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Associative algebra deformations of Connes–Moscovici's Hopf algebra $\mathcal{H}_1$

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ABSTRACT

We compute the second Hochschild cohomology space $HH^2(\mathcal{H}_1)$ of Connes–Moscovici's Hopf algebra $\mathcal{H}_1$, giving the infinitesimal deformations (up to equivalence) of the associative structure. The space $HH^2(\mathcal{H}_1)$ is shown to be one-dimensional.

Introduction

Recently, there has been much interest in Connes–Moscovici's Hopf algebra $\mathcal{H}_1$ in relation to deformation quantization. Recall that $\mathcal{H}_1$ was constructed by Connes and Moscovici in [3] in order to formalize the transverse symmetries of a codimension 1 foliation. As an associative algebra, $\mathcal{H}_1$ is the universal enveloping algebra $U\mathfrak{h}$ of a certain Lie algebra $\mathfrak{h}$ closely related to the '$ax+b$'-group. In [3,4], the authors study the Hopf cyclic cohomology of (within others) $\mathcal{H}_1$, and associate thereby characteristic classes to a codimension 1 foliation. This turns out to be closely related to Gelfand–Fuchs cohomology classes.

Notwithstanding their initial work [3], Connes and Moscovici went on remarking that Rankin–Cohen brackets, which were known to give associative deformations on spaces of modular forms, can give such deformations on all algebras $\mathcal{H}_1$ acts on [5]. The answer is given in the framework of

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universal deformation formulae (UDF), see [11]. The universality of this kind of deformation is the fact that all $\mathcal{H}_1$-module algebras inherit a deformation of the multiplication.

In further work, Bieliavsky, Tang and Yao [1] refind the Rankin–Cohen starproduct in the context of Fedosov deformation quantization and showed that the Rankin–Cohen UDF on $\mathcal{H}_1$ is related to the Moyal–Weyl starproduct on the universal enveloping algebra of the '$ax+b$'-group. Still more recently, Tang and Yao [15] showed that the $\mathcal{H}_1$-action does not have to be projective in order to define the Rankin–Cohen starproduct.

Our contribution to the subject is a computation of the space of infinitesimal deformations of the associative product of $\mathcal{H}_1$, namely the Hochschild cohomology space $HH^2(\mathcal{H}_1)$. It turns out to be 1-dimensional, and this shows the unicity (up to equivalence) of the infinitesimal term of any associative formal deformation. Note that the Rankin–Cohen deformation is an associative deformation of any algebra on which $\mathcal{H}_1$ acts (such that $h(ab) = m(\triangle(h)(a \otimes b))$ holds for all $h \in \mathcal{H}$ and all $a, b \in A$) and not of $\mathcal{H}_1$ itself, therefore our computation are a priori unrelated to the Rankin–Cohen starproduct (which relies on the coalgebra structure of $\mathcal{H}_1$, while our computations rely on the algebra structure).

The computation is performed using Lie algebra cohomology methods and spectral sequences. Indeed, Hochschild cohomology of a universal enveloping algebra boils down to Lie algebra cohomology. Here it is the Lie algebra cohomology space $H^2(\mathfrak{h}; \mathcal{H}_1^{\mathrm{ad}})$ of $\mathfrak{h}$ with values in $\mathcal{H}_1$ using the adjoint action. To go on with the computation, we use the link between $\mathfrak{h}$ and a Lie algebra called $\mathfrak{m}_0$. The algebra $\mathfrak{m}_0$ is one of the three positively graded, infinite dimensional filiform Lie algebras (up to isomorphism) [7] and has been studied intensively in [8,9]. In fact, $\mathfrak{m}_0$ is an ideal of $\mathfrak{h}$, and thus we may use the Hochschild–Serre spectral sequence to compute $H^2(\mathfrak{h}; \mathcal{H}_1^{\mathrm{ad}})$ from the various $H^p(\mathfrak{h}/\mathfrak{m}_0; H^q(\mathfrak{m}_0; \mathcal{H}_1^{\mathrm{ad}}))$ with $p+q=2$.

These latter spaces $H^p(\mathfrak{h}/\mathfrak{m}_0; H^q(\mathfrak{m}_0; \mathcal{H}_1^{\mathrm{ad}}))$ are rather easily deduced from the knowledge of the spaces $H^q(\mathfrak{m}_0; \mathcal{H}_1^{\mathrm{ad}})$, $q = 0, 1, 2$, which are computed using the Feigin–Fuchs spectral sequence – for an introduction to it, see [14].

Let us give a detailed account on the content of this paper: in Section 1.1, we define $\mathcal{H}_1$ (which we call in the following only $\mathcal{H}$) in Connes and Moscovici's original context. In 1.2, $\mathfrak{h}$ is defined by generators and relations, and 1.3 gives some background material on $\mathfrak{m}_0$. Section 1.4 gives the link between $\mathfrak{h}$ and $\mathfrak{m}_0$, commenting on other points of view which permit, for example, to associate a pro-Lie group to $\mathfrak{h}$. Section 2 gives an outline of the cohomology computations: in Section 2.1 the link between Hochschild cohomology of $U\mathfrak{g}$ and Chevalley–Eilenberg cohomology of $\mathfrak{g}$ is recalled. Section 2.2 recalls the Hochschild–Serre spectral sequence, and Section 2.3 the Feigin–Fuchs spectral sequence. Section 3 treats the computations: in Section 3.1, we compute the spaces $H^q(\mathfrak{m}_0; \mathcal{H}_1^{\mathrm{ad}})$, $q = 0, 1, 2$ corresponding to Propositions 1, 2, and 3. In Section 3.2, we deduce then $H^p(\mathfrak{h}/\mathfrak{m}_0; H^q(\mathfrak{m}_0; \mathcal{H}_1^{\mathrm{ad}}))$ with $p+q=2$ mainly by degree arguments. Observe here that while $\mathfrak{h}$ has a basis consisting of eigenvectors with respect to $Y \in \mathfrak{h}$, the grading of $\mathfrak{m}_0$ is not inner: that is the fundamental difference between $\mathfrak{h}$ and $\mathfrak{m}_0$ which renders computations easy for $\mathfrak{h}$ and difficult for $\mathfrak{m}_0$. It is the Feigin–Fuchs spectral sequence which helps out.

The main theorems of this paper are Theorems 3 and 4 in Section 3.2 which state the result of the cohomology computations of $HH^1(\mathcal{H})$ and $HH^2(\mathcal{H})$ in terms of Lie algebra cocycles whose classes generate these spaces.

Our approach is new in the sense that up to now, only the deformation theory of algebras A where $\mathcal{H}$ acts on (such that $h(ab) = m(\triangle(h)(a \otimes b))$ holds for all $h \in \mathcal{H}$ and all $a, b \in A$) has been regarded. This is the first step towards a deformation theory of the algebra $\mathcal{H}$ itself. In this context, it would be interesting to construct a starproduct whose infinitesimal term represents the generator of $HH^2(\mathcal{H})$. Such a starproduct would then be automatically the miniversal deformation of $\mathcal{H}$.

