enough to show that $\alpha_{\phi_*\eta_1} = \mu \circ a_{\lambda}.$ Let $\{h_i\}_{1 \leq i \leq n}$ be a basis of \mathbb{H} and $\{g_i\}_{1 \leq i \leq n}$ be the corresponding dual basis of $\mathbb{H}'.$ By Remarks 3.6 and 4.2 we have

$$[1 \otimes l_1, 1 \otimes l_2]_{\phi_*\eta_1} = 1 \otimes [l_1, l_2] + \sum_{i=1}^n \phi(g_i) \otimes \mu(h_i)(l_1, l_2).$$

Let $a_{\lambda'} : \mathbb{H}' \longrightarrow \mathfrak{M}$ be the dual of $a_{\lambda}.$ Then

$$a_{\lambda'}(g_j) = \sum_{i=1}^r \xi_i(a_{\lambda'}(g_j))m_i \text{ and } a_{\lambda}(\xi_i) = \sum_{j=1}^n g_j(a_{\lambda}(\xi_i))h_j.$$

$$\begin{aligned}
 \text{Thus } \alpha_{\phi_*\eta_1}(\xi_i)(l_1, l_2) &= \xi_i \otimes id[1 \otimes l_1, 1 \otimes l_2]_{\phi_*\eta_1} \\
 &= \xi_i \otimes id(1 \otimes [l_1, l_2]) + \sum_{j=1}^n \phi(g_j) \otimes \mu(h_j)(l_1, l_2) \\
 &= \xi_i \otimes id\left(\sum_{j=1}^n a_{\lambda'}(g_j) \otimes \mu(h_j)(l_1, l_2)\right) \\
 &= \sum_{j=1}^n \xi_i(a_{\lambda'}(g_j)) \otimes \mu(h_j)(l_1, l_2) \\
 &= \sum_{j=1}^n g_j(a_{\lambda}(\xi_i)) \otimes \mu(h_j)(l_1, l_2) \\
 &= \mu\left(\sum_{j=1}^n g_j(a_{\lambda}(\xi_i))h_j\right)(l_1, l_2) \\
 &= \mu \circ a_{\lambda}(\xi_i)(l_1, l_2).
 \end{aligned}$$

The uniqueness part of the theorem follows from the definition of ϕ . □

Suppose A is a local algebra with the unique maximal ideal \mathfrak{M} and $\pi : A \rightarrow A/\mathfrak{M}^2$ the corresponding quotient map. Assume $dim(A/\mathfrak{M}^2)$ is finite. The algebra A/\mathfrak{M}^2 is obviously local with maximal ideal $\mathfrak{M}/\mathfrak{M}^2$ and $(\mathfrak{M}/\mathfrak{M}^2)^2 = 0$. If λ is a deformation of L with base A then $\pi_*\lambda$ is a deformation with base A/\mathfrak{M}^2 and it is clearly infinitesimal. Therefore by the previous proposition we have a map

$$a_{\pi_*\lambda} : (\mathfrak{M}/\mathfrak{M}^2)' \rightarrow \mathbb{H}.$$

Definition 4.5 The dual space $(\mathfrak{M}/\mathfrak{M}^2)'$ is called the tangent space of A and is denoted by TA . The map $a_{\pi_*\lambda}$ is called the differential of λ and is denoted by $d\lambda$.

It follows from Proposition 4.4 that equivalent deformations have the same differential. We have constructed in this section the universal infinitesimal deformation and our goal is to extend it to a versal one. It is known that in the category of deformations of an algebraic object generally there is no universal deformation [15]. But under certain natural conditions it is possible to get a “versal” object, which still induces all non-equivalent deformations.

Definition 4.6 A formal deformation η of a Leibniz algebra L with base C is called versal, if

- (i) for any formal deformation λ of L with base A there exists a homomorphism $f : C \rightarrow A$ such that the deformation λ is equivalent to $f_*\eta$;
- (ii) if A satisfies the condition $\mathfrak{M}^2 = 0$, then f is unique.

Theorem 4.7 *If \mathbb{H} is finite dimensional, then there exists a versal deformation.*

Proof: Follows from the general theorem of Schlessinger [25], like it was shown for Lie algebras in [5]. □

5. Obstructions

The aim of this section is to study obstructions in extending deformations. For this we need the interpretation of 1- and 2-dimensional Harrison cohomology of a commutative algebra. Let us recall some definitions and results from [14].

Let A be a commutative algebra with 1 over \mathbb{K} . Let $(C_q(A), \delta)$ denote the standard Hochschild complex, where $C_q(A)$ is the A -module $A^{\otimes(q+1)}$ with A acting on the first factor by multiplication of A . Let $Sh_q(A)$ be the A -submodule of $C_q(A)$ generated by chains

$$s_p(a_1, a_2, \dots, a_q) = \sum_{(i_1, i_2, \dots, i_q) \in Sh(p, q-p)} \text{sgn}(i_1, i_2, \dots, i_q)(a_{i_1}, a_{i_2}, \dots, a_{i_q}) \in C_q(A)$$

for $a_1, a_2, \dots, a_q \in A$; $0 < p < q$.

It turns out that Sh_* is a subcomplex of $C_*(A)$ and hence we have a complex called the *Harrison complex*

$$Ch(A) = \{Ch_q(A), \delta\}; Ch_q(A) = C_q(A)/Sh_q(A).$$

For an A -module M , the Harrison cochain complex defining the Harrison cohomology with coefficients in M is given by $Ch^q(A; M) = Hom(Ch_q(A), M)$.

Definition 5.1 For an A -module M we define

$$H_{Harr}^q(A; M) = H^q(Hom(Ch(A), M)).$$

Proposition 5.2 Let A be a commutative local algebra with the maximal ideal \mathfrak{M} , and let M be an A -module with $\mathfrak{M}M = 0$. Then we have the canonical isomorphism

$$H_{Harr}^q(A; M) \cong H_{Harr}^q(A; \mathbb{K}) \otimes M.$$

Definition 5.3 An extension B of an algebra A by an A -module M is a \mathbb{K} -algebra B together with an exact sequence of \mathbb{K} -modules

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0,$$

where p is a \mathbb{K} -algebra homomorphism, and the B -module structure on $i(M)$ is given by the A -module structure of M by $i(m) \cdot b = i(mp(b))$.

Proposition 5.4 (i) *The space $H^1_{Harr}(A;M)$ is isomorphic to the space of derivations $A \rightarrow M$.*

(ii) *Elements of $H^2_{Harr}(A;M)$ correspond bijectively to isomorphism classes of extensions*

$$0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$$

of the algebra A by means of M .

(iii) *The space $H^1_{Harr}(A;M)$ can also be interpreted as the group of automorphisms of any given extension of A by M .*

Corollary 5.5 *If A is a local algebra with the maximal ideal \mathfrak{M} , then*

$$H^1_{Harr}(A; \mathbb{K}) \cong \left(\frac{\mathfrak{M}}{\mathfrak{M}^2}\right)' = TA.$$

Let λ be a deformation of a Leibniz algebra L with a finite dimensional local base A and augmentation ε . Consider $[f] \in H^2_{Harr}(A; \mathbb{K})$. Suppose

$$0 \rightarrow \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0$$

is a representative of the class of 1- dimensional extensions of A , corresponding to the cohomology class of f . Let $I = (i \otimes id) : L \cong \mathbb{K} \otimes L \rightarrow B \otimes L$,

$P = (p \otimes id) : B \otimes L \rightarrow A \otimes L$ and $E = (\hat{\varepsilon} \otimes id) : B \otimes L \rightarrow \mathbb{K} \otimes L \cong L$, where $\hat{\varepsilon} = \varepsilon \circ p$ is the augmentation of B corresponding to the augmentation ε of A . Fix a section $q : A \rightarrow B$ of p in the above extension, then

$$b \mapsto (p(b), i^{-1}(b - q \circ p(b))) \tag{5}$$

is a \mathbb{K} - module isomorphism $B \rightarrow (A \oplus \mathbb{K})$. Let us denote by $(a, k)_q \in B$ the inverse of $(a, k) \in (A \oplus \mathbb{K})$ under the above isomorphism.

