

Leibniz algebra deformations of a Lie algebra

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In this note we compute Leibniz algebra deformations of the three-dimensional Heisenberg Lie algebra \mathfrak{n}_3 and compare it to its Lie deformations. It turns out that there are three extra Leibniz deformations. We also describe a versal Leibniz deformation of \mathfrak{n}_3 with the versal base. © 2008 American Institute of Physics.

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I. INTRODUCTION

Since a Lie algebra is also a Leibniz algebra, a natural question arises. If we consider a Lie algebra as a Leibniz algebra and compute its Leibniz algebra deformations, is it true that we can get more Leibniz algebra deformations than just the Lie deformations of the original Lie algebra?

In this note we will demonstrate the problem on a three-dimensional Lie algebra example for which we completely describe its versal Lie deformation and versal Leibniz deformation. It turns out that beside the Lie deformations we get three nonequivalent Leibniz deformations which are not Lie algebras.

Our example is the following. Consider a three-dimensional vector space L spanned by $\{e_1, e_2, e_3\}$ over \mathbb{C} . Define the Heisenberg Lie algebra \mathfrak{n}_3 on it with the bracket matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here the columns are the Lie brackets $[e_1, e_2]$, $[e_1, e_3]$, and $[e_2, e_3]$. That means $[e_2, e_3] = e_1$ and all the other brackets are zero (except of course $[e_3, e_2] = -e_1$). This is the only nilpotent three-dimensional Lie algebra. We compute infinitesimal Lie and Leibniz deformations and show that there are three additional Leibniz cocycles beside the five Lie cocycles. We will show that all these infinitesimal deformations are extendable without any obstructions. We also describe the versal Leibniz deformation.

The structure of the paper is as follows. In Sec. I we recall the necessary preliminaries about Lie and Leibniz cohomology and deformations. In Sec. II we recall the classification of three-dimensional Lie algebras and describe all nonequivalent deformations of the nilpotent Lie algebra \mathfrak{n}_3 . In Sec. III we give the classification of three-dimensional nilpotent Leibniz algebras. Among those \mathfrak{n}_3 is the only nontrivial Lie algebra. Then we compute Leibniz cohomology and give explicitly all nonequivalent infinitesimal deformations. In Sec. IV we show that all infinitesimal deformations are extendable as all the Massey squares turn out to be zero. We identify our deformations with the classified objects. Finally we show that the versal Leibniz deformation is the universal infinitesimal one and describe the base of the versal deformation.

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II. PRELIMINARIES

Let us recall first the Lie algebra cohomology.

Definition 2.1: Suppose \mathfrak{g} is a Lie algebra and A is a module over \mathfrak{g} . Then a q -dimensional cochain of the Lie algebra \mathfrak{g} with coefficients in A is a skew-symmetric q -linear map on \mathfrak{g} with values in A ; the space of all such cochains is denoted by $C^q(\mathfrak{g};A)$. Thus, $C^q(\mathfrak{g};A) = \text{Hom}(\Lambda^q \mathfrak{g}, A)$; this last representation transforms $C^q(\mathfrak{g};A)$ into a \mathfrak{g} -module. The differential

$$d = d_q: C^q(\mathfrak{g};A) \rightarrow C^{q+1}(\mathfrak{g};A)$$

is defined by the formula

$$\begin{aligned} dc(g_1, \dots, g_{q+1}) = & \sum_{1 \leq s < t \leq q+1} (-1)^{s+t-1} c([g_s, g_t], g_1, \dots, \hat{g}_s, \dots, \hat{g}_t, \dots, g_{q+1}) \sum_{1 \leq s \leq q+1} (-1)^s \\ & \times [g_s, c(g_1, \dots, \hat{g}_s, \dots, g_{q+1})], \end{aligned}$$

where $c \in C^q(\mathfrak{g};A)$ and $g_1, \dots, g_{q+1} \in \mathfrak{g}$.

We complete the definition by putting $C^q(\mathfrak{g};A) = 0$ for $q < 0$, $d_q = 0$ for $q < 0$. As can be easily checked, $d_{q+1} \circ d_q = 0$ for all q , so that $(C^q(\mathfrak{g};A), d_q)$ is an algebraic complex; this complex is denoted by $C^*(\mathfrak{g};A)$, while the corresponding cohomology is referred to as the cohomology of the Lie algebra \mathfrak{g} with coefficients in A and is denoted by $H^q(\mathfrak{g};A)$.

