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COHOMOLOGY OF NILPOTENT SUBALGEBRAS OF AFFINE LIE ALGEBRAS

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ABSTRACT. We compute the cohomology of the maximal nilpotent Lie algebra of an affine Lie algebra $\hat{\mathfrak{g}}$ with coefficients in modules of functions on the circle with values in a representation space of $\mathfrak{g}$. These modules are not highest weight modules and are somewhat similar to the adjoint representation.

INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{n}_+ \subset \mathfrak{b}$ the maximal nilpotent ideal of $\mathfrak{b}$. The Bott-Kostant Theorem for Lie algebra cohomology is the following.

Theorem [K]. *Let V be an irreducible representation of $\mathfrak{g}$ with dominant highest weight and $\mathfrak{n}$ a maximal nilpotent subalgebra of $\mathfrak{g}$. Then* $\dim H^i(\mathfrak{n}\,;V)$ *is equal to the number of elements of length i in the Weyl group of $\mathfrak{g}$.*

Consider the affine infinite-dimensional graded Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ corresponding to $\mathfrak{g}$, with $\hat{\mathfrak{g}}_i = \mathfrak{g} \otimes t^i$. There are at least two analogues of the above Theorem for affine algebras. The most direct analogue is the following: if V is an irreducible representation of the current algebra $\hat{\mathfrak{g}}$ with dominant highest weight and $\hat{\mathfrak{n}}_+$ is a maximal nilpotent subalgebra of $\hat{\mathfrak{g}}$, that is,

$$\hat{\mathfrak{n}}_+ = (\mathfrak{n}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t) \oplus (\mathfrak{g} \otimes t^2) \oplus \cdots ,$$

then $\dim H^i(\hat{\mathfrak{n}}_+\,;V)$ is equal to the number of elements of length i in the Weyl group. This Theorem was proved by Garland in 1975 [G] and Garland and Lepowsky in 1976 (see [GL]). The proof is similar to that of the finite-dimensional case.

In this paper we present the proof of a different analogue of the Bott- Kostant Theorem obtained jointly with Feigin and announced in [FF]. Namely, we compute the cohomology of $\hat{\mathfrak{n}}_+$ with coefficients in modules of functions on the

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circle S^1 with values in a representation space of $\mathfrak{g}$. These modules are *not* highest weight modules and are somewhat similar to the adjoint representation.

Results

Let V be a representation of $\mathfrak{g}$, A a $\mathbb{C}$-algebra, and $\varphi: \mathbb{C}[t, t^{-1}] \to A$ a homomorphism. Let us define a representation of $\hat{\mathfrak{g}}$ on $V \otimes A$ by

$$(x \otimes f)(v \otimes a) = x(v) \otimes \varphi(f)a,$$

where $x \in \mathfrak{g}$, $v \in V$, $f \in \mathbb{C}[t, t^{-1}]$, and $a \in A$.

Consider two special cases for A and φ:

(a) $A = \mathbb{C}[t, t^{-1}]$ and $\varphi = \mathrm{id}$. In this case denote the module $V \otimes A$ by $\hat{V}$. It consists of rational functions $\mathbb{C} \to V$ that are regular outside the origin.

(b) $A = \mathbb{C}$ and $\varphi(f) = f(1)$. In this case denote the module $V \otimes A$ by V_1.

Note that the map assigning to a function $\mathbb{C} \to V$ its value at 1 defines a homomorphism $\hat{V} \to V_1$. The space $\hat{V}$ is endowed with an obvious module structure over the algebra $\mathbb{C}[t, t^{-1}]$, and it is easy to see that multiplication by an element of $\mathbb{C}[t, t^{-1}]$ is a $\mathfrak{g}$-endomorphism of the $\hat{\mathfrak{g}}$-module $\hat{V}$. Finally, note that $\hat{V}$ is a graded $\hat{\mathfrak{g}}$-module, that is, $\hat{V} = \bigoplus_{i \in \mathbb{Z}} \hat{V}_i$, with $\hat{V}_i = V \otimes t^i$.