1. Preliminaries on $\mathcal{H}_1$ and $\mathfrak{m}_0$

1.1. Connes–Moscovici's Lie and Hopf algebras

Let us recall some basics about Connes–Moscovici's Hopf algebra $\mathcal{H}_1$. It was introduced in [3] in the study of the transverse structure of foliations.

Let M be an n-dimensional smooth oriented manifold, ∇ be a flat affine connection on M, and Γ be a pseudogroup of local diffeomorphisms on M, respecting the orientation. Such a pseudogroup arises for example in the presence of an oriented foliation.

Denote by F^+ the oriented frame bundle on M. Define [3]

$$\mathcal{A} := \mathcal{C}_c^\infty(F^+) \rtimes \Gamma,$$

the crossed product of the algebra $\mathcal{C}_c^\infty(F^+)$ of smooth sections of F^+ with compact support with the pseudogroup Γ.

There are three kinds of generators of $\mathcal{A}$:

(1) Generators Y_l^i:
$Gl^+(n,\mathbb{R})$ acts on F^+, and this action, as it commutes with that of Γ, extends to $\mathcal{A}$. The infinitesimal generators (or fundamental vector fields) of this action are by definition the Y_l^i, $i,l = 1,\dots,n$.

(2) Generators X_i:
The flat connection ∇ permits to lift vector fields which are tangent to M, to F^+: we therefore get horizontal vector fields X_i on F^+, $i = 1,\dots,n$.

(3) Generators $\delta^t_{rs,i_1,\dots,i_l}$:
In case the local diffeomorphisms are affine, the X_i's commute with the action of elements of Γ, but in general, they do not. One finds [3]

$$X_i(ab) = X_i(a)b + aX_i(b) + \delta_{ij}^k Y_k^j(b),$$

for all $a, b \in \mathcal{A}$ and the vertical fields Y_k^j, infinitesimal generators of the action of $Gl^+(n,\mathbb{R})$ on F^+. Iterated brackets of the X_i and the δ^t_{rs} yield

$$\delta^t_{rs,i_1,\dots,i_l} := [X_{i_1},\dots,[X_{i_l},\delta^t_{rs}]\dots].$$

The space generated by the Y_l^i, $i,l = 1,\dots,n$, X_i, $i = 1,\dots,n$, and $\delta^t_{rs,i_1,\dots,i_l}$, $r,s,t = 1,\dots,n$, $l \in \mathbb{N}$, is closed under the Lie bracket (by construction), and yields therefore a Lie algebra $\mathfrak{h}(n)$. Its enveloping algebra is denoted by $\mathcal{H}_n$.

Connes and Moscovici [3] endow the associative algebra $\mathcal{H}_n$ with a coproduct in such a way that it acts on the algebra $\mathcal{A}$:

$$h(ab) = \sum h_0(a)h_1(b)$$

(in Sweedler notation for the coproduct) for all $a, b \in \mathcal{A}$ and all $h \in \mathcal{H}_n$. They then construct an antipode and $\mathcal{H}_n$ becomes a Hopf algebra; we call it the *Connes–Moscovici Hopf algebra* $\mathcal{H}_n$.

We will be only concerned with the 1-dimensional case, i.e. $n = 1$, in which case we will write more simply $\mathfrak{h}$ and $\mathcal{H}$ for Connes–Moscovici's Lie and Hopf algebras.

We will not recall the significance of $\mathcal{H}_n$ in foliation theory, as this would lead us too far afield.

1.2. Connes–Moscovici's Lie algebra $\mathfrak{h}$

Algebraically speaking, $\mathfrak{h}$ is a Lie algebra[1] generated by the countably infinite set of generators X, Y and δ_i, $i = 1,2,\dots,$ with the relations

[1] Because of their geometric origin, we take Lie algebras over $\mathbb{R}$ here; the algebraic results are of course valid over much more general fields.

$$[Y, X] = X, \qquad [Y, \delta_r] = r\delta_r, \qquad [X, \delta_r] = \delta_{r+1},$$

for all $r = 1, 2, \ldots$. By convention, we only write the non-trivial relations, i.e. all brackets which are not displayed, are zero. In particular, one has $[\delta_r, \delta_s] = 0$ for all $r, s = 1, 2, \ldots$.

Observe that $\mathfrak{h}$ is a graded Lie algebra: Y has degree 0, X and δ_1 have degree 1, and δ_i has degree i for $i \geqslant 1$.

1.3. The infinite dimensional filiform Lie algebra $\mathfrak{m}_0$

The significance of $\mathfrak{m}_0$ resides in the classification of infinite dimensional $\mathbb{N}$-graded Lie algebras $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ with one-dimensional homogeneous components $\mathfrak{g}_i$ and two generators (over a field of characteristic zero) such that $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$. A. Fialowski showed in [7] that any Lie algebra of this type must be isomorphic to $\mathfrak{m}_0$, $\mathfrak{m}_2$ or L_1. We call these Lie algebras infinite dimensional filiform Lie algebras in analogy with the finite dimensional case where the name was coined by M. Vergne in [16]. Here $\mathfrak{m}_0$ is given by generators e_i, $i \geqslant 1$, and relations $[e_1, e_i] = e_{i+1}$ for all $i \geqslant 2$, $\mathfrak{m}_2$ with the same generators by relations $[e_1, e_i] = e_{i+1}$ for all $i \geqslant 2$, $[e_2, e_j] = e_{j+2}$ for all $j \geqslant 3$, and L_1 with the same generators is given by the relations $[e_i, e_j] = (j-i)e_{i+j}$ for all $i, j \geqslant 1$. L_1 appears as the positive part of the Witt algebra given by generators e_i for $i \in \mathbb{Z}$ with the same relations $[e_i, e_j] = (j-i)e_{i+j}$ for all $i, j \in \mathbb{Z}$.

What matters most for the present discussion is the cohomology with trivial coefficients of $\mathfrak{m}_0$. It has been computed in [8] and will be recalled in Theorem 2.

1.4. The link between $\mathfrak{h}$ and $\mathfrak{m}_0$

Two ways of viewing $\mathfrak{h}$ are of interest: first, define a map $i : \mathfrak{m}_0 \to \mathfrak{h}$ by sending e_1 to X and e_i to δ_i for $i \geqslant 2$. A short inspection of the relation shows that via i, $\mathfrak{m}_0$ becomes a subalgebra of $\mathfrak{h}$, and one has a short exact sequence:

$$0 \to \mathfrak{m}_0 \stackrel{i}{\to} \mathfrak{h} \to \operatorname{span}(Y, \delta_1) \to 0.$$

This determines $\mathfrak{h}$ as a general extension of a 2-dimensional Lie algebra $\mathfrak{a} := \operatorname{span}(Y, \delta_1)$ (with the only non-trivial relation $[Y, \delta_1] = \delta_1$) by $\mathfrak{m}_0$. The term "general extension" means here that it is neither a central, nor an abelian extension. The algebra $\mathfrak{a}$ is the Lie algebra of the '$ax + b$'-group, see [1].