Leibniz algebras were introduced by Loday.^{7,9} Let \mathbb{K} denote a field.

Definition 2.2: A Leibniz algebra is a \mathbb{K} -module L , equipped with a bracket operation that satisfies the Leibniz identity,

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad \text{for } x, y, z \in L.$$

Any Lie algebra is automatically a Leibniz algebra, as in the presence of antisymmetry, the Jacobi identity is equivalent to the Leibniz identity. More examples of Leibniz algebras were given in Refs. 7–9, and recently, for instance, in Refs. 1 and 2.

Let L be a Leibniz algebra and M a representation of L . By definition, M is a \mathbb{K} -module equipped with two actions (left and right) of L ,

$$[-, -]: L \times M \rightarrow M \quad \text{and} \quad [-, -]: M \times L \rightarrow M \quad \text{such that}$$

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds, whenever one of the variables is from M and the two others from L .

Define $CL^n(L; M) := \text{Hom}_{\mathbb{K}}(L^{\otimes n}, M)$, $n \geq 0$. Let

$$\delta^n: CL^n(L; M) \rightarrow CL^{n+1}(L; M)$$

be a \mathbb{K} -homomorphism defined by

$$\begin{aligned} \delta^n f(x_1, \dots, x_{n+1}) := & [x_1, f(x_2, \dots, x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}), x_i] \\ & + \sum_{1 \leq i < j \leq n+1} (-1)^{j+1} f(x_1, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \hat{x}_j, \dots, x_{n+1}). \end{aligned}$$

Then $(CL^*(L; M), \delta)$ is a cochain complex, whose cohomology is called the cohomology of the Leibniz algebra L with coefficients in the representation M . The n th cohomology is denoted by $HL^n(L; M)$. In particular, L is a representation of itself with the obvious action given by the bracket in L . The n th cohomology of L with coefficients in itself is denoted by $HL^n(L; L)$.

Let S_n be the symmetric group. Recall that a permutation $\sigma \in S_{p+q}$ is called a (p, q) -shuffle, if $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$. We denote the set of all (p, q) -shuffles in S_{p+q} by $Sh(p, q)$.

For $\alpha \in CL^{p+1}(L;L)$ and $\beta \in CL^{q+1}(L;L)$, define $\alpha \circ \beta \in CL^{p+q+1}(L;L)$ by

$$\begin{aligned} &\alpha \circ \beta(x_1, \dots, x_{p+q+1}) \\ &= \sum_{k=1}^{p+1} (-1)^{q(k-1)} \\ &\quad \times \left\{ \sum_{\sigma \in Sh(q,p-k+1)} \text{sgn}(\sigma) \alpha(x_1, \dots, x_{k-1}, \beta(x_k, x_{\sigma(k+1)}, \dots, x_{\sigma(k+q)}), x_{\sigma(k+q+1)}, \dots, x_{\sigma(p+q+1)}) \right\}. \end{aligned}$$

The graded cochain module $CL^*(L;L) = \bigoplus_r CL^r(L;L)$ equipped with the bracket defined by

$$[\alpha, \beta] = \alpha \circ \beta + (-1)^{p_{\alpha} q_{\beta}} \beta \circ \alpha \quad \text{for } \alpha \in CL^{p+1}(L;L) \quad \text{and} \quad \beta \in CL^{q+1}(L;L)$$

and the differential map d by $d\alpha = (-1)^{|\alpha|} \delta\alpha$ for $\alpha \in CL^*(L;L)$ is a differential graded Lie algebra. (Here $|\alpha|$ denotes the degree of the cochain α .)

Let now \mathbb{K} a field of zero characteristic and the tensor product over \mathbb{K} will be denoted by \otimes . We recall the notion of deformation of a Lie (Leibniz) algebra $\mathfrak{g}(L)$ over a commutative algebra with identity base A with a fixed augmentation $\varepsilon: A \rightarrow \mathbb{K}$ and maximal ideal \mathfrak{M} . Assume $\dim(\mathfrak{M}^k/\mathfrak{M}^{k+1}) < \infty$ for every k (see Ref. 5).

Definition 2.3: A deformation λ of a Lie algebra \mathfrak{g} (or a Leibniz algebra L) with base (A, \mathfrak{M}) or simply with base A is an A -Lie algebra (or an A -Leibniz algebra) structure on the tensor product $A \otimes \mathfrak{g}$ (or $A \otimes L$) with the bracket $[\cdot, \cdot]_{\lambda}$ such that

$$\varepsilon \otimes id: A \otimes \mathfrak{g} \rightarrow \mathbb{K} \otimes \mathfrak{g} \quad (\text{or } \varepsilon \otimes id: A \otimes L \rightarrow \mathbb{K} \otimes L)$$

is an A -Lie algebra (A -Leibniz algebra) homomorphism.