The cocycles f representing the extension is determined by $f(a_1, a_2) = i^{-1}((a_1, 0)_q(a_2, 0)_q - (a_1 a_2, 0)_q)$. On the other hand, f determines the algebra structure of B by

$$(a_1, k_1)_q \circ (a_2, k_2)_q = (a_1 a_2, a_1 \cdot k_2 + a_2 \cdot k_1 + f(a_1, a_2))_q. \tag{6}$$

Suppose $dim(A) = r + 1$ and $\{m_i\}_{1 \leq i \leq r}$ is a basis of the maximal ideal \mathfrak{M}_A of A . Then $\{n_i\}_{1 \leq i \leq r+1}$ is a basis of the maximal ideal $\mathfrak{M}_B = p^{-1}(\mathfrak{M}_A)$ of B , where $n_j = (m_j, 0)_q$, for $1 \leq j \leq r$ and $n_{r+1} = (0, 1)_q$. Take the dual basis $\{\xi_i\}_{1 \leq i \leq r}$ of \mathfrak{M}'_A . Then by (3) and (4), we have 2-cochains $\psi_i = \alpha_{\lambda, \xi_i} \in CL^2(L; L)$ for $1 \leq i \leq r$ such that $[\cdot, \cdot]_\lambda$ can be written as

$$[1 \otimes l_1, 1 \otimes l_2]_\lambda = 1 \otimes [l_1, l_2] + \sum_{i=1}^r m_i \otimes \psi_i(l_1, l_2) \text{ for } l_1, l_2 \in L.$$

Let $\psi \in CL^2(L;L) = Hom(L^{\otimes 2};L)$ be an arbitrary element. Define a B -bilinear operation $(B \otimes L)^{\otimes 2} \rightarrow B \otimes L$,

$$\{b_1 \otimes l_1, b_2 \otimes l_2\} = b_1 b_2 \otimes [l_1, l_2] + \sum_{j=1}^r b_1 b_2 n_j \otimes \psi_j(l_1, l_2) + b_1 b_2 n_{r+1} \otimes \psi(l_1, l_2).$$

It is routine to check that the B -bilinear map $\{, \}$ satisfies

$$\begin{aligned} (i) & P\{l_1, l_2\} = [P(l_1), P(l_2)]_\lambda \text{ for } l_1, l_2 \in B \otimes L \\ (ii) & \{I(l), l_1\} = I[l, E(l_1)] \text{ for } l \in L \text{ and } l_1 \in B \otimes L. \end{aligned} \tag{7}$$

So the Leibniz algebra structure λ on $A \otimes L$ can be lifted to a B -bilinear operation $\{, \} : (B \otimes L)^{\otimes 2} \rightarrow B \otimes L$ satisfying (7).

Define $\phi : (B \otimes L)^{\otimes 3} \rightarrow B \otimes L$ by

$$\phi(l_1, l_2, l_3) = \{l_1, \{l_2, l_3\}\} - \{\{l_1, l_2\}, l_3\} + \{\{l_1, l_3\}, l_2\} \text{ for } l_1, l_2, l_3 \in B \otimes L. \tag{8}$$

It is clear that $\{, \}$ satisfies the Leibniz relation if and only if $\phi = 0$. Now from property (i) in (7) and the definition of ϕ it follows that $P \circ \phi(l_1, l_2, l_3) = 0$ for $l_1, l_2, l_3 \in B \otimes L$. Therefore ϕ takes values in $ker(P)$. Observe that $\phi(l_1, l_2, l_3) = 0$, whenever one of the arguments belongs to $ker(E)$. Suppose $l_1 = b \otimes l \in ker(E) \subseteq B \otimes L$. Since $ker(E) = ker(\hat{\varepsilon}) \otimes L = p^{-1}(ker(\varepsilon)) \otimes L = \mathfrak{M}_B \otimes L$, we can write $l_1 = \sum_{j=1}^{r+1} n_j \otimes l'_j$ with $l'_j \in L$; $j = 1, \dots, r + 1$. Then for $l_2, l_3 \in B \otimes L$, we get

$$\phi(l_1, l_2, l_3) = \phi\left(\sum_{j=1}^{r+1} n_j \otimes l'_j, l_2, l_3\right) = \sum_{j=1}^{r+1} n_j \phi(1 \otimes l'_j, l_2, l_3) = 0.$$

This is because $\phi(1 \otimes l'_j, l_2, l_3) \in ker(P) = im(I) = im(i) \otimes L = i(\mathbb{K}) \otimes L$ and for any element $k \in \mathbb{K}$ and $l \in L$,

$$n_j \cdot i(k) \otimes l = i(p(n_j)k) \otimes l = i(m_j \cdot k) \otimes l = i(\varepsilon(m_j)k) \otimes l = 0 \text{ for } 1 \leq j \leq r$$

and $n_{r+1} \cdot i(k) \otimes l = kn_{r+1}^2 \otimes l = 0$ ($m_j \in \mathfrak{M} \subset A$ and $m_j \cdot k = \varepsilon(m_j)k$).

The other two cases are similar. Thus ϕ defines a linear map

$$\tilde{\phi} : \left(\frac{B \otimes L}{ker(E)}\right)^{\otimes 3} \rightarrow ker(P),$$

$$\tilde{\phi}(b_1 \otimes l_1 + ker(E), b_2 \otimes l_2 + ker(E), b_3 \otimes l_3 + ker(E)) = \phi(b_1 \otimes l_1, b_2 \otimes l_2, b_3 \otimes l_3).$$

Moreover, the surjective map $E : B \otimes L \rightarrow \mathbb{K} \otimes L \cong L$, defined by $b \otimes l \mapsto \hat{\varepsilon}(b) \otimes l$, induces an isomorphism $\frac{B \otimes L}{ker(E)} \overset{\alpha}{\cong} L$, where

$$\alpha : L \rightarrow \frac{B \otimes L}{ker(E)} ; \alpha(l) = 1 \otimes l + ker(E).$$

Also, $ker(P) = im(I) = i(\mathbb{K}) \otimes L = \mathbb{K} i(1) \otimes L \cong^{\beta} L$ where the isomorphism β is given by $\beta(kn_{r+1} \otimes l) = kl$ with inverse $\beta^{-1}(l) = n_{r+1} \otimes l$. Thus we get a linear map $\bar{\phi} : L^{\otimes 3} \rightarrow L$, such that $\bar{\phi} = \beta \circ \tilde{\phi} \circ \alpha^{\otimes 3}$. The cochains $\bar{\phi} \in CL^3(L; L)$ and ϕ are related by

$$n_{r+1} \otimes \bar{\phi}(l_1, l_2, l_3) = \phi(1 \otimes l_1, 1 \otimes l_2, 1 \otimes l_3). \tag{9}$$

We claim that the cochain $\bar{\phi}$ is a cocycle. The coboundary $\delta\bar{\phi}$ consists of 10 terms. Let us rewrite the first term of $\beta^{-1} \circ \delta\bar{\phi}$ as follows.

$$\begin{aligned} & \beta^{-1}([l_1, \bar{\phi}(l_2, l_3, l_4)]) \\ &= n_{r+1} \otimes [l_1, \bar{\phi}(l_2, l_3, l_4)] \\ &= I([l_1, \bar{\phi}(l_2, l_3, l_4)]) \quad (i(1) = n_{r+1}) \\ &= I([l_1, E(1 \otimes \bar{\phi}(l_2, l_3, l_4))]) \\ &= \{I(l_1), 1 \otimes \bar{\phi}(l_2, l_3, l_4)\} \quad (\text{by (ii) of (7)}) \\ &= \{n_{r+1} \otimes l_1, 1 \otimes \bar{\phi}(l_2, l_3, l_4)\} \\ &= \{1 \otimes l_1, n_{r+1} \otimes \bar{\phi}(l_2, l_3, l_4)\} \\ &= \{1 \otimes l_1, \phi(1 \otimes l_2, 1 \otimes l_3, 1 \otimes l_4)\} \quad (\text{by (9)}) \\ &= \{1 \otimes l_1, \{1 \otimes l_2, \{1 \otimes l_3, 1 \otimes l_4\}\}\} - \{1 \otimes l_1, \{\{1 \otimes l_2, 1 \otimes l_3\}, 1 \otimes l_4\}\} \\ & \quad + \{1 \otimes l_1, \{\{1 \otimes l_2, 1 \otimes l_4\}, 1 \otimes l_3\}\}. \end{aligned}$$

Similarly, computing other terms and substituting in the expression of $\beta^{-1} \circ \delta\bar{\phi}$, we get $\beta^{-1} \circ \delta\bar{\phi}(l_1, l_2, l_3, l_4) = 0$.