The second way of viewing $\mathfrak{h}$ is as a trivial abelian extension of the 2-dimensional Lie algebra $\mathfrak{b} := \operatorname{span}(X, Y)$ (with the relation $[Y, X] = X$) by the infinite dimensional abelian Lie algebra $\mathfrak{c} := \operatorname{span}(\delta_i \mid i \geqslant 1)$:

$$0 \to \mathfrak{c} \to \mathfrak{h} \to \mathfrak{b} \to 0.$$

In the following, we will exploit the first point of view to compute some cohomology of $\mathfrak{h}$; let us remark here that the second point of view makes it possible to associate an infinite dimensional Lie group to the Lie algebra $\mathfrak{h}$. Indeed, truncating the infinite dimensional $\mathfrak{c}$ to a finite dimensional $\mathfrak{c}_k := \operatorname{span}(\delta_i \mid i = 1, \ldots, k)$ defines a Lie algebra $\mathfrak{h}_k$ by

$$0 \to \mathfrak{c}_k \to \mathfrak{h}_k \to \mathfrak{b} \to 0.$$

Then $\mathfrak{h}$ is obviously a projective limit of the finite dimensional Lie algebras $\mathfrak{c}_k$, and thus there is a Lie group associated to $\mathfrak{h}$ by the following theorem [13] (Lie's third theorem for pro-Lie groups):

Theorem 1 *(Hofmann–Morris). Given a profinite Lie algebra, i.e. a projective limit of finite dimensional Lie algebras, there is a connected profinite (possibly infinite dimensional) Lie group whose Lie algebra is the given one.*

It would be interesting to exploit this last remark in order to establish some relationship between the pro-Lie group associated to $\mathfrak{h}$ and $\mathcal{H}$.

2. Outline of the cohomology computation

The aim is to compute the second Hochschild cohomology $HH^2(\mathcal{H})$ of the associative algebra $\mathcal{H}$ in order to determine the different infinitesimal deformations of $\mathcal{H}$ as an associative algebra. We will do this in three steps.

2.1. From Hochschild to Chevalley–Eilenberg

Recall that *Hochschild cohomology* of an associative algebra A over a field k with values in an A-bimodule M is just

$$HH^*(A;M) := Ext^*_{A\otimes A^{\mathrm{opp}}}(k,M).$$

In case $M = A$ with the usual bimodule structure, we will write $HH^*(A)$ instead of $HH^*(A;A)$. The second cohomology group $HH^2(A)$ classifies infinitesimal deformations of the algebra structure of A [12].

Recall also that the *Chevalley–Eilenberg cohomology* of a Lie algebra $\mathfrak{g}$ over k with values in a $\mathfrak{g}$-module N is by definition

$$H^*(\mathfrak{g};N) := Ext^*_{U\mathfrak{g}}(k,N),$$

where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$.

Let M be a $U\mathfrak{g}$-bimodule, and denote by M^{ad} the right $\mathfrak{g}$-module defined by

$$m\cdot x = mx - xm.$$

It is explained in the book of Cartan and Eilenberg [2] that the change-of-rings functor associated to the map $\mathfrak{g}\to U\mathfrak{g}\otimes U\mathfrak{g}^{\mathrm{opp}}$ given by $x\mapsto x\otimes 1 - 1\otimes x$ leads to an isomorphism

$$HH^*(U\mathfrak{g};M)\cong H^*\big(\mathfrak{g};M^{\mathrm{ad}}\big)$$

in what they call the inverse process. While the homomorphism inducing this isomorphism is well-understood in one direction, its inverse does not appear in the literature. Indeed, the isomorphism is induced by the antisymmetrization map

$$x_1\wedge\cdots\wedge x_p\mapsto \sum_{\sigma\in S_p}(-1)^{\mathrm{sgn}(\sigma)}x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(p)}$$

between standard resolutions, where $x_1,\ldots,x_p\in\mathfrak{g}$.

In this way, in order to compute $HH^2(\mathcal{H})$ for the Connes–Moscovici Hopf algebra $\mathcal{H}$, it is enough to compute the Lie algebra cohomology space $H^2(\mathfrak{h};\mathcal{H}^{\mathrm{ad}})$.

2.2. From $\mathfrak{h}$ to $\mathfrak{m}_0$ via the Hochschild–Serre spectral sequence

In the second step, we will use the short exact sequence

$$0 \to \mathfrak{m}_0 \to \mathfrak{h} \to \mathrm{span}(Y, \delta_0) \to 0$$

in order to reduce the computation to one for $\mathfrak{m}_0$. This is done by the Hochschild–Serre spectral sequence: given an ideal $\mathfrak{k} \subset \mathfrak{g}$ in a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module L, there is a filtration on the space of cochains $C^*(\mathfrak{g}; L)$ which induces a spectral sequence with

$$E_2^{p,q} = H^p\big(\mathfrak{g}/\mathfrak{k}; H^q(\mathfrak{k}; L)\big)$$

converging to

$$E_\infty^{p,q} = H^{p+q}(\mathfrak{g}; L),$$

see for example [10].

In our case, we therefore have to compute the spaces $H^q(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}})$ for $q = 0, 1, 2$, and then the cohomology of $\mathfrak{a} := \mathrm{span}(Y, \delta_0)$ with values in these spaces in order to determine the E_2-term. As the latter computation is rather easy, we are left with computing $H^q(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}})$ which will be done in the third step.

2.3. The computation for $\mathfrak{m}_0$ via the Feigin–Fuchs spectral sequence

$H^q(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}})$ will be computed using the Feigin–Fuchs spectral sequence. This is a tool which is available only for $\mathbb{N}$-graded Lie algebras with values in a non-negatively graded module. It has been introduced in [6]. The preprint [14] is meant to be a readable introduction to this subject.

Let $\mathfrak{k}$ be an $\mathbb{N}$-graded Lie algebra, i.e. $\mathfrak{k} = \bigoplus_{i=1}^\infty \mathfrak{k}_i$ with $[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j}$, and let L be a non-negatively graded $\mathfrak{k}$-module, i.e. $L = \bigoplus_{i=0}^\infty L_i$ with $\mathfrak{g}_i \cdot L_j \subset L_{i+j}$. Using the gradation, one introduces a filtration on $C^*(\mathfrak{k}; L)$ which induces a spectral sequence with

$$E_1^{p,q} = H^{p+q}(\mathfrak{k}) \otimes L_p$$

converging to the cohomology $H^*(\mathfrak{k}; L)$.

The strategy of computation in this third step is thus clear: using the known results on the cohomology with trivial coefficients of $\mathfrak{m}_0$ from [8], one has to follow them through the Feigin–Fuchs spectral sequence.

3. Computation and results

Obviously, we will begin with the third step of the outline.