A deformation of the Lie (Leibniz) algebra $\mathfrak{g}(L)$ with base A is called *infinitesimal*, or *first order*, if in addition to this $\mathfrak{M}^2 = 0$. We call a deformation of *order* k if $\mathfrak{M}^{k+1} = 0$. A deformation with base is called local if A is a local algebra over \mathbb{K} , which means A has a unique maximal ideal.

Suppose A is a complete local algebra ($A = \varprojlim_{n \rightarrow \infty} (A/\mathfrak{M}^n)$), where \mathfrak{M} is the maximal ideal in A .

Then a deformation of $\mathfrak{g}(L)$ with base A which is obtained as the projective limit of deformations of $\mathfrak{g}(L)$ with base A/\mathfrak{M}^n is called a *formal deformation* of $\mathfrak{g}(L)$.

Definition 2.4: Suppose λ is a given deformation of L with base (A, \mathfrak{M}) and augmentation $\varepsilon: A \rightarrow \mathbb{K}$. Let A' be another commutative algebra with identity and a fixed augmentation $\varepsilon': A' \rightarrow \mathbb{K}$. Suppose $\phi: A \rightarrow A'$ is an algebra homomorphism with $\phi(1) = 1$ and $\varepsilon' \circ \phi = \varepsilon$. Let $\ker(\varepsilon') = \mathfrak{M}'$. Then the push-out $\phi_*\lambda$ is the deformation of L with base (A', \mathfrak{M}') and bracket

$$[a'_1 \otimes_A (a_1 \otimes l_1), a'_2 \otimes_A (a_2 \otimes l_2)]_{\phi_*\lambda} = a'_1 a'_2 \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_{\lambda}$$

where $a'_1, a'_2 \in A'$, $a_1, a_2 \in A$, and $l_1, l_2 \in L$. Here A' is considered as an A -module by the map $a' \cdot a = a' \phi(a)$ so that

$$A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L).$$

Definition 2.5: (See Ref. 3) Let C be a complete local algebra. A formal deformation η of a Lie algebra \mathfrak{g} (Leibniz algebra L) with base C is called *versal* if (i) for any formal deformation λ of $\mathfrak{g}(L)$ with base A there exists a homomorphism $f: C \rightarrow A$ such that the deformation λ is equivalent to $f_*\eta$; (ii) if A satisfies the condition $\mathfrak{M}^2 = 0$, then f is unique.

Theorem 2.6: If $H^2(\mathfrak{g}; \mathfrak{g})$ is finite dimensional, then there exists a versal deformation of \mathfrak{g} (similarly for L).

Proof: Follows from the general theorem of Schlessinger,¹¹ like it was shown for Lie algebras in Ref. 3.

In Ref. 4 a construction for a versal deformation of a Lie algebra was given and it was generalized to Leibniz algebras in Ref. 5. The computation of a specific example is given in Ref. 10.

Let us describe the universal infinitesimal deformation (see Refs. 4 and 5). For simplicity we will only discuss the Leibniz algebra case.

Assume that $\dim(HL^2(L;L)) < \infty$. Denote the space $HL^2(L;L)$ by H . Consider the algebra $C_1 = \mathbb{K} \oplus H'$ where H' is the dual of H , by setting

$$(k_1, h_1) \cdot (k_2, h_2) = (k_1 k_2, k_1 h_2 + k_2 h_1) \quad \text{for } (k_1, h_1), (k_2, h_2) \in C_1.$$

Observe that the second summand is an ideal of C_1 with zero multiplication. Fix a homomorphism

$$\mu: H \rightarrow CL^2(L;L) = \text{Hom}(L^{\otimes 2}; L)$$

which takes a cohomology class into a cocycle representing it. Notice that there is an isomorphism $H' \otimes L \cong \text{Hom}(H;L)$, so we have

$$C_1 \otimes L = (\mathbb{K} \oplus H') \otimes L \cong (\mathbb{K} \otimes L) \oplus (H' \otimes L) \cong L \oplus \text{Hom}(H;L).$$