Let us show now that the cohomology class of $\bar{\phi}$ is independent of the choice of the lifting $\{\cdot, \cdot\}$. Suppose $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$ are two B -bilinear operations on $B \otimes L$, lifting the Leibniz algebra structure λ on $A \otimes L$. Let $\bar{\phi}$ and $\bar{\phi}'$ be the corresponding cocycles. Set $\rho = \{\cdot, \cdot\}' - \{\cdot, \cdot\}$. Then $\rho : (B \otimes L)^{\otimes 2} \rightarrow B \otimes L$ is a B -linear map. Observe that

$$P \circ \rho(l_1, l_2) = [P(l_1), P(l_2)]_{\lambda} - [P(l_1), P(l_2)]_{\lambda} = 0 \quad (\text{by (i) in (7)}).$$

Thus ρ takes values in $ker(P)$ and induces a linear map

$$\tilde{\rho} : \left(\frac{B \otimes L}{ker(E)}\right)^{\otimes 2} \rightarrow ker(P),$$

$$\tilde{\rho}(l_1 + ker(E), l_2 + ker(E)) = \rho(l_1, l_2) \text{ for } l_1, l_2 \in B \otimes L.$$

Hence we get a 2-cochain $\bar{\rho} : L^{\otimes 2} \rightarrow L$ such that $\bar{\rho} = \beta \circ \tilde{\rho} \circ \alpha^{\otimes 2} \in CL^2(L; L)$. As before, for $l_1, l_2 \in L$, we have $n_{r+1} \otimes \bar{\rho}(l_1, l_2) = \rho(1 \otimes l_1, 1 \otimes l_2)$. Then a straightforward computation yields

$$\beta^{-1} \circ \delta\bar{\rho}(l_1, l_2, l_3) = \beta^{-1} \circ (\bar{\phi}' - \bar{\phi})(l_1, l_2, l_3), \text{ for } l_1, l_2, l_3 \in L.$$

Hence $(\bar{\phi}' - \bar{\phi}) = \delta\bar{\rho}$.

Suppose a B -bilinear operation $\{, \}$ is given on $B \otimes L$, lifting the Leibniz algebra structure $[\cdot, \cdot]_\lambda$ on $A \otimes L$. Then any other B -bilinear operation on $B \otimes L$, lifting $[\cdot, \cdot]_\lambda$, is determined by a 2-cochain ρ as follows. Define $\{, \}' : (B \otimes L)^{\otimes 2} \rightarrow B \otimes L$ by $\{l_1, l_2\}' = \{l_1, l_2\} + I \circ \rho(E(l_1), E(l_2))$ for $l_1, l_2 \in B \otimes L$. Then it is easy to see that $\{, \}'$ is a lifting of $[\cdot, \cdot]_\lambda$ such that the 2-cochain $\bar{\rho}$ induced by the difference $\{, \}' - \{, \}$, is the given 2-cochain ρ .

The above consideration defines a map $\theta_\lambda : H^2_{Harr}(A; \mathbb{K}) \rightarrow HL^3(L; L)$ by $\theta_\lambda([f]) = [\bar{\phi}]$, where $[\bar{\phi}]$ is the cohomology class of $\bar{\phi}$. The map θ_λ is called the *obstruction map*.

Proposition 5.6 *Let λ be a deformation of the Leibniz algebra L with base A and let B be a 1-dimensional extension of A corresponding to the cohomology class $[f] \in H^2_{Harr}(A; \mathbb{K})$. Then λ can be extended to a deformation of L with base B if and only if the obstruction $\theta_\lambda([f]) = 0$.*

Proof: Suppose $\theta_\lambda([f]) = 0$. Let

$$0 \rightarrow \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \rightarrow 0 \tag{10}$$

be a 1-dimensional extension representing the cohomology class $[f]$. Let $\{, \}$ be a lifting of the Leibniz algebra structure λ on $A \otimes L$ to a B -bilinear operation on $B \otimes L$. Let $\bar{\phi}$ be the associated cocycle in $CL^3(L; L)$ as described above. Then $\theta_\lambda([f]) = [\bar{\phi}] = 0$ implies $\bar{\phi} = \delta\rho$ for some $\rho \in CL^2(L; L)$. Now take $\rho' = -\rho$, and define a new linear map

$$\{, \}' : (B \otimes L)^{\otimes 2} \rightarrow B \otimes L \text{ by } \{l_1, l_2\}' = \{l_1, l_2\} + I \circ \rho'(E(l_1), E(l_2)).$$

If $\bar{\phi}'$ denotes the cocycle corresponding to $\{, \}'$, we have $\bar{\phi}' - \bar{\phi} = \delta\rho' = -\bar{\phi}$. Hence $\bar{\phi}' = 0$ which implies $\phi' = 0$. Therefore, $\{, \}'$ is a Leibniz algebra structure on $B \otimes L$ extending λ . The converse is clear. □

Assume $\theta_\lambda([f]) = 0$ for $[f] \in H^2_{Harr}(A; \mathbb{K})$. Let us denote by S the set of all isomorphism classes of deformations μ of L with base B such that $p_*\mu = \lambda$. The group of automorphisms \mathcal{A} of the extension (10) has a natural action σ_1 of \mathcal{A} on S , given by $\mu \mapsto u_*\mu$ for $u \in \mathcal{A}$. This is clearly well-defined, because if $\mu \cong \mu'$, then $u_*\mu \cong u_*\mu'$.

On the other hand, \mathbb{H} acts on S as follows. Suppose μ and μ' are two deformations of L with base B such that $p_*\mu = p_*\mu' = \lambda$. Let $\bar{\phi}_\mu$ be the 3-cochain determined by $[\cdot, \cdot]_\mu$ as before. If $\psi \in CL^2(L; L)$ is the cochain determined by $[\cdot, \cdot]_\mu - [\cdot, \cdot]_{\mu'}$, then we have, $\bar{\phi}_\mu - \bar{\phi}_{\mu'} = \delta\psi$. But $\bar{\phi}_\mu = 0 = \bar{\phi}_{\mu'}$ (μ, μ' are the Leibniz brackets). Hence ψ is a cocycle.

Suppose now $\mu_1 \cong \mu$ with isomorphism $\rho : (B \otimes L, [\cdot]_\mu) \longrightarrow (B \otimes L, [\cdot]_{\mu_1})$. As explained in the proof of Proposition 4.4, the 2-cochain determined by $[\cdot]_\mu - [\cdot]_{\mu_1}$ is a coboundary. Hence $[\cdot]_\mu - [\cdot]_{\mu_1} = \delta b_\rho$ for some 1-cochain b_ρ , determined by ρ . Therefore, $\psi_1 = [\cdot]_{\mu'} - [\cdot]_{\mu_1} = ([\cdot]_{\mu'} - [\cdot]_\mu) + ([\cdot]_\mu - [\cdot]_{\mu_1}) = \psi + \delta b_\rho$. This allows us to introduce a map $\sigma_2 : \mathbb{H} \times S \longrightarrow S$, $\sigma_2(\psi, \mu) = \mu'$, where $[\cdot]_{\mu'} - [\cdot]_\mu$ determines ψ . The above discussion shows that the map is well-defined. It is clear that the action is transitive.

Let us consider the relationship between the two actions σ_1 and σ_2 on S .

Proposition 5.7 *Let λ be a deformation of the Leibniz algebra L with base A and let*

$$0 \longrightarrow \mathbb{K} \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be a given extension of A . If $u : B \longrightarrow B$ is an automorphism of this extension which corresponds to an element $h \in H^1_{Harr}(A; \mathbb{K}) = TA$, then for any deformation μ of L with base B , such that $p_*\mu = \lambda$, the difference $[\cdot]_{u_*\mu} - [\cdot]_\mu$ is a cocycle in the cohomology class $d\lambda(h)$. This means that the operation σ_1 and σ_2 on S are related to each other by the differential $d\lambda : TA \longrightarrow \mathbb{H}$.

Proof: Recall that $\sigma_1 : H^1_{Harr}(A; \mathbb{K}) \times S \longrightarrow S$, $\sigma_1(\phi, \mu) = u_*\mu$ where u is the automorphism in \mathcal{A} corresponding to $h = [\phi] \in H^1_{Harr}(A; \mathbb{K})$, and

$$\sigma_2 : \mathbb{H} \times S \longrightarrow S, \sigma_2(\psi, \mu) = \mu'$$

where $[\cdot]_{\mu'} - [\cdot]_\mu$ determines $\psi \in \mathbb{H}$.