3.1. Computation of $H^q(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}})$

The Feigin–Fuchs spectral sequence will become important for $H^q(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}})$ for $q = 1, 2$, while we first compute $H^0(\mathfrak{h}; \mathcal{H}^{\mathrm{ad}})$ by elementary methods.

As it is well known, $H^0(\mathfrak{h}; \mathcal{H}^{\mathrm{ad}})$ is the subspace of $\mathfrak{h}$-invariants of $\mathcal{H}^{\mathrm{ad}}$. We have:

Proposition 1.

$$H^0\big(\mathfrak{h}; \mathcal{H}^{\mathrm{ad}}\big) = \mathbb{R}.$$

Proof. As $\mathfrak{h}$ is a graded Lie algebra and $\mathcal{H}^{\text{ad}}$ is a graded $\mathfrak{h}$-module, and we may reason degree by degree.

In degree 0, we have polynomials in Y and $\mathbb{R}$, and it is then clear by $[Y,X]=X$ that the invariants in degree 0 are $\mathbb{R}$. In degree 1, we have X and δ_1. As $[Y,X]=X$ and $[X,\delta_1]=\delta_2$, invariants are 0 in degree 1.

Now let $m \in \mathcal{H}^{\text{ad}}$ be a homogeneous element of degree $n>1$ with $m\cdot a=0$ for all $a\in\mathfrak{h}$. Suppose a non-zero δ_i occurs in m. By the Poincaré–Birkhoff–Witt theorem, there is a maximal k and a corresponding maximal $r=r(k)$ such that $m=m'\delta_k^r$. We have

$$\begin{aligned}\delta_k^r\cdot X &= \delta_k^r X - X\delta_k^r = \delta_k^{r-1}X\delta_k - \delta_k^{r-1}\delta_{k+1} - X\delta_k^r\\ &= \cdots = -r\delta_k^{r-1}\delta_{k+1},\end{aligned}$$

because the δ_i's commute with each other. Then $m\cdot X=0$ implies thus $rm'\delta_k^{r-1}\delta_{k+1}=0$ with $r\geqslant 1$. One deduces $m'=0$, and $m=0$.

The last case is the one where there is no δ_i in m. Suppose therefore $m=\sum_k Y^kX^n$ (m is supposed to be of degree $n>1$!), and again $m\cdot X=0$. This gives

$$m\cdot X=\sum_k m_kY^kX^{n+1}-\sum_k m_kXY^kX^n=0,$$

and we want to commute X to the right. We have the (binomial) formula:

$$XY^k=\big(Y^k-kY^{k-1}\pm\cdots+(-1)^k\big)X,$$

which is easily shown by induction. One deduces

$$\sum_k m_k\left(-ky^{k-1}+\binom{k}{2}Y^{k-2}\mp\cdots+(-1)^k\right)=0.$$

Looking at the highest power of Y, say $k_{\max}$, we get $m_{k_{\max}}=0$, and then $m_{k_{\max}-1}=0$ and so on. Finally, $m=0$. □

Let us now compute $H^1(\mathfrak{m}_0;\mathcal{H}^{\text{ad}})$ via the Feigin–Fuchs spectral sequence. For an introduction to this spectral sequence, see [14].

First recall the cohomology $H^*(\mathfrak{m}_0)$ of $\mathfrak{m}_0$ with trivial coefficients, see Theorem 3.4 in [8]. The generators of $\mathfrak{m}_0$ are still denoted e_i, $i\geqslant 1$, and a dual "basis" is given by the 1-cochains e^i, $i\geqslant 1$. Fialowski and Millionschikov show in [8]:

Theorem 2. *The bigraded cohomology algebra $H^*(\mathfrak{m}_0)=\bigoplus_{k,q}H_k^q(\mathfrak{m}_0)$ is spanned by the cohomology classes of the following homogeneous cocycles*:

$$e^1,e^2,\omega\big(e^{i_1}\wedge\cdots\wedge e^{i_q}\wedge e^{i_q+1}\big)=\sum_{l\geqslant 0}(-1)^l\big(\text{ad}_{e_1}^*\big)^l\big(e^{i_1}\wedge\cdots\wedge e^{i_q}\big)\wedge e^{i_q+1+l},$$

where $q\geqslant 1$, $2\leqslant i_1<\cdots<i_q$.

Fialowski and Millionschikov describe also the dimension of the cohomology spaces and the multiplicative structure in detail.

Coming back to $H^1(\mathfrak{m}_0;\mathcal{H}^{\text{ad}})$, we have (we do not distinguish cohomology classes and cocycles generating it):

Proposition 2.

$$H^1\big(\mathfrak{m}_0;\mathcal{H}^{\mathrm{ad}}\big)\cong H^1(\mathfrak{m}_0)\oplus\mathbb{R}\big(\delta_1\otimes e^1\big).$$

More precisely, in the Feigin–Fuchs spectral sequence, we have

$$E_\infty^{1,0}\cong H^1(\mathfrak{m}_0)$$

and

$$E_\infty^{0,1}=\mathbb{R}\big(\delta_1\otimes e^1\big).$$

Proof. The Feigin–Fuchs spectral sequence uses the graded structure of Lie algebra and module. The action by Lie algebra elements sends module elements necessarily in strictly upper degrees, so the first step, leading to $E_1^{p,q}$, is to exclude action terms in the Lie algebra differential. It remains the differential with trivial coefficients and the E_1-term in the Feigin–Fuchs spectral sequence is

$$E_1^{p,q}=\big(\mathcal{H}^{\mathrm{ad}}\big)_q\otimes H^{p+q}(\mathfrak{m}_0).$$

The space $H^1(\mathfrak{m}_0)$ is generated by (the cohomology classes represented by) e^1 and e^2. Therefore

$$E_1^{0,1}=\left(\bigoplus_{n\geqslant 0}\mathbb{R}Y^nX\oplus\bigoplus_{n\geqslant 0}\mathbb{R}Y^n\delta_1\right)\otimes_{\mathbb{R}}\big(\mathbb{R}e^1\oplus\mathbb{R}e^2\big),$$

and

$$E_1^{1,0}=\left(\mathbb{R}1\oplus\bigoplus_{n\geqslant 1}\mathbb{R}Y^n\right)\otimes_{\mathbb{R}}\big(\mathbb{R}e^1\oplus\mathbb{R}e^2\big).$$