Using the above identification, define a Leibniz bracket on $C_1 \otimes L$ as follows. For $(l_1, \phi_1), (l_2, \phi_2) \in L \oplus \text{Hom}(H;L)$ let

$$[(l_1, \phi_1), (l_2, \phi_2)]_{\eta_1} = ([l_1, l_2], \psi),$$

where the map $\psi: H \rightarrow L$ is given by

$$\psi(\alpha) = \mu(\alpha)(l_1, l_2) + [\phi_1(\alpha), l_2] + [l_1, \phi_2(\alpha)] \quad \text{for } \alpha \in H.$$

It is straightforward to check that $C_1 \otimes L$ along with the above bracket η_1 is a Leibniz algebra over C_1 . The Leibniz identity is a consequence of the fact that $\delta\mu(\alpha) = 0$ for $\alpha \in H$. Thus η_1 is an infinitesimal deformation of L with base $C_1 = \mathbb{K} \oplus H'$. It is proven in Ref. 5.

Proposition 2.7: *Up to an isomorphism, the deformation η_1 does not depend on the choice of μ .*

Remark: Suppose $\{h_i\}_{1 \leq i \leq n}$ is a basis of H and $\{g_i\}_{1 \leq i \leq n}$ is the dual basis. Let $\mu(h_i) = \mu_i \in CL^2(L;L)$. Under the identification $C_1 \otimes L = L \oplus \text{Hom}(H;L)$, an element $(l, \phi) \in L \oplus \text{Hom}(H;L)$ corresponds to $1 \otimes l + \sum_{i=1}^n g_i \otimes \phi(h_i)$. Then for $(l_1, \phi_1), (l_2, \phi_2) \in L \oplus \text{Hom}(H;L)$ their bracket $([l_1, l_2], \psi)$ corresponds to

$$1 \otimes [l_1, l_2] + \sum_{i=1}^n g_i \otimes (\mu_i(l_1, l_2) + [\phi_1(h_i), l_2] + [l_1, \phi_2(h_i)]).$$

In particular, for $l_1, l_2 \in L$ we have

$$[1 \otimes l_1, 1 \otimes l_2]_{\eta_1} = 1 \otimes [l_1, l_2] + \sum_{i=1}^n g_i \otimes \mu_i(l_1, l_2).$$

The main property of η_1 is the universality in the class of infinitesimal deformations with a finite dimensional base.

Proposition 2.8: *For any infinitesimal deformation λ of a Leibniz algebra L with a finite dimensional base A there exists a unique homomorphism $\phi: C_1 = (\mathbb{K} \oplus H') \rightarrow A$ such that λ is equivalent to the push-out $\phi_* \eta_1$.*

After obtaining the universal infinitesimal deformation we would like to extend it to higher order. If we have a universal infinitesimal deformation with basis cocycles $\{\phi_i\}_{i=1}^r$, the obstructions to extend it to a second order deformation are the Lie brackets $[\phi_i, \phi_j]$ in the cochain complex (see Refs. 3 and 5). These are also called first order Massey operations. If these bracket cochains are coboundaries we can extend our infinitesimal to the second order deformation. The construction of a versal deformation for Leibniz algebras is given in Ref. 5.

III. DEFORMATIONS OF THE THREE-DIMENSIONAL LIE ALGEBRA \mathfrak{n}_3

Let us recall the classification of three-dimensional complex Lie algebras (see Ref. 6). Fix a basis $\{e_1, e_2, e_3\}$. The nilpotent algebra \mathfrak{n}_3 with the commutator matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the solvable algebra $\mathfrak{r}_{3,1}$ with the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

the simple Lie algebra \mathfrak{sl}_2 with the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and the projective family of pairwise nonisomorphic algebras $d(r:s)$ with the matrix

$$\begin{pmatrix} 0 & r & 1 \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}.$$

In Ref. 6 the moduli space of these Lie algebras is described with the help of versal deformations. Let us recall the results for the nilpotent Lie algebra \mathfrak{n}_3 .

The dimensions of the cohomology spaces of the classified algebras are as follows:

Type	H^1	H^2	H^3
\mathfrak{n}_3	4	5	2
$d = \mathfrak{r}_{3,1}$	3	3	0
$d(1:1) = \mathfrak{r}_3$	1	1	0
$d(r:s), r \neq \pm s$	1	1	0
$d(1:0) = \mathfrak{r}_2 \oplus \mathbb{C}$	2	1	0
$d(1:-1) = \mathfrak{r}_{3,-1}$	1	2	1
\mathfrak{sl}_2	0	0	0

We get $H^2(\mathfrak{n}_3; \mathfrak{n}_3)$ is five dimensional. Let us give explicit representative cocycles which form the basis of $H^2(\mathfrak{n}_3; \mathfrak{n}_3)$. We give the nonzero values.