First we will compute the cohomology of $\hat{\mathfrak{n}}_+$ with coefficients in $\hat{V}$. The Lie algebra $\hat{\mathfrak{n}}_+$ is a graded subalgebra of $\hat{\mathfrak{g}}$, and $\hat{V}$ is a graded $\hat{\mathfrak{g}}$-module and $\hat{\mathfrak{n}}_+$-module. Denote by $C^*(\hat{\mathfrak{n}}_+; \hat{V})$ the cochain complex of $\hat{\mathfrak{n}}_+$ with coefficients in the $\hat{\mathfrak{n}}_+$-module $\hat{V}$. The complex $C^*(\hat{\mathfrak{n}}_+; \hat{V})$ and the cohomology $H^*(\hat{\mathfrak{n}}_+; \hat{V})$ are graded by weights. To state this, we introduce the notation

$$C^*_{(m)}(\hat{\mathfrak{n}}_+; \hat{V}) = \bigoplus_{r \in \mathbb{Z}} \mathrm{Hom}((\textstyle\bigwedge^q \hat{\mathfrak{n}}_+)_r, \hat{V}_{r+m}),$$

where

$$(\textstyle\bigwedge^q \hat{\mathfrak{n}}_+)_r = \textstyle\bigwedge^q \hat{\mathfrak{n}}_+ \cap \left(\bigotimes^q \hat{\mathfrak{n}}_+ \right)_r$$

and

$$\left(\bigotimes^q \hat{\mathfrak{n}}_+ \right)_r = \bigoplus_{r_1 + \cdots + r_q = r} ((\hat{\mathfrak{n}}_+)_{r_1} \otimes \cdots \otimes (\hat{\mathfrak{n}}_+)_{r_q}).$$

In this notation the grading is

$$C^*(\hat{\mathfrak{n}}_+; \hat{V}) = \bigoplus_{m \in \mathbb{Z}} C^*_{(m)}(\hat{\mathfrak{n}}_+; \hat{V})$$

and, similarly,

$$H^*(\hat{\mathfrak{n}}_+; \hat{V}) = \bigoplus_{m \in \mathbb{Z}} H^*_{(m)}(\hat{\mathfrak{n}}_+; \hat{V}).$$

Lemma 1. $H^*_{(m)}(\hat{\mathfrak{n}}_+; \hat{V}) \cong H^*(\hat{\mathfrak{n}}_+; V_1)$ *for all* $m \in \mathbb{Z}$.

Proof. The composition of mappings

$$C^*_{(m)}(\hat{\mathfrak{n}}_+; \hat{V}) \xrightarrow{i} C^*(\hat{\mathfrak{n}}_+; \hat{V}) \xrightarrow{s} C^*(\hat{\mathfrak{n}}_+; V_1),$$

where i is the embedding and s is induced by the homomorphism $\hat{V} \to V_1$, is obviously a complex isomorphism.

Moreover, the isomorphisms

$$\hat{V}_i = V \otimes t^i \longrightarrow V \otimes t^{i+1} = \hat{V}_{i+1}$$

define a $\hat{\mathfrak{g}}$–isomorphism $t : \hat{V} \to \hat{V}$ of degree 1, which generates an action of $\mathbb{C}[t, t^{-1}]$ in $\hat{V}$ and in $H^*(\hat{\mathfrak{n}}_+ ; \hat{V})$. Evidently, t maps $H^*_{(m)}(\hat{\mathfrak{n}}_+ ; \hat{V})$ isomorphically onto $H^*_{(m+1)}(\hat{\mathfrak{n}}_+ ; \hat{V})$. Hence we have

Lemma 1′. $H^*(\hat{\mathfrak{n}}_+ ; \hat{V}) \cong \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} H^*(\hat{\mathfrak{n}}_+ ; V_1)$.

Now let us compute the cohomology $H^*(\hat{\mathfrak{n}}_+ ; V_1)$. Introduce the subalgebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ of $\hat{\mathfrak{g}}$. In the following we will identify $\mathfrak{g}$ with $\mathfrak{g} \otimes 1 \subset \mathfrak{g}[t]$. The Lie algebra $\hat{\mathfrak{n}}_+$ is embedded into $\mathfrak{g}[t]$, and V_1 is naturally endowed with a $\mathfrak{g}[t]$-module structure.