The second step, leading to the differential $d_1^{p,q}$, is then to admit action terms with elements of degree 1. Applied to a general cochain

$$c=\sum_{n\geqslant 0}a_nY^nX\otimes e^1+\sum_{n\geqslant 0}b_nY^n\delta_1\otimes e^1+\sum_{n\geqslant 0}c_nY^nX\otimes e^2+\sum_{n\geqslant 0}d_nY^n\delta_1\otimes e^2,$$

the only non-zero term is given by the action term involving X. One gets with $i\geqslant 2$ (remember that $\mathfrak{m}_0$ is identified with $\mathbb{R}X\oplus\bigoplus_{i\geqslant 2}\mathbb{R}\delta_i$)

$$d_1^{0,1}c(X,\delta_i)=-c(\delta_i)\cdot X,$$

because the other action term $c(X)\cdot\delta_i$ does not take values in $(\mathcal{H}^{\mathrm{ad}})_1$. By the special form of the cochain c, we must have $i=2$. Using $Y^nX\cdot X=\sum_{l=0}^{n-1}Y^lXY^{n-l-1}X$ and $Y^n\delta_1\cdot X=\sum_{l=0}^{n-1}Y^lXY^{n-l-1}\delta_1-Y^n\delta_2$ (which are easily shown by induction), we get

$$d_1^{0,1}c(X,\delta_2)=\sum_{n\geqslant 1}c_n\sum_{l=0}^{n-1}Y^lXY^{n-l-1}X-\sum_{n\geqslant 0}d_n\left(\sum_{l=0}^{n-1}Y^lXY^{n-l-1}\delta_1-Y^n\delta_2\right).$$

Then the condition $d_1^{0,1}c(X,\delta_2)=0$ means that all c_n, $n\geqslant 1$, and all d_r, $r\geqslant 0$, must be zero. Indeed, the two sums must be finite, and starting with the highest power in Y, one shows one by one that all terms are zero. Therefore

$$E_2^{0,1}=\left(\bigoplus_{n\geqslant 1}\mathbb{R}Y^nX\oplus\bigoplus_{n\geqslant 0}\mathbb{R}Y^n\delta_1\right)\otimes_{\mathbb{R}}\mathbb{R}e^1\oplus\mathbb{R}X\otimes e^2,$$

because $dY=X\otimes e^1$ is a coboundary.

The next differential $d_2^{0,1}$ involves the action terms with elements of degree 2. We have to compute here the term

$$d_2^{0,1}c(X,\delta_2)=c(X)\cdot\delta_2=\left(\sum_{n\geqslant 1}a_nY^nX+\sum_{n\geqslant 0}b_nY^n\delta_1\right)\cdot\delta_2+c_0Xe^2(X).$$

In the same way as before, using this time $Y^nX\cdot\delta_2=2(1+Y+\cdots+Y^{n-1})\delta_2\delta_1$ and $Y^nX\cdot\delta_2=2(1+Y+\cdots+Y^{n-1})\delta_2X+Y^n\delta_3$, we get that b_0 is arbitrary, while all a_n for $n\geqslant 0$ and all b_r for $r\geqslant 1$ are zero. Therefore

$$E_3^{0,1}=\mathbb{R}X\otimes e^2\oplus\mathbb{R}\delta_1\otimes e^1.$$

A quick look at the area of non-zero terms in the first page of the spectral sequence shows that there are no non-zero terms strictly below the $q=0$ axis and strictly to the left of the $p=-q$ antidiagonal:

$$\begin{array}{lllll}
E_1^{-2,3} & E_1^{-1,3} & E_1^{0,3} & E_1^{1,3} & E_1^{2,3}\\
E_1^{-2,2} & E_1^{-1,2} & E_1^{0,2} & E_1^{1,2} & E_1^{2,2}\\
E_1^{-2,1}=0 & E_1^{-1,1} & E_1^{0,1} & E_1^{1,1} & E_1^{2,1}\\
E_1^{-2,0}=0 & E_1^{-1,0}=0 & E_1^{0,0} & E_1^{1,0} & E_1^{2,0}\\
E_1^{-2,-1}=0 & E_1^{-1,-1}=0 & E_1^{0,-1}=0 & E_1^{1,-1}=0 & E_1^{2,-1}=0.
\end{array}$$

Therefore, all images of $d_2^{p,q}:E_2^{p,q}\to E_2^{p-1,q+2}$, $d_3^{p,q}:E_3^{p,q}\to E_3^{p-2,q+3}$, and so on, in $E_r^{0,1}$ are zero, but one has an infinite number of outgoing non-zero differentials. It is thus more convenient at this stage to compute which combinations of $X\otimes e^2$ and $\delta_1\otimes e^1$ are actual cocycles. It turns out that $X\otimes e^2$ is not a cocycle (test on e_2 and e_i with $i\geqslant 3$), but $\delta_1\otimes e^1$ is a cocycle. Finally

$$E_\infty^{0,1}=\mathbb{R}\delta_1\otimes e^1.$$

Now let us compute $E_\infty^{1,0}$. We start from a cochain

$$c=a1\otimes e^1+b1\otimes e^2+\sum_{n\geqslant 1}c_nY^n\otimes e^1+\sum_{n\geqslant 1}d_nY^n\otimes e^2$$

in

$$E_1^{1,0}=\left(\mathbb{R}1\oplus\bigoplus_{n\geqslant 1}\mathbb{R}Y^n\right)\otimes_{\mathbb{R}}\left(\mathbb{R}e^1\oplus\mathbb{R}e^2\right).$$

As before, we have to compute $c(\delta_2)\cdot X$, and this gives here $c(\delta_2)\cdot X=\sum_{n\geqslant 1} d_n Y^n\cdot X$. The first condition is thus $d_n=0$ for all $n\geqslant 1$. For $d_2^{1,0}$, we have to compute $c(X)\cdot\delta_2$, which gives $c(X)\cdot\delta_2=\sum_{n\geqslant 1} c_n Y^n\cdot\delta_2$, and therefore the second condition is $c_n=0$ for all $n\geqslant 1$. It is clear that all combinations of $1\otimes e^1$ and $1\otimes e^2$ are actual cocycles, and we get therefore

$$E_\infty^{1,0}\cong H^1(\mathfrak{m}_0),$$

$H^1(\mathfrak{m}_0)$ being generated by e^1 and e^2. □

We go on computing $H^2(\mathfrak{m}_0;\mathcal{H}^{\mathrm{ad}})$ with the Feigin–Fuchs spectral sequence.

Proposition 3.

$$H^2\big(\mathfrak{m}_0;\mathcal{H}^{\mathrm{ad}}\big)\cong H^2(\mathfrak{m}_0).$$