- (1) $f_1: f_1(e_2, e_3) = e_3$.
- (2) $f_2: f_2(e_1, e_2) = e_2, f_2(e_1, e_3) = -e_3$.
- (3) $f_3: f_3(e_1, e_2) = e_3$.
- (4) $f_4: f_4(e_1, e_3) = e_1$.
- (5) $f_5: f_5(e_1, e_3) = e_2$.

It is easy to check that all the Massey brackets are zero. So the universal infinitesimal deformation is versal and is given by the matrix

$$\begin{pmatrix} 0 & t_4 & 1 \\ t_2 & t_5 & 0 \\ t_3 & -t_2 & t_1 \end{pmatrix}.$$

Let us check how our infinitesimal deformation (which are real deformations) fit in the moduli space of three-dimensional Lie algebras.

The first deformation with cocycle f_1 has the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{pmatrix},$$

which is equivalent to $\mathfrak{r}_2 \oplus \mathbb{C}$.

The second deformation with cocycle f_2 has the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ t & 0 & 0 \\ 0 & -t & 0 \end{pmatrix},$$

which is equivalent to \mathfrak{sl}_2 .

The third deformation with cocycle f_3 has the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix},$$

which is equivalent to $\mathfrak{r}_{3,-1}$.

The fourth deformation with cocycle f_4 has the matrix

$$\begin{pmatrix} 0 & t & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is equivalent to $\mathfrak{r}_2 \oplus \mathbb{C}$.

The fifth deformation with cocycle f_5 has the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which is equivalent to $\mathfrak{r}_{3,-1}$.

The first deformation is equivalent to \mathfrak{sl}_2 , the second and fourth deformations give the Lie algebra $\mathfrak{r}_{3,-1}$ that means that the Lie algebra \mathfrak{n}_3 deforms to the family in two different ways. The third and fifth deformations are equivalent to $\mathfrak{r}_2 \oplus \mathbb{C}$, which means that \mathfrak{n}_3 deforms to $\mathfrak{r}_2 \oplus \mathbb{C}$ in two different ways.

IV. LEIBNIZ DEFORMATIONS OF \mathfrak{n}_3

The classification of three-dimensional nilpotent Leibniz algebras is known. Let us recall the definition.

We take $L^1=L, L^{k+1}=[L^k, L]$ for $k \in \mathbb{N}$.

Definition 4.1: A Leibniz algebra L is called nilpotent if there exists an integer $n \in \mathbb{N}$ such that

$$L^1 \supset L^2 \supset \dots \supset L^n = 0.$$

The smallest integer n for which $L^n=0$ is called the nilindex of L .

The classification of complex nilpotent Leibniz algebras up to isomorphism for dimensions of 2 and 3 is in Refs. 7 and 1. In dimension of 3 there are five nonisomorphic algebras and one infinite family of pairwise nonisomorphic algebras. The list of this classification is given below.

- (i) λ_1 : Abelian.
- (ii) λ_2 : $[e_1, e_1] = e_2$.
- (iii) λ_3 : $[e_2, e_3] = e_1$, $[e_3, e_2] = -e_1$.
- (iv) λ_4 : $[e_2, e_2] = e_1$, $[e_3, e_3] = \alpha e_1$, $[e_2, e_3] = e_1$; $\alpha \in \mathbb{C}$.
- (v) λ_5 : $[e_2, e_2] = e_1$, $[e_3, e_2] = e_1$, $[e_2, e_3] = e_1$.
- (vi) λ_6 : $[e_3, e_3] = e_1$, $[e_1, e_3] = e_2$.

Let us mention that in this list only λ_3 is a Lie algebra. This is the one which is denoted by \mathfrak{n}_3 and this is the case we are computing.

Now we consider the Leibniz algebra $L = \lambda_3 = \mathfrak{n}_3$ and compute its second Leibniz cohomology space.

Our computation consists of the following steps.