Theorem 1. *We have the following isomorphism of cohomology spaces*:

$$H^i(\hat{\mathfrak{n}}_+ ; V_1) \xrightarrow{\cong} \bigoplus_{p+q=i} H^p(\hat{\mathfrak{n}}_+ ; \mathbb{C}) \otimes H^q(\mathfrak{g}[t], \mathfrak{g} ; V_1).$$

Proof. We begin the proof, which will take most of this paper, by introducing two subalgebras of $\hat{\mathfrak{g}}$. The first is

$$\bar{\mathfrak{g}} = (t-1)\mathfrak{g} \oplus (t-1)^2\mathfrak{g} \oplus \cdots$$

consisting of loops $\varphi(t)$ which vanish at 1, and the second is

$$\bar{\mathfrak{n}} = \hat{\mathfrak{n}}_+ \cap \bar{\mathfrak{g}}.$$

Note that $\mathfrak{g}[t] = \hat{\mathfrak{n}}_+ + \bar{\mathfrak{g}}$. Let G be the compact connected, simply connected Lie group, corresponding to the compact real form of $\mathfrak{g}$. Next we need the following theorem.

Theorem 2. *We have*

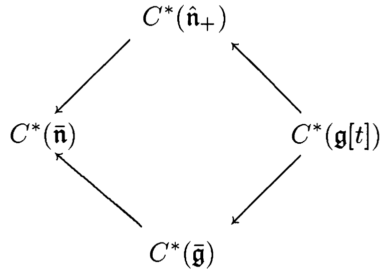
$$H^*(\bar{\mathfrak{n}}) \cong H^*(\hat{\mathfrak{n}}_+) \otimes H^*(\bar{\mathfrak{g}}) \otimes H^*(\Omega G),$$

where ΩG is the loop space of G.

Proof. Since we have the embeddings

$$\begin{array}{ccc} & \hat{\mathfrak{n}}_+ & \\ \nearrow & & \searrow \\ \bar{\mathfrak{n}} & & \mathfrak{g}[t] \\ \searrow & & \nearrow \\ & \bar{\mathfrak{g}} & \end{array}$$

with $\bar{\mathfrak{g}} \mapsto \mathfrak{g}[t] \mapsto \mathfrak{g}$, we also have the diagram:



Consequently,

$$C^*(\bar{\mathfrak{n}}) = C^*(\hat{\mathfrak{n}}_+) \otimes_{C^*(\mathfrak{g}[t])} C^*(\bar{\mathfrak{g}}),$$

where the tensor product is taken in the category of differential algebras. In such a situation there exists an Eilenberg-Moore spectral sequence, connecting these four differential algebras. Its second term is

$$E_2 = \mathrm{Tor}_{H^*(\mathfrak{g}[t])}(H^*(\hat{\mathfrak{n}}_+), H^*(\bar{\mathfrak{g}})),$$

and its limit term is $H^*(\bar{\mathfrak{n}})$. We know that $H^*(\mathfrak{g}[t]) \cong H^*(\mathfrak{g})$ (see, for example, [F]). On the other hand, since $H^*(\mathfrak{g})$ acts trivially on $H^*(\bar{\mathfrak{g}})$ and also on $H^*(\hat{\mathfrak{n}}_+)$, we conclude that the composition $H^*(\mathfrak{g}) \to H^*(\mathfrak{n}_+) \to H^*(\hat{\mathfrak{n}}_+)$ is trivial. So we have

$$E_2 = \mathrm{Tor}_{H^*(\mathfrak{g})}(H^*(\hat{\mathfrak{n}}_+), H^*(\bar{\mathfrak{g}})) \cong H^*(\hat{\mathfrak{n}}_+) \otimes H^*(\bar{\mathfrak{g}}) \otimes \mathrm{Tor}_{H^*(\mathfrak{g})}(\mathbb{C}, \mathbb{C}).$$

On the other hand,

$$\mathrm{Tor}_{H^*(\mathfrak{g})}(\mathbb{C}, \mathbb{C}) \cong H^*(\Omega G).$$