We need to show that the 2-cocycle determined by the difference $[\cdot]_{u_*\mu} - [\cdot]_\mu$ can be represented by $d\lambda(h)$. One can easily prove this by choosing a basis $\{\tilde{m}_i\}_{1 \leq i \leq l}$ of $\mathfrak{M}/\mathfrak{M}^2$ and writing out $[\cdot]_{u_*\mu} - [\cdot]_\mu$ in terms of \tilde{m}_i . Namely, using the notations of Section 4, one gets

$$[\cdot]_{u_*\mu} - [\cdot]_\mu = \sum_{i=1}^l h(\tilde{m}_i) \otimes \psi_i(l_1, l_2).$$

Thus the cocycle determined by this difference is $\sum_{i=1}^l h(\tilde{m}_i)\psi_i(l_1, l_2)$. On the other hand for the dual basis $\{\tilde{\xi}_i\}_{1 \leq i \leq l}$, if $h = \sum_{i=1}^l x_i \tilde{\xi}_i$ and $x_i = h(\tilde{m}_i)$, then $d\lambda(h) = a_{\pi_*\lambda}(h) = \sum_{i=1}^l h(\tilde{m}_i)[\psi_i]$. This completes the proof. \square

Corollary 5.8 *Suppose that for a deformation λ of the Leibniz algebra L with base A , the differential $d\lambda : TA \longrightarrow \mathbb{H}$ is onto. Then the group of automorphisms \mathcal{A} of the extension (10) operates transitively on the set of equivalence classes of deformations μ of L with base B such that $p_*\mu = \lambda$. In other words, if μ exists, it is unique up to an isomorphism and an automorphism of this extension.*

Suppose now that M is a finite dimensional A -module satisfying the condition $\mathfrak{M}M = 0$, where \mathfrak{M} is the maximal ideal in A . The previous results can be generalized from the 1-dimensional extension (10) to a more general extension

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0.$$

If we try to extend a deformation with base A to a deformation with base B , as in the beginning of the this section, then an analogous computation yields

$$\tilde{\phi} : \left(\frac{B \otimes L}{\ker(E)}\right)^{\otimes 3} \longrightarrow \ker(P) = \text{im}(I) \cong M \otimes L.$$

It will give rise to $\bar{\phi} \in CL^3(L; M \otimes L)$ with the cohomology class

$$[\bar{\phi}] \in HL^3(L; M \otimes L) = M \otimes HL^3(L; L).$$

The obstruction map for this extension is

$$\theta_\lambda : H^2_{Harr}(A; M) \longrightarrow M \otimes HL^3(L; L) \text{ defined by } \theta_\lambda([f]) = [\bar{\phi}].$$

Then, as in the case of 1-dimensional extension, we have the following.

Proposition 5.9 *Let λ be a deformation of a Leibniz algebra L with base (A, \mathfrak{M}) and let M be a finite dimensional A -module with $\mathfrak{M}M = 0$. Consider an extension B of A*

$$0 \longrightarrow M \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

corresponding to some $[f] \in H^2_{Harr}(A; M)$. A deformation μ of L with base B such that $p_\mu = \lambda$ exists if and only if the obstruction $\theta_\lambda([f]) = 0$. If $d\lambda : TA \longrightarrow \mathbb{H}$ is onto, then the deformation μ , if it exists, is unique up to an isomorphism and an automorphism of the above extension.*

We end this section with the following naturality property of the obstruction map.

Proposition 5.10 *Suppose A_1 and A_2 are finite dimensional local algebras with augmentations ε_1 and ε_2 , respectively. Let $\phi : A_2 \longrightarrow A_1$ be an algebra homomorphism with $\phi(1) = 1$ and $\varepsilon_1 \circ \phi = \varepsilon_2$. Suppose λ_2 is a deformation of a Leibniz algebra L with base A_2 and $\lambda_1 = \phi_*\lambda_2$ is the push-out via ϕ . Then the following diagram commutes.*

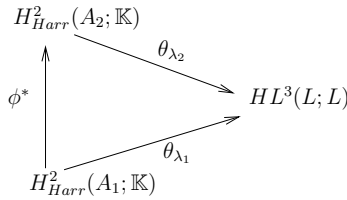


Figure 1:

Proof: Let $[f_{A_1}] \in H^2_{Harr}(A_1; \mathbb{K})$ and $[f_{A_2}] = \phi^*([f_{A_1}]) \in H^2_{Harr}(A_2; \mathbb{K})$ correspond to the classes of 1- dimensional extensions of A_1 and A_2 , represented by

$$0 \longrightarrow \mathbb{K} \xrightarrow{i_k} A'_k \xrightarrow{p_k} A_k \longrightarrow 0, \quad k = 1, 2.$$

Fix some sections $q_k : A_k \longrightarrow A'_k$ of p_k for $k = 1, 2$. Then, as in (5), we get \mathbb{K} -module isomorphisms $A'_k \cong A_k \oplus \mathbb{K}$. Let $(b, x)_{q_k}$ denote the inverse of (b, x) under the above isomorphisms. The algebra structures on A'_k are determined as in (6). Define $\psi : A'_2 \cong (A_2 \oplus \mathbb{K}) \longrightarrow A'_1 \cong (A_1 \oplus \mathbb{K})$ by $\psi((a, x)_{q_2}) = (\phi(a), x)_{q_1}$ for $(a, x)_{q_2} \in A'_2$. It is clear that ψ is a \mathbb{K} -algebra homomorphism. Thus we get a homomorphism between the two extensions given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{K} & \xrightarrow{i_2} & A'_2 & \xrightarrow{p_2} & A_2 & \longrightarrow & 0 \\ & & \parallel & & \downarrow \psi & & \downarrow \phi & & \\ 0 & \longrightarrow & \mathbb{K} & \xrightarrow{i_1} & A'_1 & \xrightarrow{p_1} & A_1 & \longrightarrow & 0 \end{array}$$

Figure 2:

Let $I_k = i_k \otimes id$, $P_k = p_k \otimes id$ and $E_k = \hat{\varepsilon}_k \otimes id$, where $\hat{\varepsilon}_k = \varepsilon_k \circ p_k$ for $k = 1, 2$. Suppose \mathfrak{M}_{A_k} is the unique maximal ideal in A_k . Then $\mathfrak{M}_{A'_k} = p_k^{-1}(\mathfrak{M}_{A_k})$ is the unique maximal ideal of A'_k . Denote by $\{m_{ki}\}_{1 \leq i \leq r_k}$ a basis of \mathfrak{M}_{A_k} and $\{n_{ki}\}_{1 \leq i \leq r_k+1}$ a basis of $\mathfrak{M}_{A'_k}$ for $k = 1, 2$. Here $n_{kj} = (m_{kj}, 0)_{q_k}$ for $1 \leq j \leq r_k$ and $n_{k(r_k+1)} = (0, 1)_{q_k}$. By (4), the Leibniz bracket on $A_2 \otimes L$ is defined by

$$[1 \otimes l_1, 1 \otimes l_2]_{\lambda_2} = 1 \otimes [l_1, l_2] + \sum_{i=1}^{r_2} m_{2i} \otimes \psi_i^2(l_1, l_2) \text{ for } l_1, l_2 \in L$$

and $\psi_i^2 = \alpha_{\lambda_2} \xi_{2i}$, where $\{\xi_{2i}\}$ is the dual basis of $\{m_{2i}\}$. Let $\phi(m_{2i}) = \sum_{j=1}^{r_1} c_{i,j} m_{1j}$ where $c_{i,j} \in \mathbb{K}$ for $1 \leq i \leq r_2$ and $1 \leq j \leq r_1$. Then the push-

out $\lambda_1 = \phi_*\lambda_2$ on $A_1 \otimes L$ is defined by

$$\begin{aligned} [1 \otimes l_1, 1 \otimes l_2]_{\lambda_1} &= 1 \otimes [l_1, l_2] + \sum_{i=1}^{r_2} \left(\sum_{j=1}^{r_1} c_{i,j} m_{1j} \right) \otimes \psi_i^2(l_1, l_2) \\ &= 1 \otimes [l_1, l_2] + \sum_{j=1}^{r_1} m_{1j} \otimes \psi_j^1(l_1, l_2) \text{ for } l_1, l_2 \in L. \end{aligned}$$

Here $\psi_j^1 \in CL^2(L; L)$ is defined by $\psi_j^1(l_1, l_2) = \sum_{i=1}^{r_2} c_{i,j} \psi_i^2(l_1, l_2)$ for $l_1, l_2 \in L$. For any 2-cochain $\chi \in CL^2(L; L)$, let $\{, \}_k : (A'_k \otimes L)^{\otimes 2} \rightarrow A'_k \otimes L$ be the A'_k -bilinear operation on $A'_k \otimes L$ lifting λ_k , defined by

$$\{1 \otimes l_1, 1 \otimes l_2\}_k = 1 \otimes [l_1, l_2] + \sum_{j=1}^{r_k} n_{kj} \otimes \psi_j^k(l_1, l_2) + n_{k(r_k+1)} \otimes \chi(l_1, l_2)$$

for $k = 1, 2$ and $l_1, l_2 \in L$. We know that $\{, \}_k$ satisfies properties (i) and (ii) of (7).