- (i) To determine a basis of the space of cocycles $ZL^2(L; L)$.
 - (ii) To find out a basis of the coboundary space $BL^2(L; L)$.
 - (iii) To determine the quotient space $HL^2(L; L)$.
- (i) Let $\psi \in ZL^2(L; L)$. Then $\psi: L \otimes 2 \rightarrow L$ is a linear map and $\delta\psi = 0$, where

$$\begin{aligned} \delta\psi(e_i, e_j, e_k) &= [e_i, \psi(e_j, e_k)] + [\psi(e_i, e_k), e_j] - [\psi(e_i, e_j), e_k] - \psi([e_i, e_j], e_k) + \psi(e_i, [e_j, e_k]) \\ &\quad + \psi([e_i, e_k], e_j) \quad \text{for } 0 \leq i, j, k \leq 3. \end{aligned}$$

Suppose $\psi(e_i, e_j) = \sum_{k=1}^3 a_{i,j}^k e_k$ where $a_{i,j}^k \in \mathbb{C}$ for $1 \leq i, j, k \leq 3$. Since $\delta\psi = 0$ equating the coefficients of e_1, e_2 , and e_3 in $\delta\psi(e_i, e_j, e_k)$, we get the following relations.

- (i) $a_{1,1}^1 = a_{1,1}^2 = a_{1,1}^3 = 0$.
- (ii) $a_{1,2}^1 = -a_{2,1}^1$; $a_{1,2}^2 = -a_{2,1}^2$; $a_{1,2}^3 = -a_{2,1}^3$.
- (iii) $a_{1,3}^1 = -a_{3,1}^1$; $a_{1,3}^2 = -a_{3,1}^2$; $a_{1,3}^3 = -a_{3,1}^3$.
- (iv) $a_{2,2}^2 = a_{2,2}^3 = 0$.
- (v) $a_{2,3}^2 = -a_{3,2}^2$; $a_{2,3}^3 = -a_{3,2}^3$.
- (vi) $a_{3,3}^2 = a_{3,3}^3 = 0$.
- (vii) $a_{1,2}^2 = -a_{1,3}^3$.

Observe that there is no relation among $a_{2,2}^1, a_{2,3}^1, a_{3,2}^1$, and $a_{3,3}^1$. Therefore, in terms of the ordered basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_2, e_3 \otimes e_3\}$ of $L^{\otimes 2}$ and $\{e_1, e_2, e_3\}$ of L , the matrix corresponding to ψ is of the form

$$M = \begin{pmatrix} 0 & x_9 & x_4 & -x_9 & x_6 & x_7 & -x_4 & x_{10} & x_{11} \\ 0 & x_2 & x_5 & -x_2 & 0 & x_8 & -x_5 & -x_8 & 0 \\ 0 & x_3 & -x_2 & -x_3 & 0 & x_1 & x_2 & -x_1 & 0 \end{pmatrix},$$

where $x_1 = a_{2,3}^3, x_2 = a_{1,2}^2, x_3 = a_{1,2}^3, x_4 = a_{1,3}^1, x_5 = a_{1,3}^2, x_6 = a_{2,2}^1, x_7 = a_{2,3}^1, x_8 = a_{2,3}^2, x_9 = a_{1,2}^1, x_{10} = a_{3,2}^1$, and $x_{11} = a_{3,3}^1$ are in \mathbb{C} . Let $\phi_i \in ZL^2(L; L)$ for $1 \leq i \leq 11$ be the cocycle with $x_i = 1$ and $x_j = 0$ for $i \neq j$ in the above matrix of ψ . It is easy to check that $\{\phi_1, \dots, \phi_{11}\}$ forms a basis of $ZL^2(L; L)$.

- (ii) Let $\psi_0 \in BL^2(L; L)$. We have $\psi_0 = \delta g$ for some 1-cochain $g \in CL^1(L; L) = \text{Hom}(L; L)$. Sup-

pose the matrix associated with ψ_0 is same as the above matrix M . Let $g(e_i) = g_i^1 e_1 + g_i^2 e_2 + g_i^3 e_3$ for $i=1, 2, 3$. The matrix associated with g is given by

$$\begin{pmatrix} g_1^1 & g_2^1 & g_3^1 \\ g_1^2 & g_2^2 & g_3^2 \\ g_1^3 & g_2^3 & g_3^3 \end{pmatrix}.$$

From the definition of coboundary we get

$$\delta g(e_i, e_j) = [e_i, g(e_j)] + [g(e_i), e_j] - \psi([e_i, e_j])$$

for $1 \leq i, j \leq 3$. The matrix δg can be written as

$$\begin{pmatrix} 0 & -g_1^3 & g_1^2 & g_1^3 & 0 & -(g_1^1 - g_2^2 + g_3^3) & -g_1^2 & (g_1^1 - g_2^2 + g_3^3) & 0 \\ 0 & 0 & 0 & 0 & 0 & -g_1^2 & 0 & g_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -g_1^3 & 0 & g_1^3 & 0 \end{pmatrix}.$$