More precisely, in the Feigin–Fuchs spectral sequence, we have

$$E_\infty^{2,0}\cong H^2(\mathfrak{m}_0)$$

and

$$E_\infty^{1,1}=E_\infty^{0,2}=0.$$

Proof. The assertion $E_\infty^{2,0}\cong H^2(\mathfrak{m}_0)$ is clear in the same way that we computed $E_\infty^{1,0}\cong H^1(\mathfrak{m}_0)$: indeed, the $E_1^{2,0}$-term is a sum of terms of the form $\mathbb{R}Y^n\otimes_{\mathbb{R}} H^2(\mathfrak{m}_0)$ for all n. The action with X on some Y^n gives once again a sum like $Y^n\cdot X=\sum_{l=0}^{n-1} Y^l X Y^{n-l-1}$. As the evaluation of the cochain on some elements from $\mathfrak{m}_0$ must be a finite sum, there is a term of highest degree in Y. By induction, we show as before that all terms involving Y must be zero. The terms $\mathbb{R}1\otimes_{\mathbb{R}} H^2(\mathfrak{m}_0)$ remain and give

$$E_\infty^{2,0}\cong H^2(\mathfrak{m}_0).$$

Now let us look at $E_1^{1,1}=\mathcal{H}_1^{\mathrm{ad}}\otimes H^2(\mathfrak{m}_0)$ which may be written like

$$E_1^{1,1}=\left(\bigoplus_{n\geqslant 0}\mathbb{R}Y^n X\oplus\bigoplus_{n\geqslant 0} Y^n\delta_1\right)\otimes_{\mathbb{R}} H^2(\mathfrak{m}_0),$$

and $H^2(\mathfrak{m}_0)$ is a countably infinite dimensional space with generators $e^2\wedge e^3$, $e^2\wedge e^5-e^3\wedge e^4$, and so on (cf. [8]). Taking a cochain c in $E_1^{1,1}$, it may be written like

$$\begin{aligned}c=&\sum_{n\geqslant 0} a_n^1 Y^n X\otimes e^2\wedge e^3+\sum_{n\geqslant 0} b_n^1 Y^n\delta_1\otimes e^2\wedge e^3\\&+\sum_{n\geqslant 0} a_n^2 Y^n X\otimes\big(e^2\wedge e^5-e^3\wedge e^4\big)+\sum_{n\geqslant 0} b_n^2 Y^n\delta_1\otimes\big(e^2\wedge e^5-e^3\wedge e^4\big)+\cdots.\end{aligned}$$

Then obviously

$$d_1^{1,1}c(e_1,e_2,e_3)=\sum_{n\geqslant 0} a_n^1 Y^n X\cdot X+\sum_{n\geqslant 0} b_n^1 Y^n\delta_1\cdot X,$$

and the sums in the previous expression are finite. We already computed this kind of sum in the previous proposition, and in the same way as there, it turns out that $a_n^1 = 0$ for all $n \geqslant 1$ and $b_n^1 = 0$ for all $n \geqslant 0$. With identical reasoning, $d_1^{1,1}c(e_1, e_2, e_5)$ shows that $a_n^2 = 0$ for all $n \geqslant 1$ and $b_n^2 = 0$ for all $n \geqslant 0$. Going on like this, our cochains looks finally like

$$c = a_0^1 X \otimes e^2 \wedge e^3 + a_0^2 X \otimes \left(e^2 \wedge e^5 - e^3 \wedge e^4\right) + \cdots.$$

Now evaluating dc on (e_2, e_3, e_4) gives

$$dc(e_2, e_3, e_4) = \pm a_0^1 X \cdot \delta_4 \pm a_0^2 X \cdot \delta_2,$$

and as c should be a cocycle, we conclude $a_0^1 = a_0^2 = 0$, because $X \cdot \delta_4 = \delta_5$ and $X \cdot \delta_2 = \delta_3$. Similarly, we can conclude that all a_0^i for $i \geqslant 1$ are zero by evaluating dc on the different e-triples. Finally

$$E_\infty^{1,1} = 0.$$

Now let us compute $E_\infty^{0,2}$. By Feigin–Fuchs, we have

$$E_1^{0,2} = \left(\bigoplus_{n \geqslant 0} \mathbb{R} Y^n X^2 \oplus \bigoplus_{n \geqslant 0} \mathbb{R} Y^n \delta_2 \oplus \bigoplus_{n \geqslant 0} \mathbb{R} Y^n X \delta_1 \oplus \bigoplus_{n \geqslant 0} \mathbb{R} Y^n \delta_1^2\right) \otimes_{\mathbb{R}} H^2(\mathfrak{m}_0),$$

and a general cochain $c \in E_1^{0,2}$ looks like

$$\begin{aligned} c = {} & \sum_{n \geqslant 0} a_n^1 Y^n X^2 \otimes e^2 \wedge e^3 + \sum_{n \geqslant 0} b_n^1 Y^n \delta_2 \otimes e^2 \wedge e^3 + \sum_{n \geqslant 0} c_n^1 Y^n X \delta_1 \otimes e^2 \wedge e^3 \\ & + \sum_{n \geqslant 0} d_n^1 Y^n \delta_1^2 \otimes e^2 \wedge e^3 + \sum_{n \geqslant 0} a_n^2 Y^n X^2 \otimes \left(e^2 \wedge e^5 - e^3 \wedge e^4\right) \\ & + \sum_{n \geqslant 0} b_n^1 Y^n \delta_2 \otimes \left(e^2 \wedge e^5 - e^3 \wedge e^4\right) + \sum_{n \geqslant 0} c_n^1 Y^n X \delta_1 \otimes \left(e^2 \wedge e^5 - e^3 \wedge e^4\right) \\ & + \sum_{n \geqslant 0} d_n^1 Y^n \delta_1^2 \otimes \left(e^2 \wedge e^5 - e^3 \wedge e^4\right) + \cdots. \end{aligned}$$

We compute once again

$$d_1^{0,2} c(e_1, e_2, e_3) = \sum_{n \geqslant 0} a_n^1 Y^n X^2 \cdot X + \sum_{n \geqslant 0} b_n^1 Y^n \delta_2 \cdot X + \sum_{n \geqslant 0} c_n^1 Y^n X \delta_1 \cdot X + \sum_{n \geqslant 0} d_n^1 Y^n \delta_1^2 \cdot X.$$

It is clear that the first sum cannot mix with the others, as there are no δ's in it. Thus $a_n^1 = 0$ for all $n \geqslant 1$. For the other terms, we have

$$\begin{aligned} Y^n \delta_2 \cdot X &= -Y^n \delta_3 + Y^{n-1} X \delta_2 + \cdots + Y X Y^{n-2} \delta_2 + X Y^{n-1} \delta_2, \\ Y^n X \delta_1 \cdot X &= -Y^n X \delta_2 + Y^{n-1} X^2 \delta_1 + \cdots + Y X Y^{n-2} X \delta_1 + X Y^{n-1} X \delta_1, \end{aligned}$$

and

$$Y^n \delta_1^2 \cdot X = -2 Y^n \delta_2 \delta_1 + Y^{n-1} X \delta_1^2 + \cdots + Y X Y^{n-2} \delta_1^2 + X Y^{n-1} \delta_1^2.$$

All sums over n are finite, therefore let us consider only the highest order in Y. In the fourth sum, there are always two δ's in the highest order term, they cannot match with the others and therefore $d_n^1 = 0$ for all $n \geqslant 0$. The terms in the second and third sum can match, but there is then a term coming from commuting one term into the other, which makes one highest coefficient zero. But then the other coefficient must be zero, too. Finally, $b_n^1 = c_n^1 = 0$ for all $n \geqslant 0$. The only term which can possibly be non-zero is thus a_0^1. The same reasoning applies to the other a_n^i, b_n^i, c_n^i and d_n^i with $i > 1$. We are then left with a cochain of the form

$$c = a_0^1 Y^n X^2 \otimes e^2 \wedge e^3 + a_0^2 Y^n X^2 \otimes \left(e^2 \wedge e^5 - e^3 \wedge e^4\right) + \cdots.$$

As before, we can then apply dc to other e_i-triples in order to show that the a_0^j must all be separately zero. Finally

$$E_\infty^{0,2} = 0. \qquad \square$$

Remark 1. The result from the previous three propositions can be interpreted as follows: The short exact sequence of augmentation

$$0 \to \left(\mathcal{H}^{\mathrm{ad}}\right)^+ \to \mathcal{H}^{\mathrm{ad}} \to \mathbb{R} \to 0$$

induces a long exact sequence in cohomology, and by the previous results, we deduce $H^0(\mathfrak{m}_0; \mathcal{H}^+) = 0$, $H^1(\mathfrak{m}_0; \mathcal{H}^+) = \mathbb{R}\delta_1 \otimes e^1$, and $H^2(\mathfrak{m}_0; \mathcal{H}^+) = 0$.