We claim that $\psi \otimes id$ preserves the liftings. It is enough to show that $(\psi \otimes id)(\{1 \otimes l_1, 1 \otimes l_2\}_2) = \{\psi \otimes id(1 \otimes l_1), \psi \otimes id(1 \otimes l_2)\}_1$ for $l_1, l_2 \in L$. Now

$$\begin{aligned} &(\psi \otimes id)(\{1 \otimes l_1, 1 \otimes l_2\}_2) \\ &= \psi(1) \otimes [l_1, l_2] + \sum_{j=1}^{r_2} \psi(1) \psi(n_{2j}) \otimes \psi_j^2(l_1, l_2) + \psi(1) \psi(n_{2(r_2+1)}) \otimes \chi(l_1, l_2) \\ &= 1 \otimes [l_1, l_2] + \sum_{j=1}^{r_2} \left(\sum_{i=1}^{r_1} c_{j,i} m_{1i} \right) \otimes \psi_j^2(l_1, l_2) + n_{1(r_1+1)} \otimes \chi(l_1, l_2) \\ &(\phi(m_{2j})) = \sum_{i=1}^{r_1} c_{j,i} m_{1i} \text{ and } \psi(n_{2(r_2+1)}) = \psi((0, 1)_{q_2}) = (\phi(0), 1)_{q_1} = n_{1(r_1+1)} \\ &= \psi(1) \otimes [l_1, l_2] + \sum_{i=1}^{r_1} \psi(1) m_{1i} \otimes \psi_i^1(l_1, l_2) + \psi(1) n_{1(r_1+1)} \otimes \chi(l_1, l_2) \\ &= \{\psi(1) \otimes l_1, \psi(1) \otimes l_2\}_1 \\ &= \{\psi \otimes id(1 \otimes l_1), \psi \otimes id(1 \otimes l_2)\}_1, \text{ which proves our claim.} \end{aligned}$$

Let ϕ_k be defined by $\{, \}_k$ as in (8) and $\bar{\phi}_k$ the corresponding cocycle as in (9). As $\psi(n_{2(r_2+1)}) = n_{1(r_1+1)}$, it follows from the definition of ϕ_k and the previous claim that $[\bar{\phi}_2] = [\bar{\phi}_1]$. Therefore, $\theta_{\lambda_1}([f_{A_1}]) = [\bar{\phi}_1] = [\bar{\phi}_2] = \theta_{\lambda_2}([f_{A_2}]) = \theta_{\lambda_2} \circ \phi^*([f_{A_1}])$. Hence $\theta_{\lambda_1} = \theta_{\lambda_2} \circ \phi^*$. □

6. Construction of a Versal Deformation

In this section we give an explicit construction of versal deformation of a given Leibniz algebra following [6].

Consider the Leibniz algebra L with $dim(\mathbb{H}) < \infty$. Set $C_0 = \mathbb{K}$ and $C_1 = \mathbb{K} \oplus \mathbb{H}'$. Consider the extension

$$0 \longrightarrow \mathbb{H}' \xrightarrow{i} C_1 \xrightarrow{p} C_0 \longrightarrow 0,$$

where the multiplication in C_1 is defined by

$$(k_1, h_1) \cdot (k_2, h_2) = (k_1 k_2, k_1 h_2 + k_2 h_1) \text{ for } (k_1, h_1), (k_2, h_2) \in C_1.$$

Let η_1 be the universal infinitesimal deformation with base C_1 as constructed in Section 4. We proceed by induction. Suppose for some $k \geq 1$ we have constructed a finite dimensional local algebra C_k and a deformation η_k of L with base C_k . Let

$$\mu : H^2_{Harr}(C_k; \mathbb{K}) \longrightarrow (Ch_2(C_k))'$$

be a homomorphism sending a cohomology class to a cocycle representing the class. Let

$$f_{C_k} : Ch_2(C_k) \longrightarrow H^2_{Harr}(C_k; \mathbb{K})'$$

be the dual of μ . By Proposition 5.4 (ii) we have the following extension of C_k :

$$0 \longrightarrow H^2_{Harr}(C_k; \mathbb{K})' \xrightarrow{\bar{i}_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}_{k+1}} C_k \longrightarrow 0. \tag{11}$$

The corresponding obstruction $\theta([f_{C_k}]) \in H^2_{Harr}(C_k; \mathbb{K})' \otimes HL^3(L; L)$ gives a linear map $\omega_k : H^2_{Harr}(C_k; \mathbb{K}) \longrightarrow HL^3(L; L)$ with the dual map

$$\omega'_k : HL^3(L; L)' \longrightarrow H^2_{Harr}(C_k; \mathbb{K})'.$$

We have an induced extension

$$0 \longrightarrow coker(\omega'_k) \longrightarrow \bar{C}_{k+1}/\bar{i}_{k+1} \circ \omega'_k(HL^3(L; L)') \longrightarrow C_k \longrightarrow 0.$$

Since $coker(\omega'_k) \cong (ker(\omega_k))'$, it yields an extension

$$0 \longrightarrow (ker(\omega_k))' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p_{k+1}} C_k \longrightarrow 0 \tag{12}$$

where $C_{k+1} = \bar{C}_{k+1}/\bar{i}_{k+1} \circ \omega'_k(HL^3(L; L)')$ and i_{k+1}, p_{k+1} are the mappings induced by \bar{i}_{k+1} and \bar{p}_{k+1} , respectively. Observe that the algebra C_k is also local. Since C_k is finite dimensional, the cohomology group $H^2_{Harr}(C_k; \mathbb{K})$ is also finite dimensional and hence C_{k+1} is finite dimensional as well.

Remark 6.1 It follows from Proposition 5.2 that the specific extension (11) has the following “universality property”. For any C_k -module M with $\mathfrak{M}M = 0$, (11) admits a unique homomorphism into an arbitrary extension of C_k :

$$0 \longrightarrow M \longrightarrow B \longrightarrow C_k \longrightarrow 0.$$

Proposition 6.2 *The deformation η_k with base C_k of a Leibniz algebra L admits an extension to a deformation with base C_{k+1} , which is unique up to an isomorphism and an automorphism of the extension*

$$0 \longrightarrow (\ker(\omega_k))' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p_{k+1}} C_k \longrightarrow 0.$$

Proof: From the above construction of the extension (12) it is clear that the corresponding obstruction map is the restriction of ω_k ,

$$\omega_k|_{\ker(\omega_k)} : \ker(\omega_k) \longrightarrow HL^3(L;L).$$

Hence, it is the zero map. Thus the result follows from Proposition 5.9. □

By induction, the above process yields a sequence of finite dimensional local algebras C_k and deformations η_k of the Leibniz algebra L with base C_k

$$\mathbb{K} \xleftarrow{p_1} C_1 \xleftarrow{p_2} C_2 \xleftarrow{p_3} \dots \xleftarrow{p_k} C_k \xleftarrow{p_{k+1}} C_{k+1} \dots$$

such that $p_{k+1} \eta_{k+1} = \eta_k$. Thus by taking the projective limit we obtain a formal deformation η of L with base $C = \varprojlim_{k \rightarrow \infty} C_k$.

Next, we give an algebraic description of the base C of the versal deformation. For that we need the following Proposition from [14].

Proposition 6.3 *Let $A = \mathbb{K}[x_1, x_2, \dots, x_n]$ be the polynomial algebra, and let \mathfrak{M} be the ideal of polynomials without constant terms.*

- (a) *If an ideal I of A is contained in \mathfrak{M}^2 , then $H^2_{Harr}(A/I; \mathbb{K}) \cong (I/\mathfrak{M}I)'$.*
- (b) *There is an extension for $B = A/I$:*

$$0 \longrightarrow I/\mathfrak{M}I \xrightarrow{i} A/\mathfrak{M}I \xrightarrow{p} A/I \longrightarrow 0$$

where i and p are induced by the inclusions $I \hookrightarrow A$ and $\mathfrak{M}I \hookrightarrow I$.