Since $\psi_0 = \delta g$ is also a cocycle in $CL^2(L; L)$, comparing matrices δg and M we conclude that the matrix of ψ_0 is of the form

$$\begin{pmatrix} 0 & x_1 & x_4 & -x_1 & 0 & x_7 & -x_4 & -x_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_4 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & -x_1 & 0 \end{pmatrix}.$$

Let $\phi'_i \in BL^2(L; L)$ for $i=1, 4, 7$ be the coboundary with $x_i=1$ and $x_j=0$ for $i \neq j$ in the above matrix of ψ_0 . It follows that $\{\phi'_7, \phi'_8, \phi'_9\}$ forms a basis of the coboundary space $BL^2(L; L)$.

(iii) It is straightforward to check that

$$\{[\phi_1], [\phi_2], [\phi_3], [\phi_4], [\phi_5], [\phi_6], [\phi_{10}], [\phi_{11}]\}$$

span $HL^2(L; L)$, where $[\phi_i]$ denotes the cohomology class represented by the cocycle ϕ_i . Thus $\dim(HL^2(L; L))=8$. A basis of $HL^2(L; L)$ is given explicitly below (the zero values are omitted).

- (1) $\phi_1: \phi_1(e_2, e_3) = e_3, \phi_1(e_3, e_2) = -e_3.$
- (2) $\phi_2: \phi_2(e_1, e_2) = e_2, \phi_2(e_2, e_1) = -e_2, \phi_2(e_1, e_3) = -e_3, \phi_2(e_3, e_1) = e_3.$
- (3) $\phi_3: \phi_3(e_1, e_2) = e_3, \phi_3(e_2, e_1) = -e_3.$
- (4) $\phi_4: \phi_4(e_1, e_3) = e_1, \phi_4(e_3, e_1) = -e_1.$
- (5) $\phi_5: \phi_5(e_1, e_3) = e_2, \phi_5(e_3, e_1) = -e_2.$
- (6) $\phi_6: \phi_6(e_2, e_2) = e_1.$
- (7) $\phi_{10}: \phi_{10}(e_3, e_2) = e_1.$
- (8) $\phi_{11}: \phi_{11}(e_3, e_3) = e_1.$

Consider $\mu_i = \mu_0 + t\phi_i$ for $i=1, 2, 3, 4, 5, 6, 10, 11$, where μ_0 denotes the original bracket in L . This gives eight nonequivalent infinitesimal deformations of L .

Here $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ are skew symmetric, so $\phi_i \in \text{Hom}(\Lambda^2 L; L) \subset \text{Hom}(L^{\otimes 2}; L)$ for $1 \leq i \leq 5$. This are exactly the cocycles presented in the previous section.

The last three cocycles define Leibniz deformations, more precisely their infinitesimal part. The Leibniz two-cocycle ϕ_6 defines the infinitesimal deformation with matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & t & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

ϕ_{10} defines the infinitesimal deformation with matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & t-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

ϕ_{11} defines the infinitesimal deformation with matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is interesting to realize that all the three Leibniz deformations are nilpotent and they are real deformation. So they can be identified with the given list. Namely, the μ_6 is combination of λ_3 and λ_2 . μ_{10} is combination of λ_2 , λ_3 , and λ_6 . μ_{11} is combination of λ_2 and λ_3 .

Remark: Notice here that $f_i = \phi_i$ for $i = 1, \dots, 5$. So, if we consider the infinitesimal deformations of the Leibniz algebra L , that automatically contains all infinitesimal Lie algebra deformations. It is interesting to note that we get few more deformations of the original bracket μ_0 giving different Leibniz algebra structures, by considering the Leibniz algebra deformation.

In order to get a simpler expression for the nontrivial cocycles let us denote ϕ_1, \dots, ϕ_6 by $\bar{\phi}_1, \dots, \bar{\phi}_6$, ϕ_{10} by $\bar{\phi}_7$, and ϕ_{11} by $\bar{\phi}_8$.