Indeed, the cohomology algebra of $\mathfrak{g}$ with trivial coefficients coincides with the cohomology algebra of G, and by the Hopf Theorem it is commutative and free (see [S]). Using the computation of $\mathrm{Tor}_A(\mathbb{C}, \mathbb{C})$ for the free commutative algebra A ([M, Proposition 7.3] and see also [A]) and the connection between the cohomology of G and ΩG, we get the isomorphism $\mathrm{Tor}_A(\mathbb{C}, \mathbb{C}) \cong H^*(\Omega G)$:

$$H^*(\mathfrak{g}) = \bigwedge\nolimits^*(e_{\alpha_1}, \dots, e_{\alpha_k}),$$

where $e_{\alpha_i} \in H^{\alpha_i}$. So with the mapping $G \to \Omega G$ we have

$$\mathrm{Tor}_{\bigwedge^*(e_{\alpha_1}, \dots, e_{\alpha_k})}(\mathbb{C}, \mathbb{C}) = S^*(c_{\alpha_1 - 1}, \dots, c_{\alpha_k - 1}),$$

where $\deg c_{\alpha_i - 1} = \alpha_i - 1$. The generators of the cohomology are the homotopy groups. To complete the proof of Theorem 2 we will need the next proposition.

Proposition 1. *The spectral sequence degenerates, namely, its second term* E_2 *coincides with the limit term* E_∞.

Proof. We shall indicate explicit cocycles of $C^*(\bar{\mathfrak{n}})$ which represent the generators of E_2. For this we apply the continuous cohomology theory. Let $\mathfrak{n}(0, 1)$ be the Lie algebra of infinitely differentiable functions $f: [0, 1] \to \mathfrak{g}$ such that $f(0) \in \mathfrak{n}$ and $f(1) = 0$. Denote by $C_c^*(0, 1)$ the complex of cochains of $\mathfrak{n}(0, 1)$, continuous in the C^∞-topology. Let α be a generator of $H^*(\mathfrak{g})$ and

$\bar{\alpha}$ a cochain representing α. For $p \in [0, 1]$ denote by φ_p the homomorphism $\bar{\mathfrak{n}} \to \mathfrak{g}$, "the value at p":

$$\varphi_p((t-1)g_1, (t-1)^2 g_2, \dots) = \sum_m (p-1)^m g_m.$$

Let $\alpha_p = \varphi_p^* \bar{\alpha}$, $\alpha_p \in C_c^*(0, 1)$. Choose $\bar{\alpha}$ in such a way that $\alpha_0 = \alpha_1 = 0$. Let $p \neq 0, 1$; then we can define the cochain $\frac{\partial \alpha}{\partial x}(p)$, where x is the coordinate on $[0, 1]$. It is shown in [F] that $\frac{\partial \alpha}{\partial x}(p)$ is a coboundary: $\frac{\partial \alpha}{\partial x}(p) = \delta\omega(p)$, where δ is the differential in $C_c^*(0, 1)$. Indeed, let K_p $(p \neq 0)$ be the cochain complex of $\bar{\mathfrak{n}}$ with support at p. It is proved in the same paper that the cohomology of K_p is isomorphic to $H^*(\mathfrak{g})$. The complex K_p is a W_1-module, where W_1 is the Lie algebra of formal vector fields at the point p. But $H^*(\mathfrak{g})$ is finite dimensional and W_1 has no nontrivial finite-dimensional representations. We conclude that if $\nu \in K_p$ and $\delta\nu = 0$ then $\frac{\partial \nu}{\partial x}$ is the differential of some other cocycle $\bar{\nu} \in K_p$.

This means that

$$\alpha_p - \alpha_q = \delta \int_q^p \omega(x)\, dx.$$

In particular, $\delta \int_0^1 \omega(x)\, dx = 0$. Suppose that $\alpha' = \int_0^1 \omega(x)\, dx$. The cochain α' represents a nontrivial cohomology class of $\bar{\mathfrak{n}}$.