Suppose $\dim(\mathbb{H}) = n$. Let $\{h_i\}_{1 \leq i \leq n}$ be a basis of \mathbb{H} and $\{g_i\}_{1 \leq i \leq n}$ be the corresponding dual basis. Let $\mathbb{K}[[\mathbb{H}']]$ denote the formal power series ring $\mathbb{K}[[g_1, \dots, g_n]]$ in n variables g_1, \dots, g_n over \mathbb{K} . Now a typical element in $\mathbb{K}[[\mathbb{H}']]$ is of the form

$$\sum_{i=0}^{\infty} a_i f_i(g_1, \dots, g_n) = a_0 + a_1 f_1(g_1, \dots, g_n) + a_2 f_2(g_1, \dots, g_n) + \dots$$

where $a_i \in \mathbb{K}$ and f_i is a monomial of degree i in n variables g_1, \dots, g_n for $i = 0, 1, 2, \dots$. Let \mathfrak{M} denote the unique maximal ideal in $\mathbb{K}[[\mathbb{H}']]$, consisting of all elements in $\mathbb{K}[[\mathbb{H}']]$ with constant term being equal to zero.

Proposition 6.4 *For the local algebra C_k we have $C_k \cong \mathbb{K}[[\mathbb{H}']]/I_k$ for some ideal I_k , satisfying $\mathfrak{M}^2 = I_1 \supset I_2 \supset \dots \supset I_k \supset \mathfrak{M}^{k+1}$.*

Proof: By construction, $C_1 = \mathbb{K} \oplus \mathbb{H}' \cong \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^2$. Suppose we already know that $C_k \cong \mathbb{K}[[\mathbb{H}']]/I_k$ where $\mathfrak{M}^2 \supset I_k \supset \mathfrak{M}^{k+1}$. Then by specifying $A = \mathbb{K}[[\mathbb{H}']]$ and $I = I_k$ in Proposition 6.3, we get $\bar{C}_{k+1} \cong \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}I_k$. In the previous construction, C_{k+1} is the quotient of \bar{C}_{k+1} by an ideal contained in $I_k/\mathfrak{M}I_k \subset \mathfrak{M}^2/\mathfrak{M}I_k$. Hence $C_{k+1} \cong \mathbb{K}[[\mathbb{H}']]/I_{k+1}$ where $\mathfrak{M}^2 \supset I_{k+1} \supset \mathfrak{M}I_k \supset \mathfrak{M}^{k+2}$. The proof is now complete by induction. \square

Corollary 6.5 *For $k \geq 2$ the projection $p_k : C_k \rightarrow C_{k-1}$ induces an isomorphism $TC_k \rightarrow TC_{k-1}$. In particular, for every $k \geq 1$, $TC_k \cong TC_1 = \mathbb{H}$. Moreover, under the above identification of TC_k with \mathbb{H} , the differential $d\eta_k : TC_k \rightarrow \mathbb{H}$ is the identity map.*

Proof: We have $C_0 = \mathbb{K}$; $C_1 = \mathbb{K} \oplus \mathbb{H}' \cong \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^2$ and for $k \geq 2$, $C_k = \mathbb{K}[[\mathbb{H}']]/I_k$ where $\mathfrak{M}^2 = I_1 \supset I_2 \supset \dots \supset I_k \supset \mathfrak{M}^{k+1}$. The projection $p_k : C_k \rightarrow C_{k-1}$ is given by $p_k(f + I_k) = f + I_{k-1}$ for $f \in C_k$ and $k \geq 1$. The map p_k gives rise to a surjective linear map $\mathfrak{M}/I_k \rightarrow \mathfrak{M}/I_{k-1}$. Taking the quotient map $\mathfrak{M}/I_{k-1} \rightarrow \frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}$, we get an epimorphism $\mathfrak{M}/I_k \rightarrow \frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}$ with kernel \mathfrak{M}^2/I_k which corresponds to an isomorphism

$$\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k} \rightarrow \frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}.$$

As a result we get an isomorphism

$$\left(\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k}\right)' = TC_k \rightarrow TC_{k-1} = \left(\frac{\mathfrak{M}/I_{k-1}}{\mathfrak{M}^2/I_{k-1}}\right)'.$$

Observe that for any $k \geq 1$, $TC_k = \left(\frac{\mathfrak{M}/I_k}{\mathfrak{M}^2/I_k}\right)' = \left(\frac{\mathfrak{M}}{\mathfrak{M}^2}\right)' \cong TC_1$. On the other hand, since $C_1 = \mathbb{K} \oplus \mathbb{H}'$ with maximal ideal \mathbb{H}' and $(\mathbb{H}')^2 = 0$. Hence $TC_1 = (\mathbb{H}')' = \mathbb{H}$. The last assertion follows from the definition of the differential. \square

Proposition 6.6 *The complete local algebra $C = \lim_{k \rightarrow \infty} C_k$ can be described as*

$$C \cong \mathbb{K}[[\mathbb{H}']]/I, \text{ where } I \text{ is an ideal contained in } \mathfrak{M}^2.$$

Proof: Consider the map

$$\phi : \mathbb{K}[[\mathbb{H}']] \rightarrow C_k = \mathbb{K}[[\mathbb{H}']]/I_k \text{ defined by } \phi(f) = f + I_k \text{ for } f \in \mathbb{K}[[\mathbb{H}']].$$

Since $I_k \supset \mathfrak{M}^{k+1}$, the map ϕ induces an epimorphism

$$\phi_k : \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^{k+1} \longrightarrow C_k \text{ for each } k \geq 1.$$

In the limit we get an epimorphism

$$\mathbb{K}[[\mathbb{H}']] = \varprojlim_{k \rightarrow \infty} \mathbb{K}[[\mathbb{H}']]/\mathfrak{M}^{k+1} \longrightarrow \varprojlim_{k \rightarrow \infty} C_k.$$

Therefore $C \cong \mathbb{K}[[\mathbb{H}']]/I$ where $I = \bigcap_k I_k$ is the kernel of the epimorphism. □

Finally we prove the versality property of the constructed deformation η with base C . For this we use the following standard lemma.

Lemma 6.7 *Suppose $0 \longrightarrow M_s \xrightarrow{i} B_s \xrightarrow{p} A \longrightarrow 0$ is an s -dimensional extension of A . Then there exists an $(s - 1)$ -dimensional extension*

$$0 \longrightarrow M_{s-1} \xrightarrow{\bar{i}} B_{s-1} \xrightarrow{\bar{p}} A \longrightarrow 0$$

of A and a 1-dimensional extension

$$0 \longrightarrow \mathbb{K} \xrightarrow{i'} B_s \xrightarrow{p'} B_{s-1} \longrightarrow 0.$$

Theorem 6.8 *Let L be a Leibniz algebra with $\dim(\mathbb{H}) < \infty$. Then the formal deformation η with base C constructed above is a versal deformation of L .*

Proof: Suppose $\dim(\mathbb{H}) = n$. Let $\{h_i\}_{1 \leq i \leq n}$ be a basis of \mathbb{H} and $\{g_i\}_{1 \leq i \leq n}$ the corresponding dual basis of \mathbb{H}' . Let A be a complete local algebra with maximal ideal \mathfrak{M} and let λ be a formal deformation of L with base A . We want to find a \mathbb{K} -algebra homomorphism $\phi : C \longrightarrow A$ such that $\phi_*\eta = \lambda$. Denote $A_0 = A/\mathfrak{M} \cong \mathbb{K}$; $A_1 = A/\mathfrak{M}^2 \cong \mathbb{K} \oplus (TA)'$. Since A is complete, we have $A = \varprojlim_{k \rightarrow \infty} A/\mathfrak{M}^k$.

Moreover, for each k we have the following finite dimensional extension

$$0 \longrightarrow \frac{\mathfrak{M}^k}{\mathfrak{M}^{k+1}} \longrightarrow \frac{A}{\mathfrak{M}^{k+1}} \longrightarrow \frac{A}{\mathfrak{M}^k} \longrightarrow 0,$$

because $\dim(\frac{\mathfrak{M}^k}{\mathfrak{M}^{k+1}}) < \infty$.