First we describe the universal infinitesimal deformation for L . Let us denote a basis of $HL^2(L; L)'$ by $\{t_i\}_{1 \leq i \leq 8}$. By Remark the universal infinitesimal deformation of L can be written as

$$[1 \otimes e_i, 1 \otimes e_j]_{\eta_1} = 1 \otimes [e_i, e_j] + \sum_{i=1}^8 t_i \otimes \bar{\phi}_i(e_i, e_j),$$

with base $C_1 = \mathbb{C} \oplus_{1 \leq i \leq 8} \mathbb{C}t_i$.

V. EXTENSION OF THE INFINITESIMAL DEFORMATION

Let us try now to extend the universal infinitesimal deformation. All the eight infinitesimal deformations considered as one parameter deformations are real deformations as $[\bar{\phi}_i, \bar{\phi}_i] = 0$ for $i = 1, \dots, 8$. Now let us consider the mixed brackets $[\bar{\phi}_i, \bar{\phi}_j]$ for $i \neq j$. For the Lie part we get that

$$[\bar{\phi}_1, \bar{\phi}_2], [\bar{\phi}_1, \bar{\phi}_5], [\bar{\phi}_3, \bar{\phi}_4]$$

give nontrivial three cochains from which two of them are linearly independent. We get two relations on the parameter space,

$$t_1 t_2 + t_3 t_4 = 0,$$

$$t_1 t_5 = 0.$$

This way the versal Lie deformation $[\cdot, \cdot]_{v_1}$ of the Lie algebra \mathfrak{n}_3 is defined by the infinitesimal part as follows:

$$[e_1, e_2]_{v_1} = t_2 e_2 + t_3 e_3,$$

$$[e_1, e_3]_{v_1} = t_4 e_1 + t_5 e_2 - t_2 e_3,$$

$$[e_2, e_3]_{v_1} = e_1 + t_1 e_3.$$

The versal basis is the factor space

$$\mathbb{C}[[t_1, t_2, t_3, t_4, t_5]] / \{t_1 t_5, t_1 t_2 + t_3 t_4\}.$$

The cocycles corresponding to only Leibniz algebras result in trivial mixed brackets,

$$[\bar{\phi}_k, \bar{\phi}_l] = 0 \quad \text{for } k, l = 6, 7, 8.$$

If we take the bracket of a cocycle from the Lie set and take the bracket with a cocycle from the Leibniz set, it turns out that not all the Massey brackets are trivial. Namely,

$$[\bar{\phi}_2, \bar{\phi}_6], [\bar{\phi}_3, \bar{\phi}_6], [\bar{\phi}_5, \bar{\phi}_6], [\bar{\phi}_3, \bar{\phi}_{10}], [\bar{\phi}_5, \bar{\phi}_{10}], [\bar{\phi}_2, \bar{\phi}_{11}], [\bar{\phi}_3, \bar{\phi}_{11}], \quad \text{and} \quad [\bar{\phi}_5, \bar{\phi}_{11}]$$

are nontrivial three cochains. They give us second order relation on the base of the versal deformation,

$$t_5 t_7 = 0, \quad t_3 t_6 = 0, \quad t_5 t_6 = 0, \quad t_5 t_8 = 0, \quad t_3 t_8 = 0, \quad t_2 t_8, \quad t_3 t_7, \quad \text{and} \quad t_2 t_6 = 0$$

together with the relations for the Lie part we get all the second order relations for the base of the Leibniz versal deformation.

As no higher order brackets show up we get that the versal Leibniz deformation $[\cdot, \cdot]_v$ is defined as follows:

$$[e_1, e_1]_v = 0,$$

$$[e_1, e_2]_v = t_2 e_2 + t_3 e_3,$$

$$[e_1, e_3]_v = t_4 e_1 + t_5 e_2 - t_2 e_3,$$

$$[e_2, e_1]_v = -t_2 e_2 - t_3 e_3,$$

$$[e_2, e_2]_v = t_6 e_1,$$

$$[e_2, e_3]_v = e_1 + t_1 e_3,$$

$$[e_3, e_1]_v = -t_4 e_1 - t_5 e_2 + t_2 e_3,$$

$$[e_3, e_2]_v = (t_7 - 1)e_1 - t_1 e_3,$$

$$[e_3, e_3]_v = t_8 e_1.$$

The base of the versal Leibniz deformation is the factor space

$$\mathbb{C}[[t_1, \dots, t_8]] / \langle t_1 t_5, t_1 t_2 + t_3 t_4, t_2 t_6, t_5 t_6, t_3 t_6, t_5 t_7, t_3 t_8, t_2 t_8, t_3 t_7, t_5 t_8 \rangle.$$

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