As we will see below, it is this term $\delta_1 \otimes e^1$ which gives rise to the only non-zero term in $HH^2(\mathcal{H})$.

3.2. *Computation of* $H^q(\mathfrak{h}; \mathcal{H}^{\mathrm{ad}})$

Here we perform the second step of the outline, i.e. we compute $H^q(\mathfrak{h}; \mathcal{H}^{\mathrm{ad}})$ for $q = 0, 1, 2$ knowing $H^q(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}})$ for $q = 0, 1, 2$ from the previous subsection, via the Hochschild–Serre spectral sequence.

Given a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module L and an ideal $\mathfrak{k}$, the spectral sequence converges to $H^*(\mathfrak{g}; L)$. As stated before, the E_2-term in this spectral sequence is

$$E_2^{p,q} = H^p\left(\mathfrak{g}/\mathfrak{k}; H^q(\mathfrak{k}; L)\right).$$

In our case with $\mathfrak{k} = \mathfrak{m}_0$, $\mathfrak{g} = \mathfrak{h}$ and $L = \mathcal{H}^{\mathrm{ad}}$, recall the 2-dimensional quotient Lie algebra $\mathfrak{a} = \mathfrak{h}/\mathfrak{m}_0$ generated by Y and δ_1 with the only (non-trivial) relation $[Y, \delta_1] = \delta_1$. Then this gives the following spaces:

$$\begin{aligned}
E_2^{0,0} &= H^0\left(\mathfrak{a}; H^0\left(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}}\right)\right) = \mathbb{R},\\
E_2^{1,0} &= H^1\left(\mathfrak{a}; H^0\left(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}}\right)\right) = H^1(\mathfrak{a}; \mathbb{R}),\\
E_2^{0,1} &= H^0\left(\mathfrak{a}; H^1\left(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}}\right)\right) = \left(H^1(\mathfrak{m}_0) \oplus \mathbb{R}\delta_1 \otimes e^1\right)^{\mathfrak{a}},\\
E_2^{2,0} &= H^2\left(\mathfrak{a}; H^0\left(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}}\right)\right) = H^2(\mathfrak{a}; \mathbb{R}),\\
E_2^{1,1} &= H^1\left(\mathfrak{a}; H^1\left(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}}\right)\right) = H^1\left(\mathfrak{a}; H^1(\mathfrak{m}_0) \oplus \mathbb{R}\delta_1 \otimes e^1\right),\\
E_2^{0,2} &= H^0\left(\mathfrak{a}; H^2\left(\mathfrak{m}_0; \mathcal{H}^{\mathrm{ad}}\right)\right) = H^2(\mathfrak{m}_0)^{\mathfrak{a}}.
\end{aligned}$$

Remark 2. A big difference between the Lie algebras $\mathfrak{h}$ and $\mathfrak{m}_0$ is that $\mathfrak{h}$ has a so-called *Euler element* and $\mathfrak{m}_0$ does not. This means that $\mathfrak{h}$ possesses a basis consisting of eigenvectors with respect to the adjoint action of Y, while no element acts (in the adjoint action) diagonally on $\mathfrak{m}_0$. In other words, the grading of $\mathfrak{h}$ is implemented by a grading element Y, while the grading of $\mathfrak{m}_0$ is not given by an inner derivation. The existence of an Euler element in $\mathfrak{h}$ implies its existence in $\mathfrak{a}$.

The grading on both Lie algebras induces (second) gradings on cochains spaces. We will follow Fuchs' convention [10, p. 29], and write for a graded Lie algebra $\mathfrak{g}$ and a graded $\mathfrak{g}$-module A

$$C^q_\lambda(\mathfrak{g};A) = \left\{c \in C^q(\mathfrak{g};A) \;\middle|\; \forall X_i \in \mathfrak{g}_{\lambda_i}\colon c(X_1,\ldots,X_q) \in A_{\lambda_1+\cdots+\lambda_q-\lambda}\right\}. \tag{1}$$

By Theorems 1.5.2 and 1.5.2*a* in [10], the subcomplex of degree-0-cochains for $\mathfrak{g}=\mathfrak{h}$ and a graded module A admitting a basis of eigenvectors for the action of Y is homotopy equivalent to the total complex. The same is true for $\mathfrak{a}$, but is not for the Lie algebra $\mathfrak{m}_0$.

Proposition 4.

$$H^1(\mathfrak{a};\mathbb{R}) = \mathbb{R}Y^* \quad \textit{and} \quad H^2(\mathfrak{a};\mathbb{R}) = 0.$$

Proof. The 2-dimensional Lie algebra $\mathfrak{a}$ admits a grading where Y is the grading element of degree 0 and δ_1 is of degree 1. This grading induces a grading on all cochain spaces. By Theorem 1.5.2 in [10], for cohomology computations one may restrict to the degree-0-subcomplex. This means the subcomplex given by $\mathbb{R} \subset C^0(\mathfrak{a};\mathbb{R})$, $\mathbb{R}Y^* \subset C^1(\mathfrak{a};\mathbb{R})$ and $0 \subset C^2(\mathfrak{a};\mathbb{R})$.

This implies the proposition. □

Proposition 5.

$$\left(H^1(\mathfrak{m}_0) \oplus \mathbb{R}\delta_1 \otimes e^1\right)^{\mathfrak{a}} = \mathbb{R}\delta_1 \otimes e^1.$$

Proof. Y and $\delta_1 \in \mathfrak{a}$ act trivially on $\mathbb{R}\delta_1 \otimes e^1$, because the cochain is of degree 0 with respect to Y and $e^1 \cdot \delta_1 = \pm e^1([\delta_1,-]) = 0$.