Let $\dim(\frac{\mathfrak{M}^k}{\mathfrak{M}^{k+1}}) = n_{k-1}$. A repeated application of Lemma 6.7 to the extension

$$0 \longrightarrow \frac{\mathfrak{M}^2}{\mathfrak{M}^3} \longrightarrow \frac{A}{\mathfrak{M}^3} \longrightarrow \frac{A}{\mathfrak{M}^2} = A_1 \longrightarrow 0$$

yields n_1 number of 1-dimensional extensions as follows.

$$\begin{aligned} 0 \longrightarrow \mathbb{K} \longrightarrow A_2 \longrightarrow A_1 \longrightarrow 0 \\ 0 \longrightarrow \mathbb{K} \longrightarrow A_3 \longrightarrow A_2 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow \mathbb{K} \longrightarrow A_{n_1+1} = \frac{A}{\mathfrak{M}^3} \longrightarrow A_{n_1} \longrightarrow 0. \end{aligned}$$

Similarly, the extension

$$0 \longrightarrow \frac{\mathfrak{M}^3}{\mathfrak{M}^4} \longrightarrow \frac{A}{\mathfrak{M}^4} \longrightarrow \frac{A}{\mathfrak{M}^3} = A_{n_1+1} \longrightarrow 0$$

splits into n_2 number of 1-dimensional extensions and so on. Thus we get a sequence of 1-dimensional extensions

$$0 \longrightarrow \mathbb{K} \xrightarrow{j_{k+1}} A_{k+1} \xrightarrow{q_{k+1}} A_k \longrightarrow 0 \quad ; k \geq 1.$$

Since $A = \varprojlim_{k \rightarrow \infty} A/\mathfrak{M}^k$, it is clear that $A = \varprojlim_{k \rightarrow \infty} A_k$. Let $Q_k : A \rightarrow A_k$ be the projection map for the inverse system $\{A_k, q_k\}_{k \geq 1}$ with the limit A , where $Q_1 : A \rightarrow A_1 = A/\mathfrak{M}^2$ is the natural projection. Let $Q_{k*}\lambda = \lambda_k$, then λ_k is a deformation of L with base A_k . Thus $\lambda_k = Q_{k*}\lambda = (q_{k+1} \circ Q_{k+1})*\lambda = q_{k+1*}\lambda_{k+1}$. Now we will construct inductively homomorphisms $\phi_j : C_j \rightarrow A_j$ for $j = 1, 2, \dots$, compatible with the corresponding projections $C_{j+1} \rightarrow C_j$ and $A_{j+1} \rightarrow A_j$, along with the conditions $\phi_{j*}\eta_j \cong \lambda_j$. Define

$$\phi_1 : C_1 \rightarrow A_1 \text{ as } id \oplus (d\lambda)' : \mathbb{K} \oplus \mathbb{H}' \rightarrow \mathbb{K} \oplus (TA)'.$$

From Proposition 4.4 we have $\phi_{1*}\eta_1 \cong \lambda_1$.

Suppose we have constructed a \mathbb{K} -algebra homomorphism $\phi_k : C_k \rightarrow A_k$ with $\phi_{k*}\eta_k \cong \lambda_k$. Consider the homomorphism $\phi_k^* : H^2_{Harr}(A_k; \mathbb{K}) \rightarrow H^2_{Harr}(C_k; \mathbb{K})$ induced by ϕ_k . Let

$$0 \longrightarrow \mathbb{K} \xrightarrow{i_{k+1}} B \xrightarrow{p_{k+1}} C_k \longrightarrow 0$$

represent the image under ϕ_k^* of the isomorphism class of extension

$$0 \longrightarrow \mathbb{K} \xrightarrow{j_{k+1}} A_{k+1} \xrightarrow{q_{k+1}} A_k \longrightarrow 0$$

(see Proposition 5.4). Then we have the following commutative diagram

where ψ is given by $\psi((x, k)_q) = (\phi_k(x), k)_{q'}$, for some fixed sections q and q' of p_{k+1} and q_{k+1} respectively. Observe that by Proposition 5.10 the obstructions

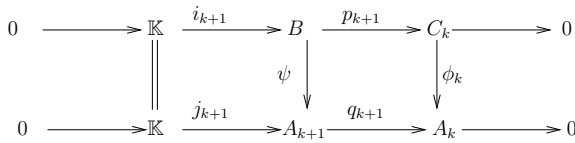


Figure 3:

in extending λ_k to the base A_{k+1} and that of η_k to the base B coincide. Since λ_k has an extension λ_{k+1} , the corresponding obstruction is zero. Hence there exists a deformation ξ of L with base B which extends η_k with base C_k such that $\psi_*\xi = \lambda_{k+1}$. By Remark 6.1 we get the following unique homomorphism of extensions.

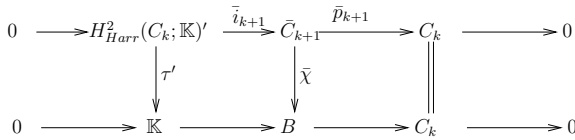


Figure 4:

Since the deformation η_k has been extended to B , the obstruction map

$$\omega_k : H^2_{Harr}(C_k; \mathbb{K}) \longrightarrow HL^3(L; L)$$

is zero. Therefore the composition $\tau' \circ \omega'_k : HL^3(L; L)' \longrightarrow \mathbb{K}$ is zero. So τ' will induce a linear map $\tau : H^2_{Harr}(C_k; \mathbb{K})' / \omega'_k(HL^3(L; L)') \longrightarrow \mathbb{K}$. Also the map $\bar{\chi} : \bar{C}_{k+1} \longrightarrow B$ will induce a linear map

$$\chi : C_{k+1} = \bar{C}_{k+1} / \bar{i}_{k+1} \circ \omega'_k(HL^3(L; L)') \longrightarrow B.$$

Since $coker(\omega'_k) \cong (ker(\omega_k))'$, the last diagram yields the following commutative diagram.

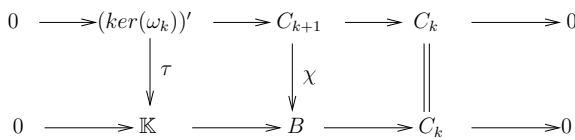


Figure 5:

By Corollary 6.5, the differential

$$d\eta_k : TC_k \longrightarrow \mathbb{H}$$

is onto, so by Corollary 5.8, the deformations $\chi_*\eta_{k+1}$ and ξ are related by some automorphism $u : B \rightarrow B$ of the extension

$$0 \rightarrow \mathbb{K} \rightarrow B \rightarrow C_k \rightarrow 0$$

with $u_*(\chi_*\eta_{k+1}) = \xi$. Now set $\phi_{k+1} = (\psi \circ u \circ \chi) : C_{k+1} \rightarrow A_{k+1}$, where ψ is as in Figure 3. Then

$$\phi_{k+1*}\eta_{k+1} = \psi_* \circ u_* \circ \chi_*\eta_{k+1} = \psi_*\xi = \lambda_{k+1}.$$

Thus by induction we get a sequence of homomorphisms $\phi_k : C_k \rightarrow A_k$ with $\phi_{k*}\eta_k = \lambda_k$. Consequently, taking the limit, we find a homomorphism $\phi : C \rightarrow A$ such that $\phi_*\eta = \lambda$. If $\mathfrak{M}^2 = 0$, then the uniqueness of ϕ follows from the corresponding property in Proposition 4.4. □

7. Examples

In this final section we discuss two examples. We begin with a computation of versal deformation of a three dimensional Leibniz algebra.

Example 7.1 Let L be the vectorspace over \mathbb{C} with basis $\{e_1, e_2, e_3\}$. Define a bilinear map $[\ , \] : L \times L \rightarrow L$, by $[e_1, e_3] = e_2$, $[e_3, e_3] = e_1$, and, all other products of basis elements being zero. Then $(L, [\ , \])$ is a Leibniz algebra. This is an example of a nilpotent Leibniz algebra [3]. Using the method developed in the present paper, a versal deformation for this Leibniz algebra is constructed in [23] as follows. A basis of $HL^2(L;L)$ is given by $\{[\mu_1], [\mu_2]\}$ where μ_1 , and μ_2 are 2-cocycles represented by the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the ordered basis

$$\{e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_2, e_3 \otimes e_3\}$$

of $L^{\otimes 2}$ and $\{e_1, e_2, e_3\}$ of L . Let $\{t, s\}$ be the dual basis in $HL^2(L;L)'$. By Remark 4.2, the universal infinitesimal deformation of L can be written as

$$[1 \otimes e_i, 1 \otimes e_j]_{\eta_1} = 1 \otimes [e_i, e_j] + t \otimes \mu_1(e_i, e_j) + s \otimes \mu_2(e_i, e_j)$$

with base $C_1 = \mathbb{C} \oplus \mathbb{C} t \oplus \mathbb{C} s$. The possible obstructions at each stage in extending η_1 to a versal deformation with base $\mathbb{C}[[t,s]]$ can be realized as the Massey brackets of μ_1 and μ_2 . It turns out that all these Massey brackets are zero and represented by 0-cocycles. Thus η_1 itself is a Leibniz bracket with base $\mathbb{C}[[t,s]]$ and represents a versal deformation of L . Explicitly, the versal deformation is given by

$$[e_1, e_3]_{t,s} = e_2 + e_1s, \quad [e_3, e_3]_{t,s} = e_1, \quad [e_2, e_3]_{t,s} = -e_1t,$$

with all other products of basis elements being zero.