The Lie algebras $\hat{\mathfrak{n}}_+$ and $\bar{\mathfrak{g}} = (t-1)\mathfrak{g} \oplus (t-1)^2 \mathfrak{g} \oplus \cdots$ are graded. Similarly, the cochain complexes are also graded. Note that the cochain complex K_0 of $\mathfrak{n}(0, 1)$ with support in 0 is isomorphic to $\bigoplus_i C_i^*(\hat{\mathfrak{n}}_+)$ and the cochain complex K_1 with support in 1 is isomorphic to $\bigoplus_i C_i^*(\bar{\mathfrak{g}})$. It follows from this that the cohomologies of K_0 and K_1 are isomorphic to $H^*(\hat{\mathfrak{n}}_+)$ and $H^*(\bar{\mathfrak{g}})$, respectively.

Recall that $H^*(\mathfrak{g})$ is isomorphic to the free graded commutative algebra with generators $\xi_1, \xi_2, \dots$, with $\deg \xi_k = 2k+1$. Using the above construction, let us assign to each ξ_i a representative cocycle ξ_i'.

Lemma 2. *The space $H^*(\bar{\mathfrak{n}})$ is generated by the cohomology classes of cochains of the form $u \wedge v \wedge P(\xi_1', \xi_2', \dots)$, where $u \in K_0$ and $v \in K_1$ are cocycles corresponding to the elements of $H^*(\hat{\mathfrak{n}}_+)$ and $H^*(\bar{\mathfrak{g}})$, respectively, and P is an arbitrary polynomial with generators $\xi_1', \xi_2', \dots$.*

The proof of Lemma 2 follows from the construction above for continuous cohomology (a similar argument in a more difficult situation was used in [FR]). In particular, we have an explicit construction of cochains, representing the generators of E_2 in the proof of Theorem 2, surviving until E_∞. Thus, Theorem 2 is proved.

We now return to complete the proof of Theorem 1. We want to prove the isomorphism

$$H^*(\mathfrak{g}[t], \mathfrak{g}; V_1) \otimes H^*(\hat{\mathfrak{n}}_+; \mathbb{C}) \xrightarrow{\sim} H^*(\hat{\mathfrak{n}}_+; V_1).$$

Consider the Serre-Hochschild spectral sequence associated with the algebra $\hat{\mathfrak{n}}_+$, its ideal $\bar{\mathfrak{n}}$, and the module V_1. The Lie algebra $\bar{\mathfrak{n}}$ acts on V_1 trivially. The second term of this spectral sequence is

$$E_2^{ij} = H^i(\hat{\mathfrak{n}}_+/\bar{\mathfrak{n}}; H^j(\bar{\mathfrak{n}}; V_1)) = H^i(\hat{\mathfrak{n}}_+/\bar{\mathfrak{n}}; H^j(\bar{\mathfrak{n}}; \mathbb{C}) \otimes V_1).$$

But by Theorem 2, this is the same as

$$H^i\left(\hat{\mathfrak{n}}_+/\bar{\mathfrak{n}}, \bigoplus_{p+q+r=j} H^p(\hat{\mathfrak{n}}_+)\otimes H^q(\bar{\mathfrak{g}})\otimes H^r(\Omega G)\otimes V_1\right).$$

Since $\hat{\mathfrak{n}}_+/\bar{\mathfrak{n}}\cong\mathfrak{g}$, we get that E_2 is then isomorphic to

$$H^*(\mathfrak{g}, \mathbb{C})\otimes[H^*(\hat{\mathfrak{n}}_+)\otimes H^*(\bar{\mathfrak{g}})\otimes H^*(\Omega G)\otimes V_1]^{\mathfrak{g}},$$

where $[\ \ ]^{\mathfrak{g}}$ denotes the invariant space.

Let us note the following facts:

(a) $\mathfrak{g}$ acts on $H^*(\hat{\mathfrak{n}}_+)$ trivially (this action is extended by the projection $\hat{\mathfrak{n}}_+\to\mathfrak{g}$ to the canonical action of $\hat{\mathfrak{n}}_+$, and a Lie algebra acts trivially on the cohomology of itself).

(b) $\mathfrak{g}$ acts trivially on $H^*(\Omega G)=\mathrm{Tor}_{H^*(\mathfrak{g})}(\mathbb{C}, \mathbb{C})$.