On the other hand, Y acts non-trivially on $1 \otimes e^1$ and $1 \otimes e^2 \in H^1(\mathfrak{m}_0)$, because $[Y,X]=X$ and X corresponds to e^1 and $[Y,\delta_2] = 2\delta_2$ in $\mathfrak{h}$ (and δ_2 corresponds to e_2). Thus only $\delta_1 \otimes e^1$ is invariant. □

Proposition 6.

$$H^2(\mathfrak{m}_0)^{\mathfrak{a}} = 0.$$

Proof. Observe that Y acts as a grading element on $H^2(\mathfrak{m}_0)$, i.e. every element of $H^2(\mathfrak{m}_0)$ is Y-eigenvector with non-trivial eigenvalue. It is clear from this and from the explicit description of $H^2(\mathfrak{m}_0)$ in Theorem 2 that no non-trivial combination of the generators will be Y-invariant. □

Proposition 7.

$$H^1\left(\mathfrak{a}; H^1(\mathfrak{m}_0) \oplus \mathbb{R}\delta_1 \otimes e^1\right) = \mathbb{R}Y^* \otimes \left(\delta_1 \otimes e^1\right).$$

Proof. Once again, Lie algebra and module admit a basis consisting of eigenvectors with respect to Y. In this situation, as before, one may restrict to the subcomplex of cochains of degree 0, see [10], Theorem 1.5.2*a*.

Let us compute the degrees of $1 \otimes e^1$ and $1 \otimes e^2$ using formula (1): $1 \otimes e^1$ is of degree 1, because $1 \cdot e^1(e_1) \in \mathbb{R} \subset (\mathcal{H})_0$. In the same way, $1 \otimes e^2$ is of degree 2 and $(1 \otimes e^2) \cdot Y = 2(1 \otimes e^2)$.

Now it is clear, as Y^* is of degree 0 and δ_1^* of degree 1, that there can be built no cochain of degree 0 from $\mathrm{Hom}(\mathfrak{a}; H^1(\mathfrak{m}_0))$.

On the other hand, $\delta_1 \otimes e^1$ is of degree 0 with respect to Y, and $Y^* \otimes (\delta_1 \otimes e^1)$ is obviously a cocycle in $C^1(\mathfrak{a}; H^1(\mathfrak{m}_0) \oplus \mathbb{R}\delta_1 \otimes e^1)$. □

Now the table looks like:

$$E_2^{0,0} = \mathbb{R},$$
$$E_2^{1,0} = \mathbb{R}Y^*,$$
$$E_2^{0,1} = \mathbb{R}\delta_1 \otimes e^1,$$
$$E_2^{2,0} = 0,$$
$$E_2^{1,1} = \mathbb{R}Y^* \otimes \big(\delta_1 \otimes e^1\big),$$
$$E_2^{0,2} = 0.$$

As a corollary, we have:

Theorem 3.

$$H^1\big(\mathfrak{h}; \mathcal{H}^{\mathrm{ad}}\big) = HH^1(\mathcal{H})$$

is 2-dimensional, generated in terms of Lie algebra cocycles by Y^* and $\delta_1 \otimes e^1$.

As the differential

$$d_2^{1,1} : E_2^{1,1} \to E_2^{3,0}$$

is zero by dimensional reasons (because $E_2^{3,0} = H^3(\mathfrak{a}; \mathbb{R})$ is the degree 3 cohomology of a 2-dimensional Lie algebra), we have as another corollary:

Theorem 4.

$$H^2\big(\mathfrak{h}; \mathcal{H}^{\mathrm{ad}}\big) = HH^2(\mathcal{H})$$

is 1-dimensional, generated in terms of Lie algebra cocycles by $X^* \wedge Y^* \otimes \delta_1$.

Remark 3. Observe that all $p = \text{const}$ columns with $p > 3$ on the second page of the Hochschild–Serre spectral sequence are zero as $\mathfrak{a}$ is 2-dimensional. But even knowing this, the computation of higher dimensional cohomology spaces is much more complicated.

Remark 4. Computing the Hochschild cohomology $HH^*(\mathcal{H}; \mathbb{R})$ with trivial coefficients is rather easy along the lines of the above computation of $HH^2(\mathcal{H})$. Indeed, the third step (involving the Feigin–Fuchs spectral sequence) is trivial, thus it remains to explore the Hochschild–Serre spectral sequence. Putting in the previous computations, its second page looks in this case (in part) like:

$$\begin{array}{lll}
E_2^{0,3} = H^3(\mathfrak{m}_0)^{\mathfrak{a}}, & E_2^{1,3} = H^1\big(\mathfrak{a}; H^3(\mathfrak{m}_0)\big), & E_2^{2,3} = H^2\big(\mathfrak{a}; H^3(\mathfrak{m}_0)\big), \\
E_2^{0,2} = H^2(\mathfrak{m}_0)^{\mathfrak{a}} = 0, & E_2^{1,2} = H^1\big(\mathfrak{a}; H^2(\mathfrak{m}_0)\big), & E_2^{2,2} = H^2\big(\mathfrak{a}; H^2(\mathfrak{m}_0)\big), \\
E_2^{0,1} = H^1\big(\mathfrak{m}_0\big)^{\mathfrak{a}} = 0, & E_2^{1,1} = H^1\big(\mathfrak{a}; \mathbb{R}Y^*\big), & E_2^{2,1} = H^2\big(\mathfrak{a}; \mathbb{R}Y^*\big), \\
E_2^{0,0} = \mathbb{R}, & E_2^{1,0} = \mathbb{R}Y^*, & E_2^{2,0} = H^2(\mathfrak{a}; \mathbb{R}) = 0.
\end{array}$$

By degree arguments, $H^p(\mathfrak{m}_0)^{\mathfrak{a}} = 0$ for all $p \geqslant 1$ and $H^2(\mathfrak{a}; H^q(\mathfrak{m}_0)) = 0$ for all $q \geqslant 2$. Indeed, Theorem 2 shows that $H^p(\mathfrak{m}_0)$ for $p \geqslant 1$ is generated by cocycles which are never Y-invariant. On the other hand, the cochains computing $H^2(\mathfrak{a}; H^q(\mathfrak{m}_0))$ must be of the form $Y^* \wedge \delta_1^* \otimes H^q(\mathfrak{m}_0)$, and are therefore of degree 0 only if the cochain in $H^q(\mathfrak{m}_0)$ is of degree 1. This can only happen for $q = 1$ by Theorem 2. The second page becomes thus:

$$\begin{array}{lll} E_2^{0,3} = 0, & E_2^{1,3} = 0, & E_2^{2,3} = 0, \\ E_2^{0,2} = 0, & E_2^{1,2} = 0, & E_2^{2,2} = 0, \\ E_2^{0,1} = 0, & E_2^{1,1} = \mathbb{R}\delta_1^* \wedge e^1, & E_2^{2,1} = \mathbb{R}Y^* \wedge \delta_1^* \wedge e^1, \\ E_2^{0,0} = \mathbb{R}, & E_2^{1,0} = \mathbb{R}Y^*, & E_2^{2,0} = 0. \end{array}$$

As a consequence, the spectral sequence collapses at the second page, and one obtains therefore the result is

$$HH^l(\mathcal{H}; \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{if } l = 0, 1, 2, 3, \\ 0 & \text{if } l \geqslant 4. \end{cases}$$

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