Since any Lie algebra L is a Leibniz algebra it is natural to investigate whether one recovers the same deformation picture of L if it is seen as a Leibniz algebra. The following example illustrates that a Lie algebra L when viewed as a Leibniz algebra may admit new deformations which are Leibniz algebras but not Lie algebras. Moreover, the versal deformation of L as Lie algebra and that of L when viewed as a Leibniz algebra may differ.

Example 7.2 Let L be the vectorspace over \mathbb{C} with basis $\{e_1, e_2, e_3\}$. Define a bilinear map $[\ , \] : L \times L \rightarrow L$, by $[e_1, e_3] = e_2$, $[e_3, e_1] = -e_2$, and, all other products of basis elements being zero. Then $(L, [\ , \])$ is the complex three-dimensional Heisenberg Lie algebra.

Let us first determine the universal infinitesimal Leibniz deformation of L . For this, we need to compute $HL^2(L; L)$. Let $\psi : L^{\otimes 2} \rightarrow L$ be a 2-cocycle. Suppose $\psi(e_i, e_j) = \sum_{k=1}^3 a_{i,j}^k e_k$ where $a_{i,j}^k \in \mathbb{C}$; for $1 \leq i, j, k \leq 3$. Since ψ is a cocycle, we have

$$\begin{aligned} \delta\psi(e_i, e_j, e_k) &= [e_i, \psi(e_j, e_k)] + [\psi(e_i, e_k), e_j] - [\psi(e_i, e_j), e_k] - \psi([e_i, e_j], e_k) \\ &\quad + \psi(e_i, [e_j, e_k]) + \psi([e_i, e_k], e_j) = 0 \quad \text{for } 0 \leq i, j, k \leq 3. \end{aligned}$$

Using the expression of $\psi(e_i, e_j)$ above we get some relations between the coefficients $a_{i,j}^k$. If we use the resulting relations then the matrix of ψ with respect to the ordered basis

$$\{e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_2, e_3 \otimes e_3\}$$

of $L^{\otimes 2}$ and $\{e_1, e_2, e_3\}$ of L takes the form

$$\begin{pmatrix} 0 & x_2 & x_5 & -x_2 & 0 & x_8 & -x_5 & -x_8 & 0 \\ x_1 & x_3 & x_6 & -x_3 & 0 & x_9 & x_{10} & -x_9 & x_{11} \\ 0 & x_4 & x_7 & -x_4 & 0 & x_2 & -x_7 & 0 & 0 \end{pmatrix}.$$

where $x_1 = a_{1,1}^2$; $x_2 = a_{1,2}^1$; $x_3 = a_{1,2}^2$; $x_4 = a_{1,2}^3$; $x_5 = a_{1,3}^1$; $x_6 = a_{1,3}^2$;
 $x_7 = a_{1,3}^3$; $x_8 = a_{2,3}^1$; $x_9 = a_{2,3}^2$; $x_{10} = a_{3,1}^2$; and $x_{11} = a_{3,3}^2$

are in \mathbb{C} .

Let ϕ_i for $1 \leq i \leq 11$, be the cocycle with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of ψ . It is easy to check that $\{\phi_1, \dots, \phi_{11}\}$ forms a basis of the subspace of 2-cocycles in $CL^2(L; L)$.

On the other hand, let ψ_0 be a 2-coboundary so that $\psi_0 = \delta g$ for some 1-cochain g . Let $g(e_i) = g_i^1 e_1 + g_i^2 e_2 + g_i^3 e_3$ for $i = 1, 2, 3$. The coboundary formula gives

$$\delta g(e_i, e_j) = [e_i, g(e_j)] + [g(e_i), e_j] - g([e_i, e_j])$$

for $1 \leq i, j \leq 3$. From this we write down the matrix of δg . But $\delta g = \psi_0$ is also a cocycle and we know the form of the matrix for a 2-cocycle as given above. If we use this information then the matrix of ψ_0 takes the form

$$\begin{pmatrix} 0 & 0 & x_5 & 0 & 0 & 0 & -x_5 & 0 & 0 \\ 0 & x_3 & x_6 & -x_3 & 0 & -x_5 & -x_6 & x_5 & 0 \\ 0 & 0 & -x_3 & 0 & 0 & 0 & x_3 & 0 & 0 \end{pmatrix}.$$

Let $\phi_i' \in BL^2(L; L)$ for $i = 3, 5, 6$ be the coboundary with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of ψ_0 . It follows that $\{\phi_3', \phi_5', \phi_6'\}$ forms a basis for the subspace of 2-coboundaries in $CL^2(L; L)$. It is then straightforward to check that

$$\{[\phi_1], [\phi_2], [\phi_4], [\phi_7], [\phi_8], [\phi_9], [\phi_{10}], [\phi_{11}]\}$$

forms a basis of $HL^2(L; L)$ where $[\phi_i]$ denotes the cohomology class represented by the cocycle ϕ_i . Thus $\dim(HL^2(L; L)) = 8$. Let $\{t_i\}_{1 \leq i \leq 8}$ denote the dual basis. Then by Remark 4.2, the universal infinitesimal deformation of L with base $\mathbb{C} \oplus HL^2(L; L)'$ is given by

$$\begin{aligned} [e_i, e_j]_{\eta_1} = & 1 \otimes [e_i, e_j] + t_1 \otimes \phi_1(e_i, e_j) + t_2 \otimes \phi_2(e_i, e_j) + t_3 \otimes \phi_4(e_i, e_j) \\ & + t_4 \otimes \phi_7(e_i, e_j) + t_5 \otimes \phi_8(e_i, e_j) + t_6 \otimes \phi_9(e_i, e_j) \\ & + t_7 \otimes \phi_{10}(e_i, e_j) + t_8 \otimes \phi_{11}(e_i, e_j). \end{aligned}$$

In particular, we get 8 non-equivalent infinitesimal deformations of L given by $\mu_i = \mu_0 + t\phi_i$ for $i = 1, 2, 4, 7, 8, 9, 10, 11$, where μ_0 denotes the original bracket in L . Observe that ϕ_j is skew-symmetric for $j = 2, 4, 7, 8, 9$ and hence the infinitesimal deformations μ_j for $j = 2, 4, 7, 8, 9$ are Lie algebras.

A similar computation yields that 2-dimensional Chevalley-Eilenberg cohomology space is 5-dimensional and spanned by the cohomology classes of ϕ_j for $j = 2, 4, 7, 8, 9$. Hence as before one can write down the universal infinitesimal deformation of L as a Lie algebra. It follows that the universal infinitesimal deformation of L as Lie algebra is not the same as the one when we view it as Leibniz algebra. Thus we see that even at the infinitesimal level the universal deformation of a Lie algebra differs from that when the Lie algebra is deformed as a Leibniz algebra. See [7] for some computation of versal deformations of 3-dimensional Lie algebras. This example shows that by deforming a Lie algebra L in

the category of Leibniz algebras not only one recovers its Lie algebra deformations but can get new deformations of L which are only Leibniz algebras as one might expect.

Conclusions: In this work we gave a constructive method for Leibniz algebras for the solution of the main deformation question, suitable for specific computations. The main feature of this method is that it completely describes all non-equivalent deformations – a problem which did not have a satisfactory solution for a long time. For this we had to consider deformations with complete local algebra base, which was necessary for the existence of a versal deformation. The construction presented here is an inductive procedure, which consists of extending the base of deformation at each step. The specific description of the base of the versal deformation is useful for computations. The examples at the end illustrate the advantages of using our method.

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