These imply that

$$E_2^{ij}=H^i(\mathfrak{g})\otimes\left(\bigoplus_{p+q+r=j}(H^p(\hat{\mathfrak{n}}_+)\otimes H^q(\Omega G)\otimes[H^r(\bar{\mathfrak{g}})\otimes V_1]^{\mathfrak{g}})\right)$$

and

$$[H^*(\bar{\mathfrak{g}})\otimes V_1]^{\mathfrak{g}}=H^*(\mathfrak{g}[t], \mathfrak{g}; V_1).$$

The differentials act in the following way.

(i) Differentials on $H^*(\hat{\mathfrak{n}}_+)\otimes H^*(\mathfrak{g}[t], \mathfrak{g}; V_1)$ are zero. We have the map

$$H^*(\hat{\mathfrak{n}}_+)\otimes H^*(\mathfrak{g}[t], \mathfrak{g}; V_1)\longrightarrow H^*(\hat{\mathfrak{n}}_+; V_1).$$

Thus, elements of the left side survive in E_∞.

(ii) Differentials on

$$H^*(\mathfrak{g})\otimes H^*(\Omega G)=\bigwedge^*\{e_{\alpha_i}\}\otimes S^*\{c_{\alpha_i-1}\}$$

map the generators of the algebra $H^*(\Omega G)$ into the generators of $H^*(\mathfrak{g})$ (the differential maps $c_{\alpha_i-1}\mapsto e_{\alpha_i}$).

Consider the Serre path fibration $EG\to G$. Since the paths are contractible, $H^0(EG)=\mathbb{C}$ and $H^i(EG)=0$ for $i>0$. In addition, we have $H^*(\mathfrak{g})=H^*(G)$, so

$$H^*(G)\otimes H^*(\Omega G)\quad\text{converges to}\quad H^*(EG)\cong\mathbb{C}.$$

Then it follows that the spectral sequence converges to $H^*(\hat{\mathfrak{n}}_+)\otimes H^*(\mathfrak{g}[t], \mathfrak{g}; V_1)$. Thus our spectral sequence is the product of $H^*(\hat{\mathfrak{n}}_+)\otimes H^*(\mathfrak{g}[t], \mathfrak{g}; V_1)$ with the spectral sequence of the Serre path fibration $EG\to G$. This completes the proof of Theorem 1.

Theorem 3. $H^i(\hat{\mathfrak{n}}_+; \hat{\mathfrak{g}})\cong\mathbb{C}[t, t^{-1}]\otimes_{\mathbb{C}}H^{i-1}(\hat{\mathfrak{n}}_+)$ *for any nonnegative integer* i.

Proof. Let $V=\mathfrak{g}$ as in Lemma $1'$. Then Lemma $1'$ implies that $H^i(\hat{\mathfrak{n}}_+; \hat{\mathfrak{g}})$ is a free $\mathbb{C}[t, t^{-1}]$-module of rank equal to $\dim H^i(\hat{\mathfrak{n}}_+; V_1)$.

The cohomology of $\hat{\mathfrak{n}}_+$ with trivial coefficients is known (see [GL]). Using this result, it is not difficult to find the cohomology $H^*(t\mathfrak{g}[t])$ of the algebra $t\mathfrak{g}[t]=(\mathfrak{g}\otimes t)\oplus(\mathfrak{g}\otimes t^2)\oplus\cdots$ (see also [G] and [GL]; the proof in [GL] is basically the same as the proof of the cohomology result of $\hat{\mathfrak{n}}_+$ with trivial coefficients). We need only the following fact. The space $H^*(t\mathfrak{g}[t])$ is a $\mathfrak{g}$-module and $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\,;H^i(t\mathfrak{g}[t]))=0$ if $i\neq 1$ and is equal to $\mathbb{C}$ if $i=1$ (see [L]). Since $H^i(\mathfrak{g}[t]\,,\mathfrak{g}\,;V)\cong \mathrm{Hom}_{\mathfrak{g}}(V\,;H^i(t\mathfrak{g}[t]))$, this gives us that $H^i(\mathfrak{g}[t]\,,\mathfrak{g}\,;\mathfrak{g})=0$ for $i\neq 1$ and is equal to $\mathbb{C}$ for $i=1$. After this, it is enough to apply Theorem 1 to find that $H^i(\hat{\mathfrak{n}}_+\,;V_1)=H^{i-1}(\hat{\mathfrak{n}}_+)$